RESEARCHES IN RELATIVITY

I. CRITICISM AND MODIFICATION OF EINSTEIN'S LATEST MANIFOLD

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Art.1. Introductory.

This is the first of a series of investigations concerning which details will be given at the end of the paper.

In "Nature" of September 1923 p.448 ("The Theory of the Affine Field") Einstein describes a remarkable mathematical discovery. He develops from Eddington's previous work a perfectly satisfactory basis for the Riemann manifold required for mechanics and gravi- -tation in the general theory of Relativity. His main object, however, was to include provision also for the electro-magnetic field, but it would appear that his conclusions on this part of the development are irreconcilable with observation. If he had examined in detail the mechanical stresses, which he could have done by the conditions imposed by the fact that the action is an invariant density, he would have found that his constant Υ cannot be "practically in- -definitely small" as he states it should be for an- -other reason. The truth of this remark will appear in the subsequent work.

It is, however, easy to modify the principle from which he works by increasing the 40 scalars from which he starts to 41. It is only necessary to ass- -ume that in a parallel displacement α of a four di- -mensional element $db = dx_1 dx_2 dx_3 dx_4$ the element suffers a definite intrinsic increase in bulk. If at a selected point the measure of the bulk of an element db is taken to be $dB = e^x db$, then the measure of bulk of a second element db' will be $dB' = e^x db'$. Our addition to Einstein's prin- -ciple is that the proportional increment of bulk namely $(d_\alpha^{PD}. dB)/dB = V_0\alpha(\Delta x - E_\epsilon \epsilon) = V_0 \alpha^\epsilon \lambda$

(1)

is intrinsic, that is $\Delta x - E_\epsilon \epsilon$ is covariant (and therefore put $= {}^c\lambda$), and that the action remains stationary when x as well as the other forty scalars is arbitrarily varied.

Art. 2. Statement of the results.

The notation here used will be found in three pap--ers by the present writer (1) to which the reader is referred.

We shall use $E_\alpha, F_\alpha, C_\omega, \Omega$ for our former $E_\alpha', F_\alpha', C_\omega', \Omega'$, also we shall write $-\Delta = \nabla$ in place of ∇, and (ϵ, ϵ) in place of $-(\zeta, \zeta)$; also ${}^lQ, {}^dQ, {}^cQ$ in place of $Q^{\cdot}, Q^{\cdot\times}, Q^{\times}$. The rea--sons for these changes in notation will be found in Art. 5. below.

Einstein's tensor symbols $\mathfrak{H}, \mathfrak{H}^*, \Gamma^\sigma_{\mu\nu}, R_{\mu\nu}, \gamma_{\mu\nu}, \phi_{\mu\nu}, g^{\mu\nu}, \mathfrak{f}^{\mu\nu}, \mathfrak{i}^\mu$,

are the equivalents of our multenion symbols

${}^lH, {}^lH^*, C_\omega, \Omega, \psi, {}^c\omega, {}^d\psi, {}^l\omega, {}^l\kappa$.

$-E_\alpha'\beta$ is the increment in β (contravariant) due to a paralell displacement α. From it the invariant--ive $C_\omega {}^c\gamma$ is formed expressing the increment in ${}^c\gamma$ when it is moved by parallel displacements round an infinitesimal circuit $\omega = V_1\alpha\beta$. C_ω gives by the process known as contraction the coexcontra $\Omega\alpha$ where

$$\left.\begin{aligned}\Omega\alpha &= -E_{g\alpha}\Delta_g + V_0\alpha\Delta . E_\epsilon\epsilon \\ &\quad + E_\epsilon E_\alpha\epsilon - E_\alpha E_\epsilon\epsilon \\ &= \psi\alpha - \tfrac{1}{2}V_1{}^c\omega\alpha\end{aligned}\right\} \qquad (2)$$

where ψ is the self-conjugate part of Ω. ${}^lH$ is a function of the 41 scalars of (E_α, x) and in par--ticular is assumed to be a function of Ω and ${}^c\lambda$. The 41 scalars (E_α, x) are given arbitrary variations which are zero at the boundary of any arbitrarily chosen region, and it is assumed that $\int^n\!\int {}^lH db$ remains stationary. Thus 41 equations are furnished

for the original scalars (E_α, x).

Define ${}^d\Omega, {}^d\psi, {}^l\omega, {}^c\lambda$ by the equations

$$\left.\begin{aligned} d^lH &= V_0{}^d\Omega e d\Omega e + V_0{}^l\lambda d{}^c\lambda \\ &= V_0{}^d\psi e d\psi e + V_0{}^l\omega d{}^c\omega \\ &\quad + V_0{}^l\lambda d{}^c\lambda \end{aligned}\right\} \qquad (3)$$

The notation (e, e) is defined in Art. 5 below as $\Sigma(i, i^{-1})$.

By (3) $\qquad {}^d\Omega{}^c\alpha = {}^d\psi{}^c\alpha + V_1{}^l\omega{}^c\alpha \qquad (4)$

and ${}^d\psi$ is the self-conjugate part of ${}^d\Omega$.

