(Papers and Proceedings of the Royal Society of Tasmania, 1912.)

THE THEORY OF THE QUOTA IN PROPORTIONAL REPRESENTATION—I.

Errata.

§25.—The last three lines should read :

" $x, y, z, \ldots$ the seats obtained by the parties with the method of apportionment actually used.

$p', q', r' \ldots$ respectively equal to xQ, yQ, $zQ \ldots$, so that—

$$p' + q' + r' + \ldots = p + q + r + \ldots = v."$$

§26.—The second sentence should read:

" The method of apportionment actually used gives x, y, $z \ldots$ seats to the parties, and this allotment is equivalent to taking the strengths of the parties to be $p', q', r' \ldots$, instead of the actual $p, q, r \ldots$, and allotting seats by the rule of three in proportion to $p', q', r' \ldots$"

§36.—The second paragraph should read:

" Regarded thus, the allotment may be considered ideal if the number of members divided by the number of votes is as nearly as possible the same for each party. This condition is expressible in the form that

$$\Sigma \left(\frac{x}{p} - \frac{m}{v}\right)^2$$

shall be a minimum. This expression can be written in the form

$$\Sigma \left(\frac{p' - p}{p}\right)^2 \equiv k^2 \quad \ldots \quad (7)."$$

Note.

In the comparison in §§9-21 of the Hare and Droop quotas in a contest between two parties it is supposed that all transfers of votes are made exactly in accordance with the rule of three This is so in the rules of the Tasmanian Electoral Act of 1907 (subject to the unimportant detail that fractional remainders are neglected), and consequently the argument of §§9-21 is correct for these rules. The argument is not necessarily correct for rules such as those of the English Municipal Representation Bill of 1907, in which exact proportion is not used in all the transfers.

THE THEORY OF THE QUOTA IN PROPORTIONAL REPRESENTATION—I.

By E. L. Piesse, B.Sc., LL.B.

(Read July 8, 1912.)

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1. It is the object of the various systems of proportional representation to secure, in the words of J. S. Mill, that "every or any section shall be represented, not disproportionately, but proportionately."

If it is necessary to divide a country into a number of constituencies, the only way to secure proportional representation with certainty is to ensure that in every constituency each party is represented in proportion to the number of its supporters in that constituency. The adoption of any other electoral system must make the representation depend on the accident of the distribution of the parties among the constituencies. (1)

The division of a country into single-member constituencies will usually produce disproportionate representation, even if the member for each constituency is elected by a majority of the voters in the constituency, (2) for the representation of either party will depend on the number of constituencies in which it is in a majority, and this

(1) J. Rooke Corbett, *Recent Electoral Statistics* (a paper read before the Manchester Statistical Society in 1906, and re-printed with additional statistics by the Proportional Representation Society in 1910).

(2) The results of the seven General Elections held in the United Kingdom from 1885 to 1910 are shown by the following table. Uncontested constituencies are allowed for by assuming that the strength of each party varied in them from one election to another in the same ratio as in the contested constituencies in the same county. Liberals include Labour and Irish Nationalist members, Conservatives include Liberal Unionists :—

General Elections, United Kingdom, 1885-1910.

	1885	1886	1892	1895	1900	1906	1910
Liberals—							
Actual	414	283	357	260	268	513	397
Proportionate share..	378	331	369	329	327	387	363
Conservatives—							
Actual	256	387	313	410	402	157	273
Proportionate share..	292	339	301	341	343	283	307

(See J. Rooke Corbett (1), Table I.). There are a few double constituencies in England, and there were three-party contests in some constituencies, but this table gives a reliable view of the possibilities of single-member constituencies.

number will not usually be proportionate to the strength of the party throughout the country. (3)

2. Proportionate representation accordingly cannot be ensured except from constituencies returning many members. In such constituencies the disproportion of the representation will not be diminished, but will rather be exaggerated, if each elector votes for all the candidates. (4) It is accordingly necessary to use some system which will produce proportionate representation; and in each system the quota has an important influence in determining how exact shall be the proportion of representation to strength of party.

3. In this paper the quota is considered in respect of the two following classes of proportional representation systems:—

(*a*) Single transferable-vote systems (varieties of which are used in Tasmania, Denmark, and South Africa);

(*b*) Party-list systems (varieties of which are used in Belgium, Sweden, Finland, Switzerland, and other European countries).

4. A representative assembly has usually to govern as well as to represent; and government is usually carried on by the party system, which requires that the party in power shall have more than a nominal majority in the representative body. Electoral statistics show that when

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(3) This was seen in America about a hundred years ago, when the Republican party in Massachusetts introduced the plan known as the "gerrymander," since practised with much success in other countries. "The gerrymander is simply such a thoughtful construction of districts as will economise the votes of the party in power by giving it small majorities in a large number of districts, and coop up the opposing party with overwhelming majorities in a small number of districts." (Commons, *Proportional Representation*, p. 50.) It may be noted that it would be easy to divide Tasmania into single-member constituencies which, on the voting at the General Election of 30th April, 1912, would return 23 members for one party and 7 for the other; the representation to which their respective strengths entitled them was 16·3 and 13·7, and these numbers were produced as nearly as possible (16 and 14) at the election, which was held under the single transferable-vote system of the Electoral Act, 1907, in five districts each returning 6 members.

(4) The most striking example in recent years is the election for the Australian Senate held on 13th April, 1910, each of the six States being a single constituency returning three members, and each elector voting for three candidates. The Labor party, which polled 2,021,000 votes out of 4,018,000, secured all 18 seats, although in proportion to its strength it was not entitled to more than 10.

there are two parties they are frequently nearly equal in strength. (5) It is therefore important to compare the various systems and quotas in regard to the size of the majority they are likely to produce in close contests.

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(5) Mr. J. Rooke Corbett, ((1), Table I.), has tabulated the results of the seven General Elections held in the United Kingdom according to the four kingdoms, and, in England, according to the ten divisions used by the Registrar-General. These divisions contain constituencies returning from 29 to 70 members. From this tabulation I have computed that in England for the 10 divisions at the 7 elections (70 occasions in all), the larger party exceeded 60 °/o of the votes on only five occasions, and its greatest strength was 64 °/o; the average for the 70 occasions was 55 °/o. In Wales the strength of the larger party varied from 58 °/o to 68 °/o, and the average was 63 °/o. In Scotland the strength varied from 50 °/o to 61 °/o, and the average was 56 °/o. In Ireland the strength varied from 67 °/o to 74 °/o, and the average was 70 °/o. For the whole of the United Kingdom, the strength of the larger party at the seven elections was 56, 51, 52, 51, 51, 58, 54 °/o; the average was 53 °/o.

Mr. Corbett states on the authority of Mr. J. H. Humphreys that at the General Election of 1910, there were majorities of under 500 in 144 constituencies. The average number of voters in a constituency was about 10,000.

In Tasmania the strengths of the parties at the elections of 1909, 1910 (House of Representatives) and 30th April, 1912 in the five six-member constituencies, and in the whole of Tasmania, were :

Tasmania—Strength of Parties, 1909, 1910, 1912.

	Bass.			Darwin.			Denison.		
	1909.	1910.	1912.	1909.	1910.	1912.	1909.	1910.	1912.
Non-Labour	64	43	52	42	37	47	72	42	51
Labour	36	57	48	58	63	53	28	58	49

	Franklin.			Wilmot.			Tasmania.		
	1909.	1910.	1912.	1909.	1910.	1912.	1909.	1910.	1912.
Non-Labour	66	53	57	59	56	66	61	45	54
Labour	34	47	43	41	44	34	39	55	46

At the election for the House of Representatives held throughout Australia on 13th April, 1910, the larger party was between 50 °/o and 60 °/o in 30 of the 64 electorates in which the principal contest was between single candidates of the two parties; its average strength was 61 °/o.

