Dicing Decimal Digits

Neville Holmes
School of Computing
University of Tasmania
Launceston 7250 Australia

(Appeared *The Mathematical Gazette*, Vol.81, No.491 (1997 July), 288–293)

Abstract

It is sometimes useful to be able to generate truly random integers without any but the most basic equipment. Coins and cubic dice are the most basic equipment for reliably generating random numbers. This note describes some simple methods of using dice to generate random decimal digits, and so, random decimal numbers.

Introduction

Random numbers are useful for many purposes, both recreational and professional.

As for many other kinds of numbers, electronic computers are useful for producing what are taken to be random numbers. However, as is, or should be, well known, digital computers cannot produce truly random numbers. To be useful for digital calculation, computers must be completely deterministic, unable to produce truly random behaviour except by fault, or unless they are fitted with special equipment which reads in something random from the environment—white noise of some kind. Furthermore, computers are not always available or convenient to use, for example recreationally, if only because they are liable to malfunction when tossed or thrown.

While there are procedures to produce entirely by hand (or by finger) allegedly random numbers[1], people are not reliably random in their behaviour, so a more acceptable method would involve throwing into the air an inert object which will assume one of a finite number of equiprobable distinguishable positions when it comes to rest on the ground. The simplest commonly available inert objects for generating random digits in such a manner are coins or cubic dice. Although neither cubic dice[2] nor coins [3, and see the Editor's note to 4] necessarily behave exactly equiprobably, in practice they are equiprobable enough.

The Principle of Least Work

A close approximation to equiprobability will almost always be accepted. Inefficiency will not, and comes in several guises.

It is inefficient to need special equipment. This is why either coins or cubic dice, both of which most people already own, are the equipment of choice, except maybe in Japan where the Japanese Standards Association is reported to sell decimally marked icosahedral dice in three different colours so that three decimal digits can be thrown at once[5].

It is inefficient to need computational competence because it is rare, because it is slow, and because it is unreliable. This is why the principle of least work rules out the use of coins to generate random binary digits, groups of which then have to be converted by calculation to decimal digits.

It is inefficient to have to remember things. The rules for generating the digits have to be remembered anyway, so extra feats of memory must be unwelcome. A single cubic die cannot produce a decimal digit in a single throw. Therefore the result of one throw has to be carried on to a second, and this carrying on is extra work, and inefficient unless the computational work can be simplified in compensation.

The computational work is typically large because a multiplication is involved, for example if base 6 digits are being collected for conversion to decimal digits. Also, people educated in advanced countries often cannot do multiplication reliably[6]. There is an interesting procedure available, pointed out to me by David Blest, which involves throwing a single die at least twice per decimal digit, and which uses no pesky multiplication. The first throw determines a bias for the subsequent throw or throws, `-1` for say an odd first throw, `4` for an even. This bias must be committed to memory, and the die is then rethrown until a non-`6` appears, whereat the bias is added to the throw to give the decimal digit.

This is a delightfully simple approach, but requires on average 2.2 throws per decimal digit.[1] This high effort per digit raises the final aspect of the principle of least work—that the number of physical throws must be kept as low as consistent with other inefficiences.

One way to reduce the physical work of throwing is to throw more than one die at a time. If this is done in the bias method just described, for example, there has to be a way of distinguishing one die from the other, so that the die that determines the bias is distinguished from the die that determines the digit to be biassed. The obvious way to do this is with contrastingly coloured dice, such as are used in playing backgammon[7].

There are two drawbacks in using coloured dice. Firstly, not that many people play backgammon. Secondly, two rules are required, one for each colour, and which is which has to be remembered.

The simplest approach, then, is to throw undistinguished dice in multiples, and find as simple an algorithm as possible to produce one or more decimal digits. In the following, indistinguishable dice are assumed.[2]

Using Two Dice

In practice, cubic dice are most commonly used, and sold, in identical pairs. The dots on each die, as they rest after being thrown, are counted together in most games using them. Because the faces of most such dice are labelled from `1` to `6`, the simple sums are not directly usable for producing decimal digits. However a decimal digit may most easily be produced by taking the throw modulo 10, an operation which is relatively easy to learn, even for school graduates of advanced countries.

The following table shows what digits are produced in this way, where each cell of the table has an equal probability of occurring.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	0
5	6	7	8	9	0	1
6	7	8	9	0	1	2

This table shows that there is an uneven distribution of the digits. If these digits are to be used directly, then at most two can be used of each, and the problem is to find a rule to select the twenty cells to be used. Since there are only two `2`s anyway, that's that, and an obvious pattern of pairs of the other digits suggests itself, as shown in the next table.

	1	2	3	4	5	6
1	2	3	4			
2	3		5	6		
3	4	5		7	8	
4		6	7		9	0
5			8	9		1
6				0	1	2

The rule here might be worded, *If you have a* `2` *take it, otherwise only use the throw if the dice are one or two apart*. Quite simple really.

However, this uses only twenty out of the thirty six cells, a work rate of 1.8 throws/digit. A rule which uses thirty out of the thirty six cells is impossible as long as the two dice are indistinguishable,

[1]There are several ways to improve the yield per throw, however. One way is to extract two biasses from the first die thrown. For instance, if `1` or `6` are rethrown, then `2`→`-1 -1`, `3`→`4 -1`, `4`→`4 4` and `5`→`-1 4` have useful mnemonicity, and the procedure will yield 1.76 throws per digit on average, though with more load on short term memory.

