APPENDIX.

Researches in Relativity

II.—The Basis of the Physical World as indicated by carrying as far as possible the Tenets of Relativity.

BY

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(Read 15th April, 1925)

NOTE.—Paging and numbering of Articles continued from "Researches in Relativity, I.," P. and P., 1924, Appendix.

Additional Errata

In the first paper an unfortunate error of sign occurred early and has necessitated the corrections below. ${}^{c}\lambda$ and ${}^{c}\omega$ were correctly defined, the first in the last line of page 1 and the second in equation (2) on page 2, but the minus sign is incorrectly put before ${}^{c}\omega$ on line 24 of page 3; and the following corrigenda are the result:-

Page 3, line 24 For $-{}^{c}\omega$, read ${}^{c}\omega$
Page 4, equation (7) For $V_2\Delta{}^{c}\lambda$, read $-V_2\Delta{}^{c}\lambda$
Page 4, equation (10) For $-{}^{l}\kappa$, read $+{}^{l}\kappa$
Page 5, equation (14) Delete first minus sign.
Page 6, line 17 For

$${}^{c}\lambda V_0\Delta^{l}\lambda + V_1{}^{l}\lambda{}^{c}\omega = {}^{c}\lambda V_0\Delta^{l}\lambda + V_1{}^{l}\lambda V_2\Delta^{c}\lambda,$$

read

$$-{}^{c}\lambda V_0\Delta^{l}\lambda + V_1{}^{l}\lambda{}^{c}\omega = -{}^{c}\lambda V_0\Delta^{l}\lambda - V_1{}^{l}\lambda V_2\Delta^{c}\lambda$$

Page 6, equation (15) For $V_1{}^{l}\lambda{}^{c}\omega$ read $-V_1{}^{l}\lambda{}^{c}\omega$

I hope this completes the list of necessary corrections.

RESEARCHES IN RELATIVITY. II.

(Art.7 - 12)

THE BASIS OF THE PHYSICAL WORLD AS INDICATED BY CARRYING AS FAR AS POSSIBLE THE TENETS OF RELATIVITY

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(Read 15th. April, 1925)

Art. 7. On the kind of invariantive relations to be expected in the physical manifold.

A summary of the present paper will be found in the opening paragraphs of Art. 12 below.

In the last paragraph but one of Art. 6 it was stated that researches already in existence were to occupy our attention. But the writer has found much of interest to add to the foundations. The present paper will be occupied with these additions.

To begin with, it may have seemed to the reader that Einstein's original basis and the writer's addition thereto given in the first paper — the basing of all Physics on the fundamental affine linity E_α — is somewhat artificial. I propose on the contrary to show that it is scarcely possible to conceive any simpler principle that can be reconciled with the beliefs of all relativists when one tries to put those beliefs down in a definite mathematical form. We all believe that the foundation is some reality (called action) situated at each element

$$db = dx_1dx_2dx_3dx_4$$

of a four-dimensional continuum, and the relative values of the portions of action in neighbouring elements. In a word we believe that there is no "action at a distance", but that everything is to be deduced in invariantive form from ${}^{l}W$ and its derivatives of all orders, where ${}^{l}W$ (henceforth to be used in place of ${}^{l}H$ of the first paper) is the action density. The existence of such invariantive form is not obvious: much less is it obvious that we can obtain relations which are so far invariantive that they do not even depend on the particular value given to ${}^{l}\mathrm{W}$ but are true simply and solely because ${}^{l}\mathrm{W}$ and its derivatives are of their respective classes. But this is precisely what the principle of our first paper ensures, though the reader may not be prepared for this statement, and our first task must be to establish it.