At the election for the Senate held the same day the strengths of the parties in each State were :

	Victoria.	N.S.W.	Queensland.	S. Australia.	W.A.	Tasmania
Non-Labour ...	52	50	50	46	46	45
Labour............	48	50	50	54	54	55

From this point of view the most important matter is that the constituencies shall each return an odd number of members. This may be illustrated by the following table, based on the Tasmanian General Election of 30th April, 1912, which shows, (*A*) the actual representation given by the six-member districts, and (*B*) what would probably have been the representation if the districts had returned five members each, or (*C*) seven members each.

Tasmania—General Election, 30th April, 1912.

District.	(*A*) Actual Result from Six-member Districts.		(*B*) Probable Result from Five-member Districts.		(*C*) Probable Result from Seven-member Districts.	
	Liberal.	Labour.	Liberal.	Labour.	Liberal.	Labour.
Bass	3	3	3	2	4	3
Darwin	3	3	2	3	3	4
Denison	3	3	3	2	4	3
Franklin	3	3	3	2	4	3
Wilmot	4	2	3	2	5	2
	16	14	14	11	20	15
Majority	2		3		5	
Representation proportional to strength	16·3	13·7	13·6	11·4	19·1	15·9
Majority	2·6		2·2		3·2	

5. In single transferable-vote systems the quota is the number of votes necessary to secure the election of a candidate. ([5a])

In party-list systems the quota is the number by which the total votes for each list is divided in order to ascertain the number of members to be elected from the list.

Single Transferable-vote Systems.

6. In 1855 a system of proportional representation based upon the single transferable vote was introduced in Denmark. The rules for conducting the election were the work of a mathematician, M. Andrae, at that time Min-

([5a]) In §§ 9-20 cases are discussed in which candidates can be returned with less than the quota.

ister of Finance. (6) At about the same time, Mr. Thomas Hare, whose well-known book, *The Election of Representatives, Parliamentary and Municipal*, was published in 1857, advocated a similar system in England.

Both Andrae and Hare used as the quota the number obtained on dividing the number of votes polled in the con-constituency by the number of members to be elected from the constituency. This quota is commonly called the Hare quota. It was used in the Tasmanian Electoral Act of 1896, under which elections were held in the constituencies of Hobart and Launceston for the House of Assembly in 1897 (7) and 1900, and throughout Tasmania for the Senate and House of Representatives in 1900. (8)

7. In 1868 Mr. H. R. Droop, in his pamphlet *On Methods of Electing Representatives*,(9) proposed as the quota the number obtained by dividing the number of votes by one more than the number of members to be elected, and adding 1.

This quota is used in the Tasmanian Electoral Act of 1907, under which the General Elections for the House of Assembly on 30th April, 1909,(10) and 30th April, 1912,(11) have been held. It is also used in the election of members of the Senate of the Union of South Africa and in municipal elections in the Transvaal. (12)

8. In the case of a constituency of 4200 voters, electing six members, the Hare quota is one-sixth of 4200 (or 700); the Droop quota is one more than one-seventh of 4200 (or 601).

Considering an election as a contest between individual candidates, it is clear that a candidate who obtains, in

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(6) See *Reports from His Majesty's Representatives in Foreign Countries and in British Colonies respecting the Application of the Principle of Proportional Representation to Public Elections.* (Cd. 3501—House of Commons, Miscellaneous, No. 3, 1907), p. 17.

(7) See R. M. Johnston, *Observations on the Working Results of the Hare System of Election in Tasmania*, Pap. and Proc. Roy. Soc. Tas., 1897, pp. 69-96.

(8) See *Report on the Hare-Clark System of Voting*, by J. G. Davies and R. M. Johnston, presented to the Senate, 13th December, 1901 (reprinted in *Reports from His Majesty's Representatives, &c.*, pp. 96-105).

(9) London, Macmillan, 1868; see also the same author's paper in *Journal of the Statistical Society*, June, 1881.

(10) See P. C. Douglas, E. L. Piesse, and W. A. B. Birchall, *Tasmania—General Election for House of Assembly, April 30, 1909—Report on the System of Proportional Representation used in accordance with the Electoral Act, 1907* (Tasmania, No. 34 of 1909).

(11) See H. E. Packer, E. L. Piesse, and J. F. Daly, *Report on the General Election of 30th April, 1912* (Tasmania, No. 11 of 1912).

(12) See John H. Humphreys, *Transvaal—Proportional Representation—Report on the Municipal Elections held at Pretoria and Johannesburg on 27th October, 1909* (Transvaal, T.G. 5-'10).

the instance just given, the Droop quota (601 votes) has more votes than each of six other candidates can obtain, and therefore he has sufficient votes to entitle him to election. Even if the Hare quota is used, any candidate who obtains the Droop quota must be elected, and a candidate who obtains the Hare quota receives an excess of votes which are not really required by him, and which are therefore wasted. Hence it is clear that considering an election as a contest between candidates the Droop quota is to be preferred to the Hare quota. ([13])

([13]) See Douglas, Piesse, and Birchall, (10), p. 4, where the following passage is quoted from *Proportional Representation in Large Constituencies* by Walter Baily (London, Ridgway, 1872) :—

"We have still to consider what is the sufficient number of votes to be retained for each candidate. The rule in use in Denmark (and adopted by Mr. Hare, for finding this number, which is called the quota), is to divide the number of votes by the number of members to be elected. This is simple, but still it is wrong. For example, if we apply Mr. Hare's plan to an election of two members, in which 100 votes are given—70 for *A* first, and then *B*, and 30 for *C*—we should obtain the quota by dividing 100 by 2 ; and then retaining this quota of 50 votes for *A*, we should hand over 20 votes to *B* ; and the votes would then stand, *A* 50, *C* 30, *B* 20, and therefore we should have *A* and *C* elected. And yet it is clear that, as 70 is more than twice 30, *A* and *B* should have been the candidates elected.

"The number of votes to be retained for a candidate must be enough to make his election certain, whatever combination may be made of the other votes given in the election. The smallest number which will suffice for this is the true quota ; all votes retained beyond this number are wasted. There is no difficulty in finding this number. Suppose that two members have to be elected, we must retain for a candidate votes enough to insure his being one of the first two, and this we shall do if we retain for him just over a third of the whole number of votes given. It is impossible for three persons each to have more than one-third of the votes, so that any candidate who has more than one-third by ever so little is certain to be one of the first two, in whatever way the rest of the votes may be distributed. In the same way, we see that if five members have to be elected, a candidate who has more than one-sixth of the votes will certainly be one of the first five, and therefore elected ; and so for any other number of members. The rule, then, for finding the true quota is this : Divide the number of votes by the number just above that of the members to be elected, and take as a quota the number just above the quotient.

"In the example given above, the true quota just exceeds one-third of 100. It is therefore 34. The 70 votes given to *A*, *B*, will then be divided into 34 for *A*, 34 for *B*, and 2 over. *C* has only 30 votes ; and the result is that *A* and *B* are elected, and it is clear they should be.

"It will be observed that some votes are wasted. This must needs be, whatever mode of election is adopted. It a constituency has only one member, a candidate who gets a bare majority will be elected, and it will be of no moment whether the remaining votes are for him or against him. All except the bare majority can have no effect upon the election, and may be considered as wasted. But as the number of members is increased, the unavoidable waste is diminished. With five members the effective votes for each will just exceed one-sixth, and therefore the waste votes will just fall short of the remaining sixth ; in fact, the unavoidable waste will always just fall short of the true quota."

9. We have next to compare the two quotas in a contest between parties. Messrs. Douglas, Piesse, and Birchall, in their report on the Tasmanian General Election of 30th April, 1909, (14), discuss this aspect as follows:—

But these arguments do not decide the superiority of one quota over the other if an election is considered, not as a contest between candidates, but as a contest between parties. For here we have to consider the possibility of one or more candidates of a party securing election on less than a quota, and so obtaining for their party an amount of representation in excess of its proportional share. With the Hare quota it is very easy for a party to secure excessive representation by returning several candidates with less than the quota. With the Droop quota this is impossible in a two-party contest (except when papers become exhausted through the neglect of voters to give a preference to each candidate of their party), and in a contest between more than two parties disproportional representation would probably occur much less frequently with the Droop quota than with the Hare quota.