I will now mention the identifications which we make of these symbols in the combined gravitational and electro-magnetic field, then state the general results that follow from the principle that has been laid down. In the next article an outline of the quite simple reduction which leads to these results is given.

${}^d\psi$ is identified with $l\theta^{-1} = {}^d\theta$ where $l = |\theta^{\frac{1}{2}}|$ and θ is looked on as the linity furnishing a standard or fundamental quadratic form $V_0 d\rho\theta d\rho$

${}^l\omega$ is the magnetic force cum displacement ${}_nV_2$: (we shall consider our manifold to be of n dimensions from henceforth). ${}^l\lambda$ is the momentum-mass-energy vector. ψ is the curvature linity; the total energy linity ${}^dT$ has exactly the same relation with this as is usually postulated for the gravitational field. $-{}^c\omega$ (our former $-\omega^\times$) is the magnetic induction cum electrostatic force. Finally ${}^c\lambda$ is the potential vector. Note very particularly that the results, about to be given, in no wise require us to state what is the form of ${}^lH$ in terms of Ω and ${}^c\lambda$, and yet they contain practically the whole of what is be--lieved in the general theory of relativity.

The results that follow from our action-principle are

$${}^l\kappa = -V_1\Delta{}^l\omega, \; V_0\Delta{}^l\lambda = V_0\Delta{}^l\kappa = 0 \qquad (5)$$

(The first of these is to be regarded as a definition of the symbol ${}^l\kappa$.)

$$({}^l\lambda + {}^l\kappa) + \tfrac{1}{2}(n-1)(n-2)\,{}^d\theta({}^c\lambda - \Delta y) = 0 \qquad (6)$$

The statement $V_0\Delta\,{}^l\lambda = 0$ expresses the conservation of mass-energy, a highly satisfactory deduction from the present theory.

$${}^c\omega = V_2\Delta{}^c\lambda\,, \qquad V_3\Delta\,{}^c\omega = 0 \qquad (7)$$

${}^l\mathrm{H}^*$ is the function reciprocal to ${}^l\mathrm{H}$ with regard to Ω and ${}^c\lambda$ and is therefore taken as an explicit function of ${}^d\Omega$ and ${}^l\lambda$. Thus

$$\left.\begin{aligned}
{}^l\mathrm{H} + {}^l\mathrm{H}^* &= V_0\Omega e\,{}^d\Omega e + V_0{}^c\lambda\,{}^l\lambda \\
&= V_0\psi e\,{}^d\psi e + V_n{}^c\omega\,{}^l\omega \\
&\quad + V_0{}^c\lambda\,{}^l\lambda \\
{}^d\psi = {}_\psi\mathbb{C}\,{}^l\mathrm{H},\quad & {}^l\omega = {}_{c\omega}\Delta\,{}^l\mathrm{H}, \\
{}^l\lambda = {}_{c\lambda}\Delta\,{}^l\mathrm{H},\quad & \psi = {}_{d\psi}\mathbb{C}\,{}^l\mathrm{H}^*, \\
{}^c\omega = {}_{l\omega}\Delta\,{}^l\mathrm{H}^*,\quad & {}^c\lambda = {}_{l\lambda}\Delta\,{}^l\mathrm{H}^*
\end{aligned}\right\} \qquad (8)$$

$${}^l\mathrm{T}\,{}^c\alpha = 2\psi\,{}^d\theta - V_{0\epsilon}\psi\,{}^d\theta\epsilon \qquad (9)$$

$$V_1({}^l\lambda - {}^l\kappa)\,{}^c\omega + {}^l\mathrm{T}_\epsilon\,\epsilon = 0 \qquad (10)$$

The kind of absolute differentiation here indicated by the suffix ϵ is that which results from a Riemann manifold: it is the ordinary absolute differentiation which is meant: generally below this is not the case, the differentiation requiring the use of our present E_α in place of that portion of it, F_α, defined in equations (12),(13) below. Next we have

$${}^c\lambda = \Delta y + (\Delta \log l - \mathrm{E}_\epsilon\,\epsilon) \qquad (11)$$

where y is the invariant $x - \Delta \log l$; (x is not invariant); and

$$\left.\begin{aligned}
\mathrm{E}_\alpha'\beta &= \theta^{-1}\Gamma_\alpha'\beta, \quad \Gamma_\alpha'\beta = \Theta_\alpha'\beta + \Phi_\alpha'\beta, \\
\Theta_\alpha'\beta &= \tfrac{1}{2}(V_0\alpha\Delta\,.\,\theta\beta + V_0\beta\Delta\,.\,\theta\alpha - \Delta V_0\alpha\theta\beta), \\
(n-1)(n-2)\,\Phi_\alpha'\beta &= \theta[\alpha V_0\beta\theta(\lambda+\kappa) \\
&\quad + \beta V_0\alpha\theta(\lambda+\kappa) \\
&\quad - (n-1)(\lambda+\kappa)V_0\alpha\theta\beta]
\end{aligned}\right\} \qquad (12)$$