It is frequently more convenient to work with the logarithm of such a density as ${}^{l}\mathrm{W}$ rather than with ${}^{l}\mathrm{W}$ itself. Let x be any log-scalar-density, that is x is of the same nature as $\log{}^{l}\mathrm{W}$. We have first to consider how to arrive at invariantive relations among

$$x,\ \Delta x,\ \mathrm{V}_0\alpha\Delta \,.\, \Delta x,\ \mathrm{V}_0\alpha\Delta \,.\, \mathrm{V}_0\beta\Delta \,.\, \Delta x, \text{ etc.},$$

where, as usual, α, β, etc., are any number of contravariant vector dummies.

Now, from the history of dealing with such continuums as we have under consideration, from Riemann to the present day, it may be taken for granted that the problem as so stated necessitates that our continuum must possess structure. (It is open to argument that without such structure we have provided no physical foundation at all.) Riemann provided this structure by the quadratic differential form. Civita-Levi, Weyl, Eddington and Einstein have developed what must be regarded as a generalisation more natural to our present point of view.

The basis of their structure was an intrinsic increment (due to parallel displacement) of vectors covariant or contravariant. But from our point of view an intrinsic increment of density (a scalar) is more fundamental. Let us put, in definite forms in parallel, the meanings of these two intrinsic increments. The use of the word intrinsic in these two senses implies the following two equations

$$x_\alpha = (\mathrm{D}_\alpha - \mathrm{S}_\alpha)x = \mathrm{V}_0\alpha(\Delta - \nu)x \qquad (1)$$

$$^c\tau_\alpha = (\mathrm{D}_\alpha - \mathrm{E}_\alpha)^c\tau \qquad (2)$$

where the expressions on the left, x_α, $^c\tau_\alpha$, are absolute increments of the scalar x described above, and of a covariant vector $^c\tau$. These absolute increments x_α, $^c\tau_\alpha$, are furnished by comparison with the intrinsic increments $\mathrm{S}_\alpha x$, $\mathrm{E}_\alpha{}^c\tau$ which are dependent on what may be termed the parallel displacement α (an infinitesimal contravariant vector) and quite independent of choice of coordinates. D_α merely stands for the ordinary differential operator $\mathrm{V}_0\alpha\Delta$. . E_α is a linity of $^c\tau$, the linity itself being linear in α; similarly S_α is a linity of the scalar x, the linity itself being linear in α; that is to say S_α is of the form $\mathrm{V}_0\alpha\nu$, where ν is a vector. It follows that E_α, S_α, ν, are non-invariantive. Although non-invariantive the relations of these symbols to change of coordinates are quite simple, and it is rather surprising that, so far as I know, they have not hitherto been given in the case of E_α.

From the meaning just given to "intrinsic" it follows that x_α is a scalar density and $^c\tau_\alpha$ is a covariant vector. From this alone follow readily the relations just mentioned of E_α and ν, whether the change of coordinates be finite or infinitesimal. In the present paper infinitesimal change only will be treated of.

The reader will find no difficulty in proving that

$$\delta' x = -\mathrm{V}_0\Delta\sigma, \delta'\mathrm{V}_0\alpha\nu = -\mathrm{V}_0\alpha\Delta \,.\, \mathrm{V}_0\Delta\sigma \qquad (3)$$

$$\delta'\mathrm{V}_0\gamma\mathrm{E}_\alpha{}^c\beta = -\mathrm{V}_0\gamma\Delta \,.\, \mathrm{V}_0\alpha\Delta \,.\, \mathrm{V}_0{}^c\beta\sigma \qquad (4)$$

where δ' and σ are the δ and ϵ used in equations (2), (3), of Art. 15 of M.D.I. (2), in which place infinitesimal change of coordinates was first considered in our notation. (For σ see Art.4 on p. 14 above.)

Note that nothing whatever has been added here to the meaning (as originally introduced by Weyl and Eddington) of the affine linity E_α . I have in M.D.I. (3) already shown that absolute differentiation is a consequence of the meaning.

Note also some important pure mathematical truisms. Let x , ${}^0x$ be two scalars of type x ; ν , ${}^0\nu$ vectors of type ν ; and E_α , ${}^0E_\alpha$ two affine linities.