Take the case of an election of six members by 210 voters, 63 of whom belong to party *A*, and 147 to party *B*, and assume the Hare quota is used. Party *A*, having roughly one-third of the voters, is entitled to two members, and party *B* to four. When all candidates but seven have been excluded, the state of the poll might be that the five remaining candidates of party *B* had respectively 30, 30, 29, 29, 29 votes each (total 147); and the two remaining candidates of party *A* 35 and 28 each (total 63). The candidate lowest on the poll has now to be excluded; that is, the *A* candidate with 28 votes is excluded, and there are left six candidates—five of party *B*, and one of *A*, who are declared elected. That is, party *A*, instead of getting two members, has got only one; and party *B*, instead of four members, has got five.

Now this has happened solely because the use of the Hare quota (35) has wasted the four votes which the *A* candidate, with 35 votes, had in excess of the Droop quota (31). If the Droop quota had been used, this surplus of four would have been distributed before the exclusion of the lowest candidate. It would naturally have gone to the other candidate of the party, whose votes would thus have been raised from 28 to 32; and the candidate excluded as lowest on the poll would then have been one of the *B* candidates. Thus, the result would have been the correct result—party *A*, two members; party *B*, four members.

It is interesting to note in passing that if, in the election for Franklin, the Hare quota had been used, and if there had been no cross-voting between the candidates of the Labour Party and other candidates by voters who gave their first preferences to Non-labour candidates, and no exhaustion of the papers of such voters, the Labour Party would have secured only one member in place of the two to whom it was entitled in proportion to the number of its supporters.

There are also some cases in which the Droop quota produces representation not proportionate to the strengths

(14) See (10), p. 5.

of the two parties. Thus, in a contest in a six-member constituency between two parties, one of which polls 72 % of the votes, the other 28%, the first party has five Droop quotas, and therefore returns five members; while the second party, having less than two quotas, returns only one member. But the fairest representation (see § 52) in this case would be four members for the first party, and two for the second. ([15])

10. The comparison of the merits of the two quotas in a contest between parties therefore involves an examination of the possible number of cases in which each can give disproportionate representation.

In making this examination it must be assumed that there is no cross-voting, and that each voter votes for all the candidates of his party.

11. Fig. 1 shows the result of the comparison in a contest between two parties in a six-member constituency and in a seven-member constituency.

In each case the continuous sloping line OO' represents the strength of the parties for all values from 0 % to 100 % of the voters. The strength of party A is represented by the distance from OX, and the strength of party B by the distance from $O'Y$.

The continuous stepped line shows what is the best possible apportionment of members between the two parties for each strength of each party. This apportionment is obtained by dividing the strength of each party by the Hare quota, and, in the case of remainders, giving the last member to the party having the larger remainder (see § 52). Thus, in the case of the six-member constituency, if party A has just over $8\frac{1}{3}$ %, party B just under $91\frac{2}{3}$ %, of all the votes, on dividing by the Hare quota (which is $16\frac{2}{3}$ % of the total votes), A will have the larger remainder, and should therefore have one member. As the strength of A increases, it does not become entitled to two members until its strength reaches 25 %; from 25 % to $41\frac{2}{3}$ % it is entitled to two members; and so on.

The broken lines show the representation which may be produced by the Hare quota and the Droop quota. Where

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([15]) Mr. J. W. McCay, in one of a series of articles on Proportional Voting published some years ago in the Melbourne "Age," gave the following instance of the Droop quota as "a successful practical joker":— In a six-member constituency, let there be 500 voters for party A, 199 for party B; total 699 voters. The quota is 100, and A gets 5 seats, B 1 seat. But let A poll one more vote, making the total of the voters 700. The Droop quota is now 101, and party A now gets only 4 seats.

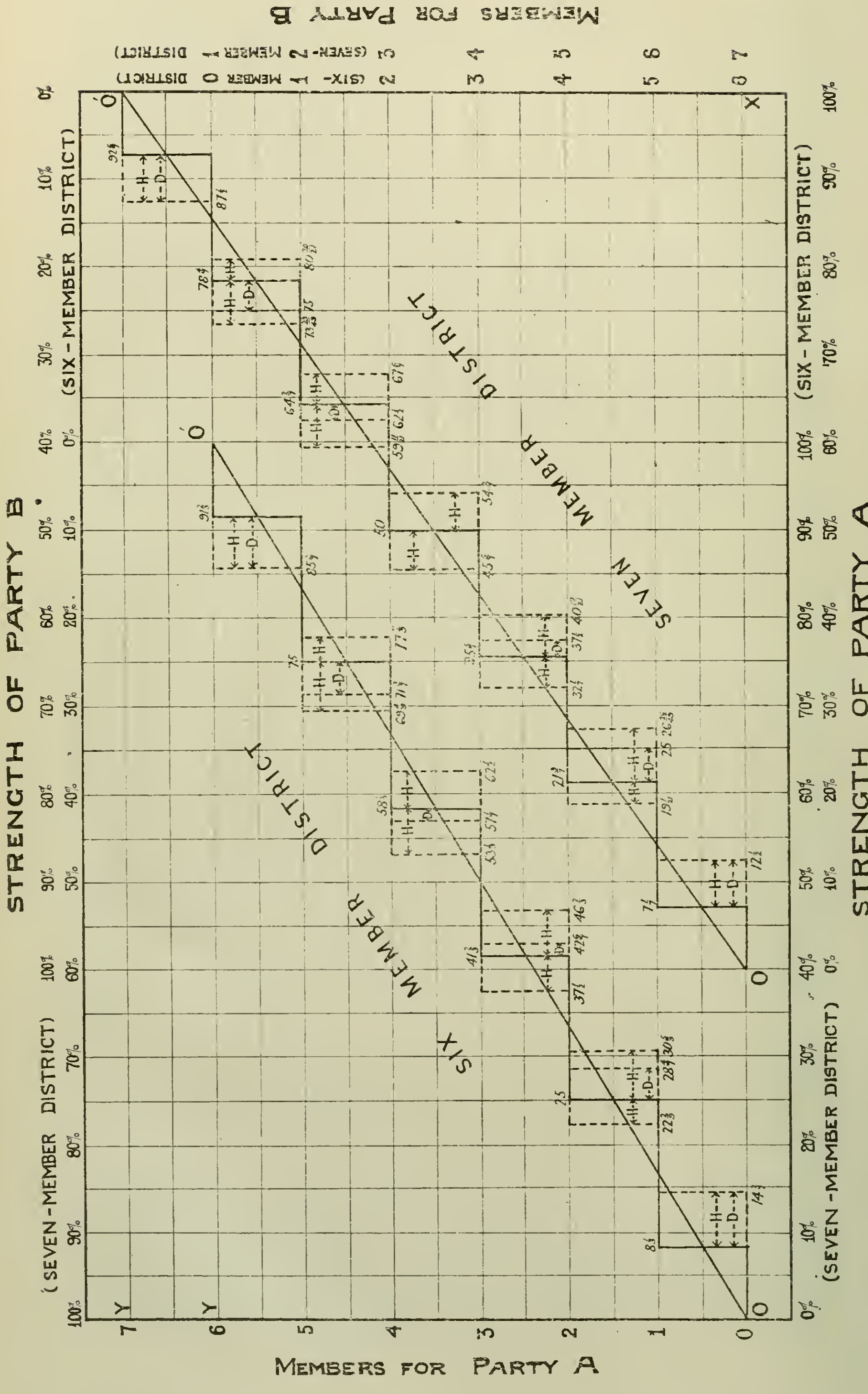

FIG. 1.—Comparison of the Hare and Droop Quotas in a Two-party contest (see §§ 11, 16, 17).

the ends of two vertical broken lines are joined by a continuous line the broken line coincides with the continuous line; and for the strengths corresponding to these coinciding lines, each quota gives the correct representation of the parties. Where there is a horizontal broken line there may be disproportionate representation.