In connection with (12) it is convenient to define $\mathrm{F}_\alpha'\beta$ and $\mathrm{G}_\alpha'\beta$ thus

$$\left.\begin{aligned}
\mathrm{F}_\alpha'\beta = \theta^{-1}\Theta_\alpha'\beta, &\quad \mathrm{G}_\alpha'\beta = \theta^{-1}\Phi_\alpha'\beta \\
\mathrm{E}_\alpha'\beta &= \mathrm{F}_\alpha'\beta + \mathrm{G}_\alpha'\beta
\end{aligned}\right\} \qquad (13)$$

of two parts of $\Gamma_\alpha'\beta$ in (12) the first $\Theta_\alpha'\beta$ is not invariantive, and the second is. Similarly for (13)

It is the results given, up to (12), of which we have postponed till next article the simple establish--ment. We will here first make some remarks on (9) and (10) and the following obtained by eliminating the potential vector ${}^c\lambda$ from the above results,

$$-\tfrac{1}{2}(n-1)(n-2)\,{}^c\omega = V_2\Delta[{}^d\theta^{-1}({}^l\lambda + {}^l\kappa)] \qquad (14)$$

Our remarks will be on (A) the nature of the con--nection of (10) with the motion of matter and the modification of its form to one more near the usual; (B) the breaking up of (10) into three separate inde--pendent equations, the first involving ${}^d\theta$ only, the second ${}^d\theta$ and ${}^l\kappa$ only, and the third involving all three ${}^d\theta$, ${}^l\kappa$, and ${}^l\lambda$; and (C) the novelty of the expression for ${}^c\lambda$ in (6) and for ${}^c\omega$ in (14).

(A). Equation (10), with (9) to explain the mean--ing of ${}^d\mathrm{T}$, is an identity which does not in the least depend on our principle of stationary action but is true simply because ${}^{l}\mathrm{H}$ is an invariant density. If, therefore, following Einstein, the reader should try some special assumption such as that ${}^{l}\mathrm{H}^{*} = -\frac{1}{2}l^{-1}\mathrm{V}_0{}^{l}\omega\theta{}^{l}\omega$ and endeavour to derive information from (10) he will be disappointed; after his reduction he will discover that (10) gives $0=0$ and nothing else. The utility of (10) lies in the very fact that he will find this whatever legitimate special form he gives to ${}^{l}\mathrm{H}$; in a word, it is an identity. The very simple term $\mathrm{V}_1{}^{l}\lambda{}^{c}\omega$ in (10) looks very different from the Newtonian md^2x/dt^2 or the usual generalisation thereof that appears in general relativity. But if we take the form in which it actually presents itself below in the next article, namely as

$$ {}^{c}\lambda\mathrm{V}_0\Delta{}^{l}\lambda + \mathrm{V}_1{}^{l}\lambda{}^{c}\omega = {}^{c}\lambda\mathrm{V}_0\Delta{}^{l}\lambda + \mathrm{V}_1{}^{l}\lambda\mathrm{V}_2\Delta{}^{c}\lambda $$

we are at once guided to

$$ \mathrm{V}_1{}^{l}\lambda{}^{c}\omega = {}^{c}\lambda_g\mathrm{V}_0\Delta_g{}^{l}\lambda_g - \Delta_g\mathrm{V}_0{}^{c}\lambda_g{}^{l}\lambda \tag{15} $$

The second term on the right is made invariantive by adding to the same side of the equation $\mathrm{E}_{l\lambda}{}^{c}\lambda$ and therefore the first term by subtracting the same expression from the same side, for the left side is invariantive; we will ignore these invariantive additions in our discussion.

The first term on the right of (15) is the accepted proper expression for the term corresponding in the general theory of relativity to Newton's $-md^2x/dt^2$ The second term is to be interpreted as a part of the radiation pressure which in the present theory is probably equal to ${}^{l}\mathrm{H}$. We reserve till our second paper a full discussion of this term $-\Delta_g\mathrm{V}_0{}^{c}\lambda_g{}^{l}\lambda$.

In anticipation, however, it is of interest to mention that I have already verified (1) that the identification of ${}^{l}\mathrm{H}$ with the radiation pressure leads to essentially the current views thereof, but (2) that the term, while leaving "large scale phenomena" unaffected indicates a "flat contradiction to the Newtonian laws" when we attempt to apply them to "small scale phenomena" (Jeans' Report on the Quantum Intro. chap.)

Another aspect of the term is that ${}^{c}\lambda$ is the vector pontential. As usual the physical interpretation of ${}^{c}\lambda$ must be sought in its effect on ${}^{c}\omega$. The expression for ${}^{c}\omega$ in (14) is peculiar to the present theory, and is interesting in itself. It suggests what we know to-day for a fact, that wheresoever is a current there also is momentum. It is ordinarily assumed that in the ether both terms on the right of (14), or at anyrate the second, are zero. Relativists hitherto have failed to provide any invariantive flux of energy which shall serve to transfer energy across interstellar space. The present theory provides ${}^{l}\lambda$ (closely associated with the vector potential ${}^{c}\lambda$) to perform this duty.