Then:-

(1) $x-{}^0x$ is an invariant. $\nu-{}^0\nu$ is a co-variant vector. $E_\alpha-{}^0E_\alpha$ is invariantive, i.e. it is a coexco vector linity whose form is linear in the contravariant vector α .

(2) Therefore the general values of x , ν , E_α , are given by

$$\left.\begin{aligned} x = {}^0x+y,\ \nu = {}^0\nu+{}^c\nu, \\ E_\alpha = {}^0E_\alpha+Y_\alpha \end{aligned}\right\} \quad (5)$$

where y is an invariant, ${}^c\nu$ a covariant vector, Y_α a coexco vector linity which is linear in α . ${}^0x$, ${}^0\nu$, ${}^0E_\alpha$ are in equation (5) any convenient particular functions of their respective types.

(3) Both Δx and $E_\epsilon\epsilon$ are of type ν. In particular we may take ${}^0\nu$ to be either Δ^0x or ${}^0E_\epsilon\epsilon$.

Thus any fundamental scalar density furnishes standard forms for the particular functions ${}^0x$, ${}^0\nu$. The early study of Riemannian Geometry provides a form for ${}^0E_\alpha$. Let ϕ be any coexcontra self-conjugate vector linity. Then defining $\phi_{\alpha)}$ by

$$2\phi_{\alpha)}\beta = V_0\alpha\Delta \cdot \phi\beta - V_0\beta\Delta \cdot \phi\alpha+\Delta V_0\beta\phi\alpha \quad (6)$$

we may put ${}^0E_\alpha = \phi_{\alpha)}\phi^{-1}$ More particularly we may define ϕ by saying that $\phi =1$ in some selected system of coordinates, not merely at a single point but at all points.

<u>Art.8.</u> Insufficiency of Structure simpler than the affine.

A world with structure ν but without structure E_α

seems possible at first sight. $V_2\Delta\nu$ is covariant. This may be verified at once from equation (5) by putting ${}^0\nu = \Delta^0x$ for $V_2\Delta^c\nu$ is known to be covariant. It may however be proved by a more familiar process, by summing the absolute increment of any x round a closed path and observing that as the sum equals the difference of two such values of x at a single point it must be invariant. Thus $V_2\Delta\nu$ is seen to be analogous to the general curvature derivable from the other kind of structure E_α .

Our present quest is for invariant relations as a basis, in our manifold, for a physical world. Can we find an invariant scalar density function ${}^lW$, of the two invariantive quantities

$${}^c\lambda = \Delta x-\nu,\ {}^c\omega = -V_2\Delta^c\lambda = V_2\Delta\nu \quad (7)$$

that must exist when structure involving an intrinsic x and an intrinsic ν exists? The necessary and sufficient condition that such a scalar function ${}^lW$ exists was found in our first paper (Art. 3, 4,) to be that

$${}^lW{}^c\alpha = {}^c\lambda V_0{}^l\lambda{}^c\alpha+V_1{}^c\omega V_1{}^l\omega{}^c\alpha \quad (8)$$

where

$${}^l\lambda = \%_{c}\lambda \cdot {}^lW,\ {}^l\omega = \%_{c}\omega \cdot {}^lW \quad (9)$$

As usual ${}^c\alpha$ is quite arbitrary. In a general n-fold equation (8) imposes n^2 scalar conditions on the $\frac{1}{2}n(n+1)$ scalar partial first derivatives of ${}^lW$ with respect to the same number of independent scalar variables. The number of conditions exceeds the number of scalars at our disposal to satisfy (8). Nevertheless in a four-fold the 16 conditions of (8) are satisfied in one case. For aught we know there may exist a class of such cases, and a four-fold world dependent on the satisfaction of (8) may be possible. The case referred to is when ${}^lW = \overset{c}{\iota}V_4{}^c\omega^2$ where $\overset{c}{\iota}$ as usual stands for $\iota_1\iota_2\iota_3\iota_4$, the product of all the primitive units.