12. As an example consider, in the six-member constituency, the case of party A having 14 %, party B 86 %, of the votes. Party A should have one member, party B five members. If the Droop quota ($14\frac{2}{7}$ %) is used, party A will not obtain a member, and party B will get all six members. If the Hare quota ($16\frac{2}{3}$ %) is used, six candidates of party B may each obtain more than 14 % of the votes, and the one candidate (or the last unexcluded candidate) of party A having only 14 % must be excluded, and the six candidates of party B will be elected.

13. As another example consider, again in the six-member constituency, the case of party A having 40 %, party B 60 %, of the votes. Using the Hare quota, suppose that three candidates of party B have been elected (thus absorbing 50 % of the votes), and that four candidates are left, one of party B with 10 %, and three of party A, each with more than 10 % (and having 40 % between them). The remaining B candidate must be excluded, and the three A candidates elected; so that A, with only 40 % of the votes will obtain half the members, instead of two, the number to which it is entitled. With strengths of 40 % and 60 % this is not possible if the Droop quota is used; although the graph shows that with A 42 %, the Droop quota would give only two members, instead of the three to which A is entitled.

14. It is to be noted that with the Hare quota the disproportionate representation is due to the possibility of one or more candidates being elected with less than the quota. The occurrence of disproportionate representation depends, therefore, on the distribution of votes among the unexcluded candidates when the last seat is filled; and the distribution may be such that no error can occur. With the Droop quota, on the other hand, no candidate (on the assumptions made) can be elected with less than the quota, and the error is due to the quota itself; and when the parties have the appropriate strengths (for instance, 14 % and 86 % in the example in § 12), it is certain that there will be an error.

15. Considering the matter in symbols, let m be the number of members to be elected; and assume that the

scrutiny has reached the stage at which the number of candidates elected or unexcluded is $m + 1$.

16. Using the Hare quota, let us examine what representation will be obtained by a party as its strength increases from 0 % to 100 %.

If the strength of the smaller party is less than $100/2m$, *i.e.*, less than half the Hare quota, it cannot obtain a member, however the votes may be distributed among the candidates of the other party. For if the smaller party has only one candidate left, the other party has m, and each of these (the surpluses of any elected candidates having been distributed) must have more than half the Hare quota; and any other case can be reduced to this.

If the strength, A, of the smaller party is greater than $100/2m$ and less than $100/m$, the smaller party is entitled to one member. At the stage under consideration, the smaller party will have only one candidate unexcluded, and his votes will be A. The larger party will have $m - 1$ candidates, who between them will have $100 - A$ votes. Consequently, if $(100 - A)/(m - 1)$ is greater than A, each of the $m - 1$ candidates of the larger party may have more votes than the one candidate of the smaller party; the latter may be excluded, and the larger party will obtain all the seats, instead of $m - 1$ seats, the number to which it is entitled. The equation—

$$A = \frac{100 - A}{m - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

therefore gives the greatest value of A for which this can occur.

From (1) we get—

$$A = \frac{100}{m + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The smaller party, then, being entitled to one member, may fail to obtain any representation, if its strength lies between $100/2m$ and $100/(m + 1)$; so that the range of strength for which disproportionate representation is possible is—

$$\frac{100}{m + 1} - \frac{1}{2} \cdot \frac{100}{m}$$

Similarly, the smaller party, being entitled to two members, may obtain only one, if its strength lies between $3/2 \times 100/m$ and $(2m - 1)/m \times 100/m$. In the same way it may be shown that if the strength of the smaller party lies between $2/3 \times (100 - (m - 2).100/m)$ and $3/2 \times 100/m$, it may obtain two members, although entitled only to one. The range of strength for which disproportionate representation is possible is therefore—

$$\frac{2m-1}{m} \cdot \frac{100}{m} - \frac{2}{3}\left(100 - (m-2)\frac{100}{m}\right)$$

Proceeding in the same way for the cases in which the smaller party is entitled to three or more members, we obtain the following expression for a measure of the range of strengths for which the Hare quota may give disproportionate representation:—

$$\left[\frac{100}{m+1} - \frac{1}{2} \cdot \frac{100}{m}\right] +$$
$$\left[\frac{2m-1}{m} \cdot \frac{100}{m} - \frac{2}{3}\left(100 - (m-2)\frac{100}{m}\right)\right] +$$
$$\left[\frac{3m-4}{m-1} \cdot \frac{100}{m} - \frac{3}{4}\left(100 - (m-3)\frac{100}{m}\right)\right] + \ldots +$$
$$\left[\left((m-1) - \frac{1}{m+1-\overline{m-2}}\right)\frac{100}{m} - \frac{m-1}{m} \cdot \left(100 - \right.\right.$$
$$\left.\left.(m - \overline{m-1})\frac{100}{m}\right)\right] +$$
$$\left[\frac{m}{m+1} \cdot \frac{100}{m} - \frac{2m-1}{2} \cdot \frac{100}{m}\right] \quad . \quad . \quad . \quad (3)$$

All such strengths are shown on the graph by the lines marked "*H*" connecting arrow-heads.

17. With the Droop quota the following expression gives a measure of the range of strengths for which there must be disproportionate representation:—

$$\left(\frac{100}{m+1} - \frac{1}{2} \cdot \frac{100}{m}\right) + \left(2 \cdot \frac{100}{m+1} - \frac{3}{2} \cdot \frac{100}{m}\right)$$
$$+ \left(3 \cdot \frac{100}{m+1} - \frac{5}{2} \cdot \frac{100}{m}\right) + \ldots +$$
$$\left((m-1) \cdot \frac{100}{m+1} - \frac{2m-1}{2} \cdot \frac{100}{m}\right) \quad . \quad . \quad (4)$$

All such strengths are shown on the graph by the lines marked "*D*" connecting arrow-heads.

18. It is to be noted that when the Droop quota gives disproportionate representation, it is the larger party that is over-represented; with the Hare quota it is equally likely that the larger party will be under-represented as that it will be over-represented.

19. From (3) and (4) we get the following results:—

In a six-member constituency, disproportionate representation may occur with the Hare quota for a range of 46 % of all the possible strengths of a party; with the Droop quota it is certain to occur with a range of 21 %.

In a seven-member constituency, disproportionate representation may occur with the Hare quota for a range of 52 %; with the Droop quota it is certain to occur with a range of 19 %.

20. From Fig. 1 we can see what will be the representation for the important cases of the larger party between 50 % and 60 %.

In the six-member district, the larger party is not entitled to four members until its strength exceeds $58\frac{1}{3}$ %; although, in the interests of party government, it might be justifiable to say that the larger party shall have four members even if its strength is only just over 50 %. The figure shows that the Hare quota may give the larger party four members if its strength exceeds $53\frac{1}{3}$ %; also that the Hare quota may fail to give more than three members even if the strength exceeds $58\frac{1}{3}$ %. The Droop quota is certain to give four members if the strength of the larger party lies between $57\frac{1}{7}$ % and $58\frac{1}{3}$ %; otherwise no anomalies are possible with this quota. (15a)

In the seven-member district, the larger party is entitled to four members as soon as its strength exceeds 50 %. The figure shows that the Hare quota may give only three members if the strength lies between 50 % and $54\frac{2}{7}$ %, and that it may give five members if the strength exceeds $59\frac{1}{2}$ %. No anomalies are possible with the Droop quota.

21. On the whole, then, the Droop quota seems to be preferable in a two-party contest; but neither the Hare nor the Droop quota is quite satisfactory. In fact a single transferable-vote system subordinates the party to the candidates, and in essence is not a system of proportional representation at all; as contrasted with list systems which subordinate the candidate to the party and have for their

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(15a) This would have occurred in the district of Franklin at the General Election of 30th April, 1912, if the Liberal party had held during the transfers of the scrutiny the first choices which its candidates obtained. The strength of the Liberal party, based on the first choices, was 57·3 %, and the strength of the Labour party was 42·7 %.

primary object the return by each party of the proportion of members to which it is entitled. ([16]).