The very direct manner in which both in (10) and in (14) the momentum vector ${}^{l}\lambda$ comes into comparison with the current vector ${}^{l}\kappa$ was no surprise to me. It seemed to me a merit and not a defect in the theory. In May 1923 I sent a series of seven letters to "Nature" which would have occupied in all less than seven pages, discussing the phenomena of quanta in the ether from the point of view of relativity. (The letters were returned as unsuitable for "Nature"). I pointed out that if relativity were true and also the conservation of energy, we were practically compelled to believe that in empty space the momentum vector ${}^{l}\lambda$ exists and satisfies precise conditions. The only simple view seemed to be that there was some universal natural constant expressing that wherever there was electric current there was also a propor--tional amount of material momentum. I went so far as tentatively to suggest that the factor of convers-ion was m_1/e where e is the charge of an electron or a proton, taken as positive, and $m_1 = mm_0/(m+m_0)$ where m and m_0 are the masses of a proton and elect--ron respectively; but I would not have it supposed that this is in any way involved in our present theory. Indeed I can see several reasons for thinking the ratio of conversion may be of quite a different order of magnitude.

(B) Notice that when we regard (2) as defining ψ in terms of E_α, the expression for ψ is linear in E_α so far as the two terms in Δ are concerned and is quadratic in the other two terms. In the same way E_α itself by (12) and ${}^c\omega$ by (14) are expressed lin--early in terms of (Δ_g, θ_g), ${}^l\kappa$ and ${}^l\lambda$. We may suppose these values of ψ and ${}^c\omega$ substituted in the identity (10) which thus comes to be expressed in terms of the three symbols (involving twenty scalars) and their derivatives up to the second order. We have shown that this identity is true for arbitrary inde--pendent values of the three. It must therefore sep--arate into three identities which are easily obtained in invariant form. If we put ${}^l\lambda = {}^l\kappa = 0$ we get the already famous identity for a Riemann mani--fold. (10) itself is a second of these three ident--ities. The third is most conveniently obtained by putting ${}^l\lambda = 0$. This is the case which should have been considered by Einstein. Although the momen--tum vector does not appear, yet material force (in our notation $V_1{}^c\omega{}^l\lambda$) due to current is brought into direct connection with the curvature, and as a conse--quence it seems to me that we ought to have $\gamma = 6\beta$ where β and γ are Einstein's constants. If this is correct, as already remarked, his theory is inconsis-tent with observation.

(C) It seems probable that in (6) Δy is as a

rule the chief term in the expression for ${}^c\lambda$ and that y is a natural measure of cosmic time, very nearly proportional to co-ordinate time when taken in a usual manner. If this is so $d\theta{}^c\lambda$ will be large compared with ${}^l\lambda$.

We may think of $V_0{}^l\lambda\Delta y$ as the true expression for material density in ordinary units; if so ${}^l\lambda\sqrt{(V_0\Delta y\theta^{-1}\Delta y)}$ would be very nearly the pro--per expression for the momentum vector density in the same units, with the same sense of "very nearly" as in the phrase two sentences above.

Further at this present moment of writing I lean towards the surmise that the total mass in the uni--verse is exactly zero (as we all surmise about the charge). From the known data of the stars in the neigh--bourhood of the solar system we may suppose there is in such neighbourhood a uniform negative density of the order 1/200 ergs per c.c. on which is superposed the positive density due to radiation, about 2 close to the sun's surface and about 1/20,000 near the earth. I would suggest that possibly we have here (in the endowment of the ether with a negative material den--sity) a hopeful method of overcoming some of the out-standing difficulties of radiation.

In this connection it may be observed that from (14) ${}^l\lambda$ and ${}^l\kappa$ cannot both in interstellar space be zero, and the former in particular must exist to account for energy flux. ${}^l\kappa$ is always assumed to be zero in em--pty space, but in my opinion without due warrant. The absence of dissipation of energy seems to me to involve only the less drastic condition

$$V_0\kappa\theta\kappa = 0$$

Art. 3. Proof of the results.

Before establishing the results of Art. 2 it is des--irable to consider how expressions of the forms (Δ_g, Q_g) and $(\Delta_g, {}^lQ_g)$ are by supplementary add--itions made invariantive. Q is an invariantive multenion or multenion linity and ${}^lQ$ a corresponding invariantive density. In the papers already referred to the six standard cases for Q have been fully dealt with by aid of our present $E_\alpha{}^c\beta$, and to deal with ${}^lQ$ is a simple matter. If ${}^lk$ is any scalar such that ${}^lk\,db$ is invariant and l is a particular case of ${}^lk$ which we take as a standard, then, of course, ${}^lk/l$ is an invariant and the difference of any two of the three vectors $E_\epsilon\epsilon$, $\Delta \log {}^lk$ and $\Delta \log l$ is a covariant vector. In the case of Q the supplementary addition may be written $(\epsilon, S_\epsilon Q)$ where $S_\epsilon Q$ is a definite bilinear function of ϵ and Q . (For instance, if $Q = \xi$ is a coexcontra multenion linity then if $({}^c\alpha, Q)$ is invariantive $(\Delta_g, Q_g) + (\epsilon, S_\epsilon Q)$ is also invariantive provided $-S_\epsilon\xi = E_\epsilon\xi + \xi E_\epsilon{}'$). In the case of ${}^lQ$ we have in addition to change Δ_g to any one of the three $\Delta_g - E_\epsilon\epsilon$, $\Delta_g - \Delta \log {}^lk$ or $\Delta_g - \Delta \log l$. This new supplement may however be incorporated with S_ϵ by changing S_ϵ to $S_\epsilon - V_0\epsilon E_{\epsilon'}\epsilon'$ and, when we please, we may use S_ϵ for this fuller form.