[2]Simple radix conversion methods using multiple dice cannot be used if the dice are indistinguishable. There has to be some way of sequencing the sexary digits. However, for someone of unusually great arithmetic ability there is a highly efficient possibility pointed out by a helpful referee who observes that 6^9 is 10 077 696, only very slightly greater than 10^7, and so throwing seven decimal digits with nine mutually distinguishable cubic dice is therefore highly conservative of physical labour, yielding a work rate of 1.2957 throws/digit/die, compared to the best possible of 1.2851.

because there are then only twenty-one distinct throws and the distinct throws are not equiprobable.

One possibility to be examined is that of throwing three dice at once. The idea here would be to find a fairly simply rule that wastes fewer throws.

In the following, it will be useful to adopt a variation on the two dice rule given above to cope with situations where the double 1 or double 6 can't be used. The simplest way out here is to adopt the 1 and 6 pair as producing a 2 instead of being discarded. Since it then uses twenty out of the thirty non-double cells, this rule can be called the *two thirds* rule.

Using Three Dice

In the following chart classifying three dice throws, the six component squares suggest a cube of equiprobable cells. A cell with a hyphen stands for a throw with no pair, with an = a throw with one pair, and a 3 three pairs.

```
3===== ==---- =-=--- =--=-- =---=- =----=
==---- =3==== -==--- -=-=-- -=--=- -=---=
=-=--- -==--- ==3=== --==-- --=-=- --=--=
=--=-- -=-=-- --==-- ===3== ---==- ---=-=
=---=- -=--=- --=-=- ---==- ====3= ----==
=----= -=---= --=--= ---=-= ----== =====3
```

Firstly, there are the entries along the main diagonal, the six triples. These are decimally intractable, so must be discarded. This is not cost effectively hopeless, as it represents only 3.6% waste in itself.

Secondly, there are the three main diagonal planes, excluding the six cells of the main diagonal itself common to each. These are for the throws with one double, with thirty cells to each plane. If we discard one of each double, this reduces the problem to a two dice situation where the two thirds rule can be applied. This group therefore uses sixty out of the ninety diagonal plane cells.

Thirdly, there are one hundred and twenty off-diagonal cells, representing the throws with no doubles. This is a nice multiple of ten, so there is a good prospect of using all such throws. There are twenty distinct equiprobable such throws, as shown in the following table complete with modulus 10 sums, so the prospects shine brightly.

		3	4	5	6
1	2	6	7	8	9
1	3		8	9	0
1	4			0	1
1	5				2
2	3		9	0	1
2	4			1	2
2	5				3
3	4			2	3
3	5				4
4	5				5

Depressingly, the decimal digits are not evenly produced. However, the table splits nicely into two halves of ten cases each, and, as shown here, one half with a 1 throw and one half without. So let's try modulus 5 for the same tabled sums, an operation not all that daunting for non-mathematicians.

		3	4	5	6
1	2	1	2	3	4
1	3		3	4	0
1	4			0	1
1	5				2
2	3		4	0	1
2	4			1	2
2	5				3
3	4			2	3
3	5				4
4	5				0

Now the quinary digits spread evenly in each half of the table, so the rule can be, *Add* 5 *to the modulo 5 sum unless there is a* 1 *to keep the digit small.* A very simple rule, with no wasted throws!

The two rules together give the following result cube, with 180 out of the 216 cells used, 1.2 throws/digit, a great improvement on 1.8 throws/digit of the two indistinguishable dice method. The cost of this improvement is the burden of remembering two rules.

```
 34   2 331234 414340  23 01  340 2 240122
331234 3 56    155956 269667 3 56 8 4 678
414340 155956 45 78  397778 458789 06 89
 23 01 269667 397778  67 90 067995 178050
 340 2 3 56 8 458789 067995   89 1 289511
240122 4 678  06 89  178050 289511 2  01
```

Conclusion

The methods described, for generating decimal digits with two or three *indistinguishable* cubic dice thrown simultaneously, are based on merely taking the sum of the values shown on the faces of the standard cubic die. While a labour-saving method for three dice has been shown, the mind-saving two dice method is probably more acceptable.

If *distinguishable* dice are available, or if only one die is available and memory of a previous throw can be relied on, the Blest derivation seems simplest and not onerously labour intensive.

Are these the simplest methods based on summing? Are there simpler methods based on some combining function other than the sum? What does *simpler* mean, here[8]?

References

1. D. Zeilberger & J. Legrange, How to Play Backgammon without Dice. *J. Recrtnl. Maths.* **11**, 3 (1978–79), p.206.

2. D. C. Blest & C. B. Hallam, The Design of Dice. *Maths Today* **32**, 1/2 (Jan./Feb. 1996), pp.8–13.

3. I. Szalkai & D. Velleman, Versatile Coins. *Am. Math. Mthly.* **100**, 1 (Jan. 1993), pp.26–33.

4. D. Singmaster, Theoretical probabilities for a cuboidal die. *Math. Gaz.* **65** 433 (1981) pp.208–210.

5. M. Gardner, On the ancient lore of dice and the odds against making a point. *Sc. American* **219**, 5 (Nov. 1968), pp.140–2,4,6.

6. J. P. Paulos, *Innumeracy*. Penguin, London, 1990.

7. R. C. Bell, *Board & Table Games*. Oxford University Press, London, 2 vols. 1960, 1969.

8. M. Gardner, Simplicity as a scientific concept. *Sc. American* **221**, 5 (Nov. 1969), pp.118–21.