However this be, such a world would not be that of natural physics. In it there would be no orthogonality or orthodromy or gravitation. There would only be bulk, inertia, and electric field.

<u>Art.9.</u> Sufficiency of the affine structure.

ν denoting, as above, intrinsic structure, and z any log-scalar-density, the covariant vector $\Delta z - \nu$ is the absolute gradient of z. To be able to compare two such gradients at neighbouring points demands the affine structure. It is desirable henceforth to limit the meaning of the affine function E_α as previous writers have done by assuming $\mathrm{E}_\alpha{}'\beta$ to be symmetrical in α and β, that is assuming E_α to be self-conjugate in α. The general E_α resolves into the two parts, self-conjugate and skew with respect to α, and the second part (which is invariantive) has no share in satisfying equation (4), the only condition demanded of the structure.

We may now take ν to be $\mathrm{E}_{\epsilon\epsilon}$, and the structure thus involves an intrinsic x and E_α. The very simplest non-singular scalar density function ${}^l\mathrm{W}$ (based on the structure) appears to be a function of ${}^c\lambda = \Delta x - \mathrm{E}_{\epsilon\epsilon}$ and of the contracted curvature (a coexcontra vector linity) denoted in our first paper by $\psi - \frac{1}{2}\mathrm{V}_1{}^c\omega(\ )$. ${}^l\mathrm{W}$ is now a function of ${}^c\lambda, {}^c\omega = -\mathrm{V}_2\Delta{}^c\lambda, \psi$, where ψ is self-conjugate. The necessary and sufficient condition for the existence of ${}^l\mathrm{W}$ now becomes

$$ {}^l\mathrm{W}{}^c\alpha = {}^c\lambda \mathrm{V}_0{}^l\lambda{}^c\alpha + \mathrm{V}_1{}^c\omega \mathrm{V}_1{}^l\omega{}^c\alpha + 2\psi\epsilon \mathrm{V}_0{}^d\psi\epsilon{}^c\alpha \quad (10) $$

where

$$ {}^d\psi = \%\psi \,.\, {}^l\mathrm{W} \quad (11) $$

Of the three ${}^c\lambda, {}^c\omega, \psi$ the first may be absent from ${}^l\mathrm{W}$, but neither the second nor the third, if we are to have the full complement of n^2 scalar first partial derivatives of ${}^l\mathrm{W}$ to satisfy (10). Physically this would seem to mean that all inertia is of electric origin and that there is but one conservation

law, the conservation of charge $\mathrm{V}_0\Delta{}^l\kappa = 0$

It is thus of considerable interest that from our a priori mode of approaching the physical problem neither electric field, ${}^c\omega$, nor gravitation ψ, can be supposed absent. Also we may note that ignoration of a mass energy(given by ${}^c\lambda$) independent of charge seems arbitrary and artificial, for from our standpoint ${}^c\lambda$ seems more fundamental than either $\mathrm{V}_2\Delta{}^c\lambda$ or ψ.

We shall return to (10) and its connection with the energy tensor later. Meanwhile we resume our a priori approach.

<u>Art.10.</u> Relativity tenets carried as far as possible.

It is open to argument that the principle "physical laws are independent of choice of coordinates" applies only to the original coordinates x_1, x_2, x_3, x_4; but it appears more natural to regard the scalars required to specify the structure as coordinates to which the principle also applies. Can this be done?

If the physical world is finite in each of its dimensions as held by De Sitter the answer is affirmative. To apply the principle in this its second aspect we have merely to vary these new coordinates, and ensure that the only physical reality namely $\int^n\!\!\int {}^l\mathrm{W}db$ taken over the whole manifold remains unchanged. This is precisely what we did in the first paper (under the name of Stationary Action). If the manifold extends indefinitely in one or more of its dimensions we are not able fully to render $\int^n\!\!\int {}^l\mathrm{W}db$ independent of choice of the new species of coordinates. The breakdown however can be pushed away to as remote a boundary as we please, and the argument for the naturalness of the process of the first paper retains much of its force.