Let us now consider the application of our principle. In the first place it greatly simplifies matters to observe that in using equation (2) above to express $\delta\,{}^lH$ in terms of $\delta\Omega$ and $\delta\,{}^c\lambda$ the third and fourth terms of Ω can be entirely ignored. By making invariantive every definite result obtained from our principle we ensure that the effect of these terms is taken account of. (If the reader does not consider this statement easy to establish he may use the ignored terms throughout his work and he will find that he is led to precisely our results). We shall therefore only write down the non-ignored terms, just as if they formed the whole of the expressions under consideration, except when we come to the definite stages of our results which we shall mark by numbered equations. In these we shall always insert the terms necessary to make the expressions invariantive. With this explanation we have:-

$$0 = \int^n\!\!\int \delta\,{}^lH\,db = \int^n\!\!\int db[V_0{}^d\Omega_\epsilon\delta\Omega_\epsilon + V_0{}^l\lambda\,\delta(\Delta x - E_\epsilon\epsilon)] \,.$$

From the term in x we at once derive from our principle, that the equation of conservation of mass-energy, $V_0\Delta\,{}^l\lambda = 0$, is true; and we may drop the x term. Thus

$$\begin{aligned} 0 &= \int^n\!\!\int db[V_0{}^d\Omega_\epsilon(-\delta E_{g.\epsilon}\Delta_g + V_0\epsilon\Delta.\delta E_{\epsilon'}\epsilon') \\ &\quad - V_0{}^l\lambda\,\delta E_\epsilon\epsilon] \\ &= \int^n\!\!\int db[V_0\delta E_\epsilon\Delta_g{}^d\Omega_g\epsilon - V_0\delta E_\epsilon\epsilon({}^d\Omega_g\Delta_g + {}^l\lambda)] \end{aligned}$$

Hence

$$\begin{aligned} 0 &= V_0\delta E_\epsilon\epsilon'[V_0\epsilon'\Delta_g{}^d\Omega_g\epsilon \\ &\quad - V_0\epsilon\epsilon'.({}^d\theta_g\Delta_g + {}^l\kappa + {}^l\lambda)] \\ 0 &= V_0\beta\Delta.{}^d\theta\,{}^c\alpha \\ &\quad - \tfrac{1}{2}V_0\beta\,{}^c\alpha.({}^d\theta_g\Delta_g + {}^l\kappa + {}^l\lambda) \\ &\quad - \tfrac{1}{2}V_0({}^d\theta_g\Delta_g + {}^l\kappa + {}^l\lambda)\,{}^c\alpha.\beta \end{aligned}$$

The last of these equations follows from the preceding one when it is observed that the forty scalars of $\delta E_\alpha{}^c\beta$ are arbitrary and that from the symmetrical condition $E_\alpha{}'\beta = E_\beta{}'\alpha$ it follows that $E_\alpha{}^c\beta$ is self-conjugate in ${}^c\beta$. Putting (ϵ, ϵ) for $(\beta, {}^c\alpha)$ in the last equation we obtain

$$\frac{{}^d\theta_g\Delta_g + E_\epsilon{}'{}^d\theta\epsilon}{1 + n} = \frac{{}^l\kappa + {}^l\lambda}{1 - n} \qquad (16)$$

$$= \frac{{}^d\theta_g\Delta_g + E_\epsilon{}'{}^d\theta\epsilon + ({}^l\kappa + {}^l\lambda)}{2}$$

Therefore
$$\begin{aligned} 0 = {} & V_0\beta\Delta.{}^d\theta\,{}^c\alpha + E_\beta{}'{}^d\theta\,{}^c\alpha \\ & + {}^d\theta E_\beta{}^c\alpha - {}^d\theta\,{}^c\alpha V_0\beta E_\epsilon\epsilon \\ & + [V_0\beta\,{}^c\alpha.({}^l\kappa + {}^l\lambda) \\ & + V_0({}^l\kappa + {}^l\lambda)\,{}^c\alpha.\beta]/(n-1) \qquad (17) \end{aligned}$$

In (17) put $d\theta{}^c\alpha = \epsilon$ and operate on the equation by $V_0\epsilon(\ )$. The equation may be put in the form $V_0\beta[\ ] = 0$ where [ ] is, of course, a covari-ant vector which our principle shows to be equal to zero. We thus obtain equation (6) above. Returning to the equation from which we derived this result express it in terms of θ where $d\theta = l\theta^{-1}$ as usual, and making it invariantive, we obtain (12) thus:- Equation (17) may be written $0 = d\theta_\beta{}^c\alpha +$ terms in $({}^l\kappa + {}^l\lambda)$ where the suffix β indicates absolute differentiation in the E_β sense. Writing $l^{-1}({}^l\kappa + {}^l\lambda)$ as $(\lambda+\kappa)$, the last equation in turn becomes $0 = \theta_\beta\alpha +$ terms in $\theta(\lambda+\kappa)$ or in full

$$\begin{aligned} 0 = {} & V_0\beta\Delta\,.\,\theta\alpha - E_\beta\theta\alpha - \theta E_\beta{}'\alpha \\ & + (1-n)^{-1}\theta[V_0\beta\theta\alpha\,.\,(\lambda+\kappa) \\ & + V_0(\lambda+\kappa)\,\theta\alpha\,.\,\beta \\ & + 2(2-n)^{-1}V_0\beta\theta(\lambda+\kappa)\,.\,\alpha] \end{aligned}$$