The method here used is not well suited for a comparison of the quotas in a contest between more than two parties.

List Systems.

22. In list systems of proportional representation, the votes obtained by the list of each party are counted, and it is required to partition the seats for the constituency among the parties in proportion to the strengths of the parties; that is, in proportion to the votes for the respective lists.

We shall use the term electoral unit for the number obtained by dividing the total of the votes for all the candidates by the number of seats for the constituency. The electoral unit corresponds to the Hare quota; if each elector has six votes, the electoral unit is six times the Hare quota.

23. If the strength of each party is an exact multiple of the electoral unit, the apportionment of seats among parties can be carried out exactly by applying the rule-of-three. This method, with the condition afterwards mentioned as to the allotment of seats to the largest remainders, is referred to as the rule-of-three method. Usually it will be found, on dividing the strengths by the

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([16]) See *Report of the Royal Commission appointed to enquire into Electoral Systems* (United Kingdom, 1910, Cd. 5163).

The statistics available of the effect of the Droop quota show that it has produced exactly proportional representation. The following are the actual results of the General Elections in Tasmania on 30th April, 1909, and 30th April, 1912 (in which, of course, exhaustion of votes and cross-voting, possibilities excluded from the argument of the paper, occurred).

Tasmania—General Elections, 30th April, 1909, and 30th April, 1912. Proportional Representation of the Parties.

	Bass.		Darwin.		Denison.		Franklin.		Wilmot.		Tasmania.	
	Proportionate Share.	Members Returned.	Proportionate Share.	Members Returned.	Proportionate Share.	Members Returned.	Proportionate Share.	Members Returned.	Proportionate Share.	Members Returned.	Proportionate Share.	Members Returned.
1909.												
Non-Labour	3·8	4	2·49	2	4·3	4	4·0	4	3·54	4	18·3	18
Labour..........	2·2	2	3·51	4	1·7	2	2·0	2	2·46	2	11·7	12
1912.												
Non-Labour	3·1	3	2·8	3	3·1	3	3·4	3	4·0	4	16·3	16
Labour..........	2·9	3	3·2	3	2·9	3	2·6	3	2·0	2	13·7	14

See (10), p. 8, and (11), p. 4.

electoral unit, that there are two or more remainders; and, accordingly, in order to complete the apportionment it will be necessary to allot one or more seats to remainders less than the electoral unit. We have then to consider how this allotment of seats to remainders is to be effected so as to give as nearly as may be an equality of representation between the various parties.

24. The following example illustrates the problem. Assume that 10 seats are to be allotted among three parties, A, B, C, whose strengths are as follows:—

Party A	16,000 votes
Party B	4,000 votes
Party C	10,000 votes
Total votes	30,000

The electoral unit is obtained by dividing 30,000, the total of all the votes, by 10, the number of seats for the constituency; it is therefore 3000.

The result of dividing the strength of each party by the electoral unit is—

Party A 5 electoral units; remainder, 1000.
Party B 1 electoral unit; remainder, 1000.
Party C 3 electoral units; remainder, 1000.

Party A, then, must have at least five members; party B, at least one; party C, at least three; but which party is to get the remaining seat? If party A gets it, each 2666 votes polled for that party return a member; each 3333 votes polled for party C return a member; but there is only one member for the 4000 votes of party B. If party B gets the remaining seat, the corresponding numbers will be: party A, 3200 votes to a member; party B, 2000; party C, 3333. If party C gets the remaining seat, the numbers will be: party A, 3200 votes to a member; party B, 4000; party C, 2500.

Which is the nearer approach to electoral equality between the supporters of the various lists? Probably the first distribution, in which the larger party is favoured in a case of doubt; but it is clear that the apportionment of seats may often present difficulties of which the solution is not at once obvious to common sense.

25. To examine the matter more accurately, we shall use the following notation:—

m, the number of members to be elected by the constituency.

p, q, r . . . . . , the strengths of the several parties A, B, C . . . .

v, the total of all the votes polled.

$x_0, y_0, z_0, \ldots$, numbers proportional to p, q, r . . . . . and such that—

$$x_0 + y_0 + z_0 + \ldots = m \quad . \quad . \quad . \quad . \quad (1)$$

Q, the electoral unit, given by—

$$Q = \frac{p + q + r + \ldots}{m} = \frac{v}{m} \quad . \quad . \quad . \quad . \quad (2)$$

X, Y, Z . . . , the integral parts of x_0, y_0, z_0 . . .

α, β, γ, . . the fractional parts of x_0, y_0, z_0 . . . . . ., so that—

$$x_0 - X = \alpha, y_0 - Y = \beta, z_0 - Z = \gamma \quad . \quad . \quad (3)$$

p', q', r' . . respectively equal to x_0Q, y_0Q, z_0Q . . .

x, y, z . . . the seats obtained by the parties with the method of apportionment actually used.

26. The apportionment would be ideal if the rule-of-three method could be used without allotment of seats to remainders; that is, if the seats obtained by the parties were x_0, y_0, z_0 . . . To allot x_0, y_0, z_0 . . seats to the parties is equivalent to taking their strengths to be p', q', r' . instead of the actual p, q, r . . .

27. Confining ourselves to three parties, we have for x, y, z the equation—

$$x + y + z = m \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is the equation of a plane which cuts the axes of x, y, z, at points A, B, C equidistant from the origin. As x, y, z are positive, the only portion of the plane to be considered is that in the octant in which all the coordinates are positive; this portion is the equilateral triangle ABC.

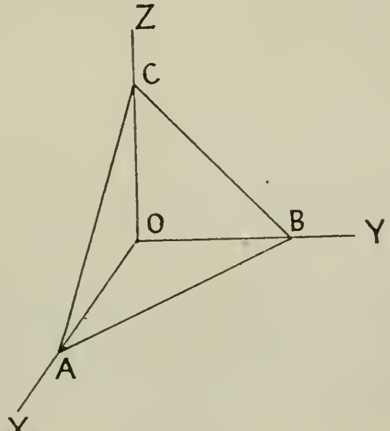

FIG. 2.

The numbers x, y, z are integral; and the positive integral solutions of (4) are represented by the points of intersection on ABC of planes drawn parallel to the coordinate planes, and at distance 0, 1, 2 . . .m from them. The solutions are therefore represented by the nodes of the equilateral triangular lattice shown in Fig. 3; and their numerical values are proportional to the distances of the nodes from the sides of ABC

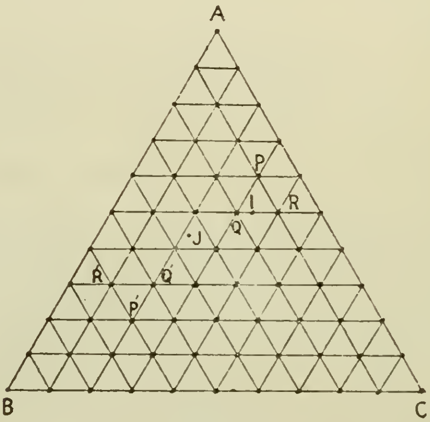

FIG. 3.

28. The ideal solution x_0, y_0, z_0, is represented by a point I (called the ideal point) in this triangle. If the rule-of-three method can be used without allotment of seats to remainders, the point I is a node. If allotment of seats to remainders is necessary, I is not a node, and we have to determine which of the neighbouring nodes gives the solution.

29. Let the triangle ABC be drawn so that the perpendiculars from A, B, C to the opposite sides are each m. The x, y, z of any point are then equal to the distances of the point from the sides of ABC, that is, they are the trilinear coordinates of the point. For instance, the point I in Fig. 3 represents the ideal solution in a case in which the strength of party A is exactly five times the electoral unit; the strength of party B, between one and two times the electoral unit; and the strength of party C between three and four times the electoral unit. Thus $x_0 = X =$ perpendicular from QR to BC; $y_0 =$ perpendicular from I to AB; $Y =$ perpendicular from PQ to AB; $\beta =$ perpendicular from I to PQ; $z_0 =$ perpendicular from I to AC; $Z =$ perpendicular from PR to AC; $\gamma =$ perpendicular from I to PR.