Here ends our a priori enquiry. Some general aspects of the results of our method will now be considered.

<u>Art. 11.</u> The fundamental identity of relativity.

From the physical side (10) has to be viewed as the stress form (or energy tensor: no longer a "pseudo" tensor) of the "laws of motion". When in (10) we replace ${}^{c}\alpha$ by Δ the laws take on their vector force form. Note that the electric field and gravitation as well as inertia are included in our meaning of the laws. Indeed a great unification of our ideas of the physical world arises from the straightforward interpretation of (10).

This interpretation was not possible earlier. It required a rearrangement of the foundation stones of general relativity, which was gradually effected by the labours of Weyl, Eddington, and Einstein (see the second sentence of the first paper). Now for the first time we have a complete parallelism of $({}^{c}\lambda, {}^{c}\omega, \psi)$ with the velocities, and of $({}^{l}\lambda, {}^{l}\omega, {}^{d}\psi)$ with the momenta, of nineteenth century holonomic dynamics. Hitherto this has not been possible in the case of ψ and ${}^{d}\psi$. A formidable obstacle to advance was left in the complexities resulting from the second differential coefficients and the non-linear form of the contracted curvature ψ.

Denote the identically zero form obtained by removing the left-hand member of (10) to the right-hand side by ${}^{l}U{}^{c}\alpha$; and, putting dc for an arbitrary infinitesimal invariant, let the differentials $d{}^{l}U$, $d{}^{c}\lambda$, etc., be replaced by corresponding fluxes ${}^{\bullet l}U$ ${}^{\bullet c}\lambda$, where $d{}^{l}U = {}^{\bullet l}U dc$, $d{}^{c}\lambda = {}^{\bullet c}\lambda dc$. ${}^{l}U$ does not naturally separate into three stresses, but the flux ${}^{\bullet l}U$ is the sum of three fluxes, kinetic ${}^{\bullet l}T$, electric ${}^{\bullet l}T'$, and gravitational ${}^{\bullet l}T''$
Thus

$$\left.\begin{aligned} {}^{\bullet l}U &= {}^{\bullet l}T + {}^{\bullet l}T' + {}^{\bullet l}T'' = 0 \\ {}^{\bullet l}T{}^{c}\alpha &= {}^{c}\lambda V_0 {}^{c}\alpha {}^{\bullet l}\lambda - V_1(V_2 {}^{c}\alpha {}^{\bullet c}\lambda){}^{l}\lambda \\ {}^{\bullet l}T'{}^{c}\alpha &= V_1(V_1 {}^{c}\alpha {}^{\bullet l}\omega){}^{c}\omega - V_1(V_3 {}^{c}\alpha {}^{\bullet c}\omega){}^{l}\omega \\ {}^{\bullet l}T''{}^{c}\alpha &= 2\psi\epsilon V_0 {}^{c}\alpha {}^{\bullet d}\psi\epsilon - 2V_1(V_2 {}^{c}\alpha {}^{\bullet}\psi\epsilon){}^{d}\psi\epsilon \end{aligned}\right\} \quad (12)$$

${}^{l}U_9\Delta_9$ is the sum of three corresponding forces ${}^{d}\nu$, ${}^{d}\nu'$, ${}^{d}\nu''$. Thus

$$\left.\begin{aligned} -{}^{d}\nu &= {}^{d}\nu' + {}^{d}\nu'' \\ {}^{d}\nu &= {}^{c}\lambda V_0 \Delta {}^{l}\lambda - V_1(V_2\Delta {}^{c}\lambda){}^{l}\lambda \\ {}^{d}\nu' &= V_1(V_1\Delta {}^{l}\omega){}^{c}\omega - V_1(V_3\Delta {}^{c}\omega){}^{l}\omega \\ {}^{d}\nu'' &= 2\psi\epsilon V_0 \Delta {}^{d}\psi\epsilon - 2V_1(V_2\Delta\psi\epsilon){}^{d}\psi\epsilon \end{aligned}\right\} \quad (13)$$