(The third term within the square brackets arises from $E_\epsilon\epsilon$ when we change from $d\theta_\beta$ to θ_β above). The establishment of (12) from this equation is a familiar process, and in multenions it takes the fol-lowing form. From this equation write down two other equations, the first by interchanging α and β, the second being the negative conjugate of the original with regard to β; then add the three equations to-gether. It may be noted that (16) is a result in addition to those given in Art. 2. We may consider that we have now established all the results of Art.2 except the identity (10).

(10) is obtained by the process fully exhibited in the paper already referred to, where the identity depending on the invariants of $l^{-1}f$ was considered. Here, however, we will treat the matter much more generally. We will first consider a general theorem of the type of (10), and afterwards enunciate a still

more general theorem.

Let ${}^lx$ be an invariant density function of any number of covariant hyper-vectors (of which ${}^cw$ is taken as the type) and of any number of coexcontra multenion linities of the type ${}_a\xi_b$, which is used to denote a linity of the form $\Sigma{}^cu\,V_0{}^cv(\ )$. (As usual in multenions $u, v, w,$ stand always for hyper-vectors ${}_nV_a$, ${}_nV_b$, ${}_nV_c$ respectively). The con-dition that ${}^lx$ is an invariant density may be stated as that

$$-{}^lx\,V_0\Delta\sigma = \delta'\,{}^lx \qquad (18)$$

The σ here used is put in place of the ϵ represent-ing an infinitesimal change of coordinates in my papers already referred to, and in the same papers δ' is explained as expressing a definite kind of in-crement in any symbol depending on this change. $\delta'\,{}^lx$ may be explicitly put in the form $-V_0\sigma_g\,{}^lU\Delta_g$ so that since all the derivatives of σ are arbitary we get the identity ${}^lU - {}^lx = 0$ or, say, ${}^lX = 0$ where ${}^lX$ stands for ${}^lU - {}^lx$. This is the necessary and sufficient condition that ${}^lx$ is an invariant density. (10) expresses that ${}^lX_g\Delta_g = 0$ In our general case we can show that the part contrib-uted to this second identity by every individual ${}^cw$ and also every individual ${}_a\xi_b$ is invariantive.

There is no necessity to cumber our work with the summation sign. It is quite sufficient to speak as though there were but a single ${}^cw$ and a single ${}_a\xi_b$. Let ${}_a\xi_b = \xi$. Define ${}^lw$ and ${}^d\xi$ by the equation

$$d\,{}^lx = V_0{}^lw\,d{}^cw + V_0{}^d\xi_e\,d\xi_e\,, \qquad (19)$$

We find that

$$\begin{aligned} {}^lX^c\alpha = {} & V_1{}^cw\,V_{c-1}{}^lw^c\alpha \\ & + V_1\xi_e\,V_{a-1}{}^d\xi_e{}^c\alpha \\ & + V_1\xi'^d\xi_e\,V_{b-1}e^c\alpha \\ & - {}^lx^c\alpha \end{aligned} \qquad (20)$$

and

$$\Delta^l x = \Delta_9 V_0{}^c w_9{}^l w + \Delta_9 V_0 \xi_9 e^d \xi e \qquad (21)$$

whence

$$^l X_9 \Delta_9 = V_1{}^c w V_{c-1}{}^l w_9 \Delta_9 - V_1{}^l w V_{c+1}{}^c w_9 \Delta_9$$
$$+ \text{ terms contributed by } \xi \qquad (22)$$

Abreviate the expression on the right of (22) as follows:-

$$\{^l w, {}^c w\} = V_1(V_{c-1}\Delta\, ^l w)^c w$$
$$- V_1(V_{c+1}\Delta\, ^c w)^l w \qquad (23)$$

Also let

$$\xi(\ ) = \Sigma^c w V_0{}^c v(\ ) \qquad (24)$$

Then (22) may be written

$$^l X_9 \Delta_9 = \{^l w, {}^c w\} + \Sigma\{^d \xi^c v, {}^c w\}$$
$$+ \Sigma\{^d \xi'^c w, {}^c v\} \qquad (25)$$

which is obviously a covariant vector density.

We will now in Art. 4. indicate the proof of these assertions, and ennunciate the more general theorem mentioned above.

Art. 4. A general theorem in identities.