I may be either in a small triangle (such as PQR) similarly situated to ABC, or in a small triangle (such as $P'Q'R'$) not similarly situated to ABC. In the first case, if we move from I to P we do not alter Y or Z, but we increase X to $X + 1$; in the second case, if we move from I to P' we do not alter X, but we increase Y to $Y + 1$, and Z to $Z + 1$; and so with the other coordinates. In the first case, the sum of the remainders α, β, γ, is 1, and there is one seat to be allotted to a remainder; in the second there are two seats to be allotted.

30. Returning to § 26, in which we saw that the seats may be considered to be allotted as if the strength of party A were p' in place of p; the strength of B, q' in place of q; and the strength of C, r' in place of r, let us first examine what will be the solution if we consider the apportionment to be ideal when the differences between p and p', q and q', r and r' respectively, are as small as possible; that is, when—

$$\Sigma\,(p' - p)^2 \equiv k^2 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

is a minimum.

31. Putting $p' = xQ$, &c., and substituting from (2), we get—

$$\left(\frac{v}{m}\cdot x - p\right)^2 + \left(\frac{v}{m}\cdot y - q\right)^2 + \left(\frac{v}{m}\cdot z - r\right)^2 = k^2 \quad . \quad (6)$$

For the minimum value of (5), $k^2 = 0$, and $x = x_0$, $y = y_0$, $z = z_0$, but in this case it will not usually be possible to satisfy the further condition of the problem that x, y, z shall each be an integer.

For other values of k^2 (6) is a sphere having its centre at (x_0, y_0, z_0), or the ideal point I, and intersecting ABC in a circle whose centre is I. As k^2 increases from 0, the sphere expands from the ideal point, and is cut by ABC in a gradually increasing circle.

32. We now have the solution of the problem. In the triangle ABC, plot the point I whose trilinear coordinates are (x_0, y_0, z_0). If x_0, y_0, z_0 are integers (that is, if the strength of each party is divisible without remainder by the electoral unit), I will be a node of the lattice and the solution is (x_0, y_0, z_0).

If x_0, y_0, z_0 are not integers, we have to choose x, y, z so that k^2 is a minimum. The minimum value of k^2 is that for which the gradually increasing circle of intersection of ABC with the expanding sphere (6) passes through the nearest node of the lattice. The solution is therefore given by the trilinear coordinates of the nearest node. If the sum of the remainders is 1, I is in a triangle similarly situated to ABC, and the one unallotted seat goes to the largest remainder; if the sum of the remainders is 2, I is in a triangle not similarly situated to ABC, and the two unallotted seats go to the two largest remainders.

33. The solution is therefore the same as that given by the rule-of-three method, with the condition, in the case of remainders, that if there is one unallotted seat it goes to the party having the largest remainder, if two they go to the two parties having the largest remainders. Our discussion shows that the rule-of-three method, with this condition, gives the correct result if the apportionment is considered ideal when the differences between p and p', q and q', r and r', are as small as possible.

34. The following example for a 10-member constituency illustrates the solution:—

Party A	13,000	votes.
Party B	10,500	votes.
Party C	6,500	votes.
Total votes ...	30,000	

The rule-of-three method gives:—

Party A	4 electoral units;	remainder, 1000
Party B	3 electoral units;	remainder, 1500
Party C	2 electoral units;	remainder, 500

With the condition that the unallotted seat goes to the party having the largest remainder, the allotment is: A, 4 members; B, 4; C, 2. The point J of Fig. 3 is the position of the ideal point for this case; and the nearest node is (4, 4, 2).

35. The solution gives no guidance where remainders are equal. In such cases the solution given by the next method shows that the unallotted seat or seats should go to the largest party or parties.

36. A second method of discussing the problem is to give our attention to the proportion between votes and seats in each party instead of (as in the first discussion) to the differences between the actual and the assumed strengths of the parties.

Regarded thus, the allotment may be considered ideal if the number of votes to a member is as nearly as possible the same for all parties. This condition is expressible in the form that

$$\Sigma \left(\frac{x}{p} - \frac{y}{q}\right)^2$$

shall be a minimum. This expression can be written in the form

$$\Sigma \left(\frac{p' - p}{p}\right)^2 = k^2 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

37. For x, y, z, we now have the equation

$$\left(\frac{x}{p} - \frac{m}{v}\right)^2 + \left(\frac{y}{q} - \frac{m}{v}\right)^2 + \left(\frac{z}{r} - \frac{m}{v}\right)^2 = k^2 \quad . \quad . \quad . \quad (8)$$

As before, the minimum value of k^2 will be zero, and then $x = x_0$, $y = y_0$, $z = z_0$; but in this case x, y, z will not usually be integers.

For other values of k^2, (8) is an ellipsoid, having its centre at I, and intersecting ABC in an ellipse whose centre is I. As k^2 increases from 0, the ellipsoid expands from I, and is cut by ABC in a gradually increasing ellipse.

38. The solution of the problem is similar to the case of the circle. As before, we plot the ideal point I; if this is a node, the co-ordinates (x_0, y_0, z_0) of the node are the numbers of members for the parties.

If I is not a node, we have to select the node which is first touched by the gradually increasing ellipse. This is not necessarily the nearest node, and to determine which it is we must ascertain the direction of the longer axis of the ellipse. The direction of the longer axis may be calculated; but a simpler way is to project the triangle ABC so that the ellipse becomes a circle.

39. To see the effect of this projection, change the centre to the ideal point I; the ellipsoid (8) is then

$$\frac{x^2}{x_0^2} + \frac{y^2}{y_0^2} + \frac{z^2}{z_0^2} = k^2 \quad . \quad . \quad . \quad . \quad (9)$$

Stretching lengths parallel to the axes in the ratios, $1 : x_0$, $1 : y_0$, $1 : z_0$, respectively, (9) becomes a circle, and, as in § 32, the solution is given by the node nearest to I. We have next to calculate the lengths of the sides a, b, c

of the projected triangle ABC; these will be found to be given by

$$\left.\begin{aligned} a^2 &= \Sigma\frac{m^2}{x_0{}^2} - \frac{m^2}{x_0{}^2} = \Sigma\frac{v^2}{p^2} - \frac{v^2}{p^2} \\ b^2 &= \Sigma\frac{m^2}{x^{02}} - \frac{m^2}{y_0{}^2} = \Sigma\frac{v^2}{p^2} - \frac{v^2}{q^2} \\ c^2 &= \Sigma\frac{m^2}{x_0{}^2} - \frac{m^2}{z_0{}^2} = \Sigma\frac{v^2}{p^2} - \frac{v^2}{r^2} \end{aligned}\right\} \quad . \; . \quad (10)$$

Writing *cos* A, *cos* B, *cos* C in terms of a, b, c, and so of m, x_0, y_0, z_0, we find that each is positive, and not greater than 1. The triangle is therefore acute; and (10) show that the greatest side is the base from which the co-ordinate of the greatest party is measured. One possible shape of the triangle is shown in Fig. 4.

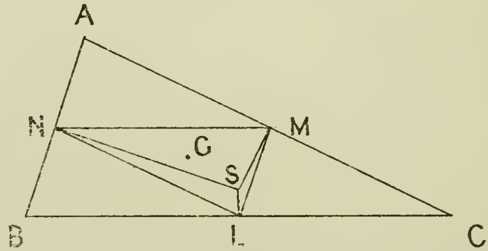

Fig. 4.

Other properties of the triangle are that the circumcentre S is within the triangle; and that if x_0 is the greatest of x_0, y_0, z_0, the centre of gravity G is within the triangle MNS.