The reader should observe that (12) and (13) follow from the mere assumption that ${}^{l}\lambda$, ${}^{l}\omega$, ${}^{d}\psi$ are the first partial derivatives of some scalar density function ${}^{l}W$ with respect to the independents ${}^{c}\lambda$, ${}^{c}\omega$, ψ. A second form of the assumption is that ${}^{c}\lambda$, ${}^{c}\omega$ ψ are the derivatives of ${}^{l}W^*$ with respect to the independents ${}^{l}\lambda$, ${}^{l}\omega$, ${}^{d}\psi$, where

$${}^{l}W^* + {}^{l}W = V_0 {}^{c}\lambda {}^{l}\lambda + V_0 {}^{c}\omega {}^{l}\omega + V_0 \psi\epsilon {}^{d}\psi\epsilon \quad (14)$$

On these results we now superpose those following out of the method we have based on Einstein's remarkable mathematical discovery. We find that ${}^{d}\nu''$ gives exactly the expression relativists demand for gravitational force; that ${}^{d}\nu'$ gives exactly the general electric field of M. D. I. (3), (the allied equation $V_3\Delta{}^{c}\omega = 0$ also following from our method); and that the conservation of energy must exist. To attain the accepted form for the matter term ${}^{d}\nu$ (as well as to interpret easily in any wanted sense the equation $V_0\Delta{}^{l}\lambda \doteq 0$ as affirming the conservation of energy), we have to make the usual assumption that ${}^{l}\lambda$ contributes to ${}^{l}W^*$ the one term $\sqrt{(V_0 {}^{l}\lambda\theta {}^{l}\lambda)}$ where

$$\theta = {}^{d}\psi^{-1} \cdot |{}^{d}\psi^{1/(n-2)}| \quad (15)$$

The last paragraph asserts the truth of a series of statements which in their entirety may seem a little astonishing or even erroneous. It has been asserted that (13) agrees symbol by symbol with the usually accepted equation of motion though based on different primary assumptions, and that the associated stress form or energy tensor is in no sense "pseudo". Why then, it will be asked, is the tensor of the usual theory "pseudo"?. The explanation is that our present method reveals two new identities which effect the simplification. From the single identity $d\nu + d\nu' + d\nu'' = 0$, from which ${}^l W$ has vanished and which involves only ${}^l\lambda$, ${}^l\omega$, ${}^d\psi$ and their Δ derivatives up to the second order, there arise three independent identities.

The facts about these three were correctly stated in the first paper, but it was not rendered clear why six instead of three do not arise. ${}^d\psi$ and ${}^l\omega$ may be given independent arbitrary values at every point, while ${}^l\lambda$ is taken to be zero. On now introducing ${}^l\lambda$ the forty-first equation $V_0\Delta^l\lambda = 0$, (required to make the integral of ${}^l W$ stationary) seems at first sight inconsistent with the previous forty equations, for ${}^l\lambda$ is expressible in terms of ${}^l\psi$, ${}^l\omega$ and E_α. Thus by a complex indirect way ${}^l\lambda$ is dependent on the previously assigned values of ${}^d\psi$, ${}^l\omega$, and the single identity is by no means an identity involving three independent symbols ${}^l\lambda$, ${}^d\psi$, ${}^l\omega$.

Let us now make a somewhat important departure from a usual procedure by supposing ${}^l\lambda$ to be involved in any way in ${}^l W^*$ instead of in the very restricted and artificial looking form $\surd(V_0{}^l\lambda\theta{}^l\lambda)$. On reflection the reader will I believe agree that the conservation equation $V_0\Delta^l\lambda = 0$, and the "hydrodynamic term" $d\nu$ in the equation of motion claim our first attention. We use Galilean co-ordinates; and find that, in the conservation equation, $\surd(V_0{}^l\lambda\theta{}^l\lambda)$ $[= {}^l m]$ appears as three-dimensional density of mass-energy; and that, in the hydrodynamic equation, (on the assumption that so far as ${}^l W^*$ depends on ${}^l\lambda$ it is