In different parts of my papers (2) already referred the following has been shown. Let ρ become $\rho + \sigma$ by infinitesimal change of coordinates. The increment $\delta'\alpha$ in a contravariant vector α will then be

$$\delta'\alpha = V_0 \alpha \Delta . \sigma = \chi_0 \alpha \qquad (26)$$

If now q, $^c q$ are a contravariant and covariant mul--tenion respectively

$$\delta' q = \chi_0 q, \quad \delta'^c q = -\chi_0'{}^c q \qquad (27)$$

To express these explicitly in terms of σ put

$$q = \Sigma w, \quad ^c q = \Sigma^c w \qquad (28)$$

where the summation sign implies that the homogeneity c of w and of $^c w$ takes all intergral values from 0 to n. Then

$$\left.\begin{aligned} \chi_0 q &= \Sigma V_c \chi_0 \epsilon V_{c-1} \epsilon w \\ &= \Sigma V_c \sigma_9 V_{c-1} \Delta_9 w, \\ -\chi_0'{}^c q &= -\Sigma V_c \Delta_9 V_{c-1} \sigma_9{}^c w \end{aligned}\right\} \qquad (29)$$

With this help the reader should find no difficulty.

The more general theorem is as follows. Let an invariantive multenion here mean $qf(^l k)$, where q is any covariant or contravariant multenion and $f(^l k)$ is any scalar function of any scalar density $^l k$. Similarly an invariantive multenion linity means $\phi . F(^l k)$ where ϕ is any coexcontra, contraexco, coexco or contraexcontra, linity; and $F(^l k)$ any scalar function of $^l k$. Let $^l x$ be any given function of any number of such invariantive multenions and multenion linities, and let $^l X$, a covariant vector linity density, be formed from it exactly as in Art. 3. Then, as in Art. 3, $^l X = 0$ and $^l X_9 \Delta_9 = 0$ and the expression $^l X_9 \Delta_9$ consists of a sum of invariantive parts contributed by the individ--ual multenions and multenion linities. Each of these parts in turn subdivides into two invariantive parts, contractile and non-contractile, whose treatment however is reserved for a later paper.

It may be suggested to the reader that to prove this more general form attention should first be paid to the separate part contributed by $^l k$, which will be found to be of the form $^l k \Delta$ (invariant).

It is interesting to observe that the invariantive expressions above are quite independent of the exist--ence of a fundamental differential quadratic form.

Art. 5. Some reforms in notation.

When first in 1907 treating of multenions I express-ed a doubt whether the sign of Hamilton's ∇ should be taken over from quaternions into multenions. To-day I am certain that it should not have been. As the present series of papers affords a suitable opp-ortunity to make some very desirable changes of notation, for all time, I will not apologise for causing the reader some little inconvenience by so doing.

Thus is caused the change expressed by $\Delta = -\nabla$, (Art. 2. above), and the allied change $(\epsilon\epsilon) = -(\zeta\zeta)$; and (ee) is a natural addendum to $(\epsilon\epsilon)$

Formally we have

$$(\epsilon\epsilon) = \Sigma(\iota_\alpha, \iota_\alpha^{-1}), \quad , \quad = \Sigma(\grave{\imath}, \grave{\imath}^{-1}) \qquad (30)$$

where ι_1, ι_2, $\ldots$, ι_n are the n primitive unit vectors and $\grave{\imath}$ is one of the 2^n primitive unit vect-oriums. ${}_q\Delta$ stands to q as does Δ to ρ, that is to say, if $q = \Sigma x\grave{\imath}$, then ${}_q\Delta = \Sigma \grave{\imath}^{-1}\partial/\partial x$. With these meanings we have

$$\left.\begin{aligned} \sigma &= \epsilon V_0 \epsilon\sigma, \quad q = e V_0 e q, \\ d. &= V_0 d\rho\Delta., \quad d. = V_0 dq\, {}_q\Delta. \end{aligned}\right\} \qquad (31)$$

which shows that the inconvenient minus sign has, by our new conventions, been banished from many express-ions.

Our former p and $q^{\times}$, that is our present p and ${}^cq$, are complementary in an invariant sense, in that $V_0 p\,{}^cq$ is an invariant whatever be the values of p and ${}^cq$. Similarly our former ϕ and $\psi^{\times}$, that is our present ϕ and ${}^c\psi$, are complementary in that $V_0\phi e\,{}^c\psi e$ is an invariant whatever be the values of ϕ and ${}^c\psi$.

In my former paper we had X and $Y^{\times}$, or say X and ${}^cY$, with a similar complementary relation name-

-ly, $V_0 X e\,{}^cY e$ is an invariant. As this use of ${}^cY$ is not absolutely required, we will abandon it for simplicity of notation.

If X is a coexco linity then X' is a contraex-contra. The dash here used may be made to supply our old use of $Y^{\times}$ thus: X shall always mean a coexco, and Y' a contraexcontra. This explains how, in Art. 2, we came to denote our old E_β by E_β' and consequently to denote old Γ_β, F_β, Θ_β, Ω, by the same letters each with a dash.