It is to be noted that areal coordinates project unaltered, and that any line from a vertex to the opposite side is divided by a line parallel to the base in the same ratio after projection as before.

The region $AMSN$ is the portion of the triangle in which each point is nearer to A than to B or C; so with $BLSN$ and $CLSM$.

All these properties are true of the projection of the small triangle in which I lies; let us now consider Fig. 4 as representing the projection of the small triangle.

40. If the small triangle is similarly situated to the large triangle, and α, or $x_0 - X$, is greater than $\frac{1}{2}$, the ideal point will be in the region ANM, and so will be nearer to A than to B or C, whatever the values of

y_0, z_0, β, γ. That is, if there is only one seat to be allotted to remainders, and one party has a remainder greater than half the electoral unit, that party will get the seat whatever the sizes of the parties. In a similar way we see that if there are two seats to be allotted to remainders, and one party has a remainder less than half the electoral unit, the other two parties each get one seat, whatever the sizes of the parties.

41. As the centre of gravity G, for which $\alpha = \beta = \gamma = \frac{1}{3}$, lies within $AMSN$, we see that when there are three remainders each equal to one-third of the electoral unit, and one unallotted seat, the largest party gets the seat; when there are three remainders, each equal to two-thirds of the electoral unit, and two unallotted seats, the two smaller parties each get a seat.

Thus Fig. 4 is drawn for the case in which $x_0 = 5\frac{1}{3}$; $y_0 = 1\frac{1}{3}$, $z_0 = 3\frac{1}{3}$ (see § 24). The ideal point is the centre of gravity of the small triangle in which it lies, and also the centre of gravity G of the projected triangle. As G is nearer to A than to B or C, A represents the solution, which is therefore (6, 1, 3).

42. Other properties of the figure can be written down from inspection, but in general to ascertain what the apportionment will be it is necessary to plot the particular case. Certain general results can however be seen without plotting each case. Thus taking the cases in which the strength of party A is $53\frac{1}{3}$ % ($x_0 = 5\frac{1}{3}$, when $m = 10$), and in which the strengths of the other parties have all possible values from O to $46\frac{2}{3}$ %, we should have a series of figures similar to Fig. 4. The distance of the ideal point from BC will always be one-third of the perpendicular from A to BC. The distance of A from BC varies as y_0 and z_0 change; but the path of the ideal point can be pictured as not very different from the line described by G as it moves across the triangle of Fig. 4 parallel to BC. This leads to the result that when the largest party is $53\frac{1}{3}$ %, and the remainders β and γ of the other parties are each less than half the electoral unit, the largest party will always get the doubtful seat, except for a very small range in which β is just under half the electoral unit, and a very small range in which γ is just under half the electoral unit. For a further discussion of this case, see § 47.

43. In the third method of discussing the problem, attention is again given to the number of votes to a member, looked at in the form of the fraction of a member returned by each vote. The apportionment is considered

ideal when the deviation from equality in this respect is as small as possible for each vote; a condition expressible in the form that

$$\Sigma p\left(\frac{x}{p} - \frac{x+y+z}{p+q+r}\right)^2$$

or

$$\Sigma\frac{(p'-p)^2}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

shall be a minimum.

As in the second method, the analysis gives us for the zero value of (11) the ideal point; and for values greater than zero an ellipsoid gradually expanding from the ideal point. As before, we project from the ideal point to obtain a circle, and the shape of the projected triangle is given by—

$$\left.\begin{aligned} a^2 &= \Sigma\frac{v}{p} - \frac{v}{p} \\ b^2 &= \Sigma\frac{v}{p} - \frac{v}{q} \\ c^2 &= \Sigma\frac{v}{p} - \frac{v}{r} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The properties of this triangle can be investigated in much the same way as before, and will be found to be similar to those of the triangle of the second method.

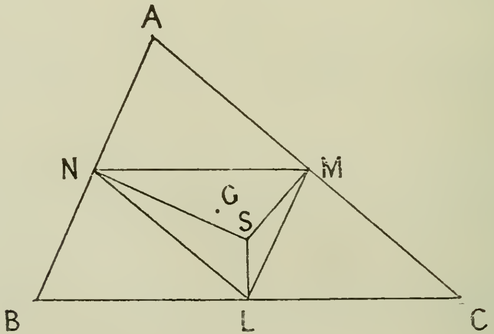

Fig. 5.

Fig 5 is drawn for the case of $x_0 = 5\frac{1}{3}$, $y_0 = 1\frac{1}{3}$, $z_0 = 3\frac{1}{3}$. Comparing this with Fig. 4, which represents this case according to the second method, we see that the result is much the same, but that the range for which β γ can be less than half the electoral unit, without the largest party getting the seat, is rather less than with the second method.

The third method is not again referred to in this paper.

44. An account of the rules which have been used in various countries for the allocation of seats to competing lists is given by Mr. John H. Humphreys in an article in *Representation* for October, 1908 (Vol. II., pp. 67-81), and also in his work *Proportional Representation* (London, 1911), Chapter VIII. It seems that the rule-of-three method, with the condition that seats not allotted to complete electoral units should go to the largest remainders, was the first to be used, when a party-list system was introduced in Switzerland in 1890. This condition was afterwards abandoned, and instead the seats not allotted to complete electoral units were given to the largest parties, without consideration of the size of the remainders.

45. Neither of these rules was found to be quite satisfactory. Of the substitutes proposed the best known is the method of Professor Victor d'Hondt of the University of Ghent. This is embodied in the following articles of the Belgian Electoral Code (17):—

Article 263.—Le bureau principal divise successivement par 1, 2, 3, etc., le chiffre électoral de chacune des listes et range les quotients dans l'ordre de leur importance jusqu' à concurrence d'un nombre total de quotients égal à celui des membres à élire. Le dernier quotient sert de diviseur electoral.

La répartition entre les listes s'opère en attribuant à chacune d'elles autant de sièges que son chiffre électoral comprend de fois ce diviseur, sauf application de l'article 264. . . .

Article 264.—Lorsqu'un siège revient à titre égal à plusieurs listes, il est attribué à celle qui a obtenu le chiffre électoral le plus élevé et, en cas de parité des chiffres électoraux, à la liste où figure le candidat dont l'élection est en cause qui a obtenu le plus de voix ou, subsidiairement, qui est le plus âgé.

46. The working of the D'Hondt rules will be clear from the following passages from Mr. J. H. Humphreys: (18)

Let it be assumed that three lists have been presented; that they have obtained 8000, 7500, and 7000 votes respectively, and that there are five vacancies to be filled. The total number of votes for each list is divided successively by the numbers 1, 2, 3, and so on, and the resulting numbers are arranged thus:—

List No. 1.		List No. 2.		List No. 3.
8000	...	7500	...	4500
4000	...	3750	...	2250
2666	...	2500	...	1500

(17) See (6), p. 15.

(18) *Representation*, Oct., 1908 (Vol. II., pp. 71-73), and *Proportional Representation*, pp. 179-180.

The five highest numbers (five being the number of vacancies to be filled) are then arranged in order of magnitude, as follows:—

8000 (List No. 1).
7500 (List No. 2).
4500 (List No. 3).
4000 (List No. 1).
3750 (List No. 2).

The lowest of these numbers, 3750, is called the "common divisor," or the "electoral quotient," and forms the base on which the seats are allotted. The number of votes obtained by each of the lists is divided by the common divisor, thus:—

8000	divided by	3750 = 2	with a remainder of	500.
7500	,,	3750 = 2.		
4500	,,	3750 = 1	,, ,,	750.

The first list contains the electoral quotient twice, the second twice, and the third once, and the five seats are allotted accordingly. Each party obtains one representative for every quota of voters which it can rally to its support; all fractions of "quotas" are disregarded, and all seats are disposed of at the first distribution.