When in our present subject we have, which is by no means always the case, a fundamental quadratic form $V_0 d\rho\theta d\rho$ then we take $|\theta^{\frac{1}{2}}| = l$ to be a funda-mental scalar invariant density, and we generally re-gard any other density as obtained by multiplying a previously defined symbol by l. Hence our use of lq, $l\xi$, etc. (our former $q^{\cdot}$, $\xi^{\cdot}$, etc.) for such densities. In our use of $d\Omega$, $d\xi$, for our former $\Omega^{\times}$; $\xi^{\times}$; the d in the pre-index position is suppos-ed in form a combination of the c and l that were also used in that position.

Art. 6. The origin of the present researches.

The series of researches in relativity, of which the present paper is the first, began in December 1921 with a paper sent to the Phil. Mag. correcting an error in Professor Eddington's paper "Relativity of Field and Matter" in the Phil. Mag. of Nov. 1921. In my paper was a first intimation of the importance of consider-ing intrinsic bulk as we do above. Professor Edding-ton later found his error and corrected it before my communication could have reached England, and no notice whatever was taken of my communication. In Jan. 1922 I sent a rather long paper suggested by the December communication to the Phil. Mag. This was entitled "A New Identity Affecting Questions on Relativity; and Some Cognate Matters." The new identity occupied but a small portion of the paper at the beginning. In the Phil. Mag. of Aug. 1922 I was surprised to find this new identity (in a less general form) credited to Mr. Harward, who dates his communication May 1922. In September 1922 I wrote a letter, intended for publi--cation, pointing out this editorial oversight. In the Phil. Mag. for July 1923, p. 153, an editorial footnote promises that the paper whose title is given above will "appear shortly", and now in April 1924 I continue to take for granted that the paper and the letter will "appear shortly".

In November and December 1922 I sent to the Roy. Soc. Lond. three papers and a single summary of the three. The first of these appeared in the Phil. Mag. of July 1923. The summary, with the exception of two concluding sentences referring to this first paper, is as follows.

"(1) The Mechanical Forces Indicated by Relativity in an Electromagnetic Field. Can their Existence be Demonstrated?

"(2) Relativity and Elasticity.

"(3) Entropy, Conduction of Heat, Viscosity and Electric Resistance Treated on the Principles of Rel--ativity.

"The papers are concerned with bringing all the fun--damental equations of the physics of matter in bulk under the domain of relativity. A paper by the author On the Mathematical Theory of Electromagnetism (Phil. Trans.) in the year 1893 had, in pre-relativity days, made a comprehensive survey of like general kind. In addition to electromagnetism the foundations of all the subjects in our present titles are reviewed. With--out exception and also without any loss of generality it has been shown in our present three papers that all the features of the 1893 paper can be copied into the relativity scheme. The results are surprising in that, so far from introducing complications into these phys--ical foundations as probably all relativists antici--pated, relativity introduces great and important simplification."

It was officially communicated to me that the ref--eree reported that these papers "cannot possibly be printed" for one and only one reason, which was worded thus:-

"You use quaternions."

The reason was expanded into the assertion that it was as inappropriate to "use quaternions" as it would be to address an English scientific audience in Sans--krit. Doubtless a copy of this curious communication has been kept by the Secretaries of the Royal Society and can be seen by any F.R.S. who cares to verify that this is an exact account of the matter.

The researches on quanta sent to "Nature" in May 1923 have been mentioned above.

It is therefore necessary to bring out these resear--ches in distant Tasmania. The whim of a gentleman who dislikes quaternions has ruled that for this de--cade at any rate that subject is verboten in London. It should be clear that one of my main objects has been to show from the fruits of the method that it is much more efficient that the current method of tensors in surmounting the formidable difficulties that rela--tivity presents to the pioneer.

The present series of papers will deal with the subjects mentioned in this article and some two other researches already in existence; but they need to be revised because the present paper has co-ordinated the previously independent elements in the basis of relativity into a consistent whole, the bearing of which on these questions remains to be looked into. (The original papers will be placed in custody of the Roy. Soc. Tas.)

Mathematicians generally should be grateful to the Royal Society of Tasmania for finding means for the publication of these papers, in spite of the fact tha the funds of the society are urgently needed in other scientific directions. Not all readers may realise that the population of Tasmania, the sole support of this Society, is considerably less than that of Cornwall, or about the same as that of Dorset. It is not usual to expect a community of that size to have the facilities for printing technical mathematics. Our difficulties in this matter of printing or some alternative method of reproduction are proving great. But for the very friendly interest taken by all the printing establishments in Hobart, and the great amount of trouble that the Government Statistician, Major L.F.Giblin, D.S.O., has taken upon himself in making enquiries as to methods and processes, the work could scarcely have been undertaken. My best thanks are also due to Mr. R.G.L. Brett who has given much of his time in the laborious work of drafting the formulae for reproduction.

References.

(1) "Multenions and Differential Invariants, I., II, III." Proc. R.S. Lond. A; Vol. 99, 1921, p.292; Vol. 102, 1922, p. 210; Vol. 103, 1923, p. 162. For the future these papers will be referred to as M.D.I. (I), (II), (III). The present paper may be taken to supersede M.D.I. (III) Art. 21.

(2) M.D.I. (II) Art. 18, p. 235 (bottom of); M.D.I. (II) Art. 15; M.D.I. (I) Art. 6 (part added Feb. 1921) eq. 11.