The method of determining the electoral quotient may appear at first sight rather empirical, but the rule is merely the arithmetical expression, in a form convenient for returning officers, of the following train of reasoning: The three lists with 8000, 7500, and 4500 supporters are competing for seats. The first seat has to be allotted; to which list is it to go? Plainly to the list with 8000 supporters. Then the second seat has to be disposed of; to which list is it to go? If it is given to the first list, then the supporters of the first list will have two members in all, or one member for each 4000 votes. This would be unfair while 7500 supporters of the second list are unrepresented, therefore the second seat is allotted to the list with 7500 supporters. Similar reasoning will give the third seat to the list with 4500 supporters, the fourth to the list with 8000 supporters (which now will rightly have one representative for each 4000), and the fifth to the list with 7500. The question in each case is to what list must the seat be allotted in such a way that no one group of unrepresented electors is larger than a represented group. The separate allotment of seats one by one in accordance with the foregoing reasoning may be shown thus:—

8000 (List No. 1).
7500 (List No. 2).
4500 (List No. 3).
4000 (List No. 1).
3750 (List No. 2).

This result, of course, agrees with that obtained by the official process of dividing the total of each list by the electoral quotient.

The d'Hondt rule certainly accomplishes its purpose. It furnishes a measuring rod by which to measure off from each total of votes the number of seats won by the list. But the d'Hondt rule is not without its critics. As in the earlier Swiss methods objection was taken to the undue favouring of certain fractions, so in Belgium, objection is taken to the fact that remainders are not taken into account at all. The Belgian rule works to the advantage of the largest party, a fact that many may consider as a point in its favour. A further simple example will show the force of this statement. Assume that 11 seats are being contested by three parties, whose votes are as follows:—

Party A	6000 votes
Party B	4800 votes
Party C	1900 votes
Total	13,700

Arrange these numbers in a line and divide successively by 1, 2, 3, and so on, thus :—

Party A.	Party B.	Party C.
6000	4800	1900
3000	2400	950
2000	1600	633
1500	1200	475
1200	960	380
1000	800	316

The eleventh highest number, which constitutes the measuring rod, will be found to be 1000; the largest party obtains 6 seats, the second party obtains 4 seats, with a remainder of 800 votes, and the third only one seat, with a remainder of 900 votes. The two smaller parties taken together poll 6700 votes but only 5 seats, as compared with the 6 seats obtained by the larger party with 6000 votes, the two remainders, 800 and 900 votes which, together, constitute more than a quota, having no influence on the result of the election. Even if, in the allotment of seats, the largest party has a remainder of votes not utilised, yet this remainder necessarily bears a smaller proportion to the total of the votes polled than is the case with a small party. Thus the system works steadily in favour of the larger party.

47. We are now in a position to compare the representation which will be given in a three-party contest by—

(*a*) The first, or rule-of-three, method;
(*b*) The second method (number of votes to a member as nearly as possible the same for each party).
(*c*) The D'Hondt method.

The following tables show the representation given by each method for cases in which the largest party has rather more than half of the votes. Other cases can be compared in the same way.

The numbers in the columns x, y, z are the members for the parties. The strengths at which the representation changes are tabulated; where there is no entry in the x, y, z columns, the representation is the same as at the next higher entry.

Party-List System in Three-party Contest in 10-member Constituency.

(Size of Largest Party, $53\frac{1}{3}$ °/₀.)

Second Party.	Third Party.	First Method (Rule-of-three).			Second Method.			D'Hondt Method.		
°/o	°/o	x		z	x	y	z		y	z
46·67	0	5	5	0	5	5	0	5	5	0
*45	*1·7	...	...	...	6	4	0	...	...	...
44·44	2·22	...	...	...	...	...	...	6	4	0
43·33	3·33	5	4	1	...	...	...	...	...	...
41·67	5	...	...	...	5	4	1	...	...	...
37·77	8·88	...	...	...	...	...	...	5	4	1
35·55	1·11	...	...	...	...	...	...	6	3	1
*34	*12·7	...	...	...	6	3	1	...	...	...
33·33	13·33	5	3	2	...	...	...	...	...	...
31·67	15	...	...	...	5	3	2	...	...	...
28·88	17·77	...	...	...	...	...	...	5	3	2
26·67	20	...	...	...	...	...	...	6	2	2
*24·7	*22	...	...	...	6	2	2	...	...	...
23·33	23·33	5	2	3	6	2	2	6	2	2

* Approximate.

(Size of Largest Party, $56\frac{2}{3}$ °/₀.)

Second Party.	Third Party.	First Method. (Rule-of-three).			Second Method.			D'Hondt Method.		
°/o	°/o	x	y	z	x	y	z	x	y	z
43·33	0	6	4	0	6	4	0	6	4	0
*38	*5·3	...	...	...	5	4	1	...	...	...
36·67	6·67	6	3	1	...	...	...	...	...	...
35	8·33	...	...	...	6	3	1	...	...	...
34·67	8·66	...	...	...	...	...	...	6	3	1
*28	*15·3	...	...	...	5	3	2	...	...	...
26·67	16·66	6	2	2	...	...	...	...	...	...
*25	*18·3	...	...	...	6	2	2	...	...	...
24·44	18·88	...	...	...	...	...	.	6	2	2
21·67	21·67	6	2	2	6	2	2	6	2	2

* Approximate.

Party-List System in Three-party Contest in 11-member Constituency.

(Size of Largest Party, $53\frac{1}{3}$ %.)

Second Party.	Third Party.	First Method (Rule-of-three).			Second Method.			D'Hondt Method.		
%	%	x	y	z	x	y	z	x	y	z
46·66	0	6	5	0	6	5	0	6	5	0
41·51	5·15	6	4	1	6	4	1	...	...	...
38·88	7·77	...	...	...	...	...	...	6	4	1
32·42	14·24	6	3	2	6	3	2	...	...	...
31·11	15·55	...	...	...	...	...	...	6	3	2
23·33	23·33	6	2	3	6	2	3	6	2	3

48. The tables indicate (as is otherwise apparent) that an odd number of seats for a constituency is to be preferred to an even number, and that all the methods are liable to give somewhat irregular results as the strength of a party changes.

49. The D'Hondt method has not the advantage that the reasons for it are obvious; and its merits are still much discussed on the Continent of Europe. (19)

50. The rule-of-three method has the advantage of simplicity. The condition on which it is based (see the first method, § § 26-35) does not seem to be as just a condition as that of the second method (see § § 36-42), but the cases investigated for a three-party contest show that the results are much the same from the two conditions, and indeed that the representation of the largest party by the rule-of-three method is not liable to the curious changes which occur with the second method. For a three-party contest, then, the rule-of-three method seems to be satisfactory; but if the number of seats for the constituency is even, it is desirable to add the further condition that a party whose strength is between 50 per cent. and 60 per cent. of all the votes shall have one more than half the number of seats.

(19) Mr. Humphreys (*Proportional Representation*, p. 202), refers to the following works, which are not accessible to me in Hobart:—

Examen Critique des Divers Procédés de Répartition Proportionnelle en Matière Electorale, by M. E. Macquart, in *Revue Scientifique*, 28th October, 1905.

La Répresentation Proportionnelle et les Partis Politiques, by M. P. G. La Chesnais.

La Vraie Représentation Proportionnelle, by M. Gaston Moch.

51. The rules for a three-party contest in a constituency returning an even number of members would then be:—

I. Divide the total of the votes for each list by the electoral unit (§ 22). For each whole electoral unit contained in the votes for a list allocate a seat to the list.

II. If the largest party has between 50 per cent. and 60 per cent. of all the votes, give it one more seat than the number it obtains under I.

III. Subject to Rule II., allocate the one seat or the two seats remaining to the party or parties having the largest remainder or remainders.

If the constituency returned an odd number of members, use Rules I. and III. only.

52. In a contest between two parties, each of the methods will give the same result; and Rules I. to III. above will be used.

53. It is to be noted in conclusion that the differences described in this paper between the various systems and the various quotas are only minor matters, though important in deciding between the various methods of securing proportional representation; and the discussion of them should not be allowed to distract attention from the substantial advantages which any system of proportional representation has over any non-proportional system.

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