some function of ${}^l m$) the density of matter-inertia appears as ${}^l m(\partial/\partial{}^l m){}^l W^*$ If these two three-dimensional densities are identified with each other in the strict mathematical sense, the usual assumption must be made that ${}^l W^*$ is linear in ${}^l m$. In the immediate neighbourhood of protons and electrons, that is where both densities are to be reckoned in many thousands of tons per c.c. the two must be identified to a very high order of numerical accuracy. Apparently at distances greater than 10^{-8} cm. the densities sink to values comparable with 10^{-7} gm. per c.c. We may well suppose that ${}^l W^*$ is of the form ${}^l m f({}^d\psi, {}^l\omega, {}^l m)$, where f is a finite invariant function of its constituents, for all values of ${}^l m$, inclusive of when ${}^l m$ is indefinitely increased.

It would seem then that we ought to call ${}^l\lambda$ the energy flux and reserve the name momentum vector for

$$-{}^l\lambda V_0{}^l\lambda{}^c\lambda/\surd(V_0{}^l\lambda\theta{}^l\lambda).$$

<u>Art.12.</u> The problem of matter: protons, electrons, and the Bohr orbits.

Starting from Einstein's illuminating article in "Nature" we have now arrived at a beautifully rounded off relativity scheme of physics. In direct contrast however to Einstein's concluding words, we appear to have obtained a very promising insight into the true nature of the problem presented by matter, and in what direction to attack the position.

Submitting ourselves with severe interpretation to the ordinance "do naught but carry relativity tenets as far as possible", we have found many detailed results harmonizing with natural (as opposed to a priori) physics, and not a little which was lacking from former presentations of relativity. Examples are the conservation of energy, the existence of a true energy tensor, and the formulation as an identity of the laws of motion, understood here to include electric field and gravitation. We shall now show that great atomic concentrations of matter and of electric charge each necessarily presuppose the other. Later, general reasons will be given for expecting such concentrations, and something very like the Bohr orbits accompanying them.

The more I reflect on the dual (matter-electric) aspect of the pair of allied vectors ${}^{l}\lambda$, ${}^{e}\lambda$, the more I realise that a long-felt want has here been supplied, a basic natural and simple unification of the three great physical entities matter, electricity, and energy.

Though we may affirm that the dual aspect is prima facie evidence that a rotating mass should be a magnet, the remarks at the end of Art.11. render it improbable that any numerical deduction is possible from present-day knowledge.

We are on safer, and very interesting, ground when we observe that very high electrostatic potential (irrespective of sign) and very high material density necessarily go together. This, of course, admirably accords with our knowledge of protons and electrons.

Consider the case of a hydrogen atom, where we have very high positive and negative potentials at the proton and electron and an intervening locus at which the potential sinks to zero. This suggests that in our theory we may have to recognise the existence of negative mass. On this point one is inclined at first to argue somewhat as follows. (1) There is no a priori diffculty in supposing mass, either as energy or inertia, to be negative. (2) The total apparent mass of a proton, or of an electron, includes a term duo to its charge because of the conservation of energy (though in the absence of such conservation the argument for this electric inertia seems to fail). (3) Observation shows that this total mass is, in each case, positive (for otherwise the two particles would separate), but in the case of the electron the result may be due to the positive electric term masking a negative term contributed by ${}^{l}\lambda$.

There seems, however, a very real reason preventing us from recognising negative mass. We seem instead to be impelled to assume that when we reach a point at which ${}^{l}m = 0$ we ipso facto reach a boundary of the physical world. In the arguments (1), (2), (3) above, we tacitly assumed that the single scalar condition expressed by saying that the electrostatic potential is zero gives rise to the four scalar conditions expressed by saying that ${}^{l}\lambda = 0$. For we assumed that, on each side of the locus ${}^{l}m = 0$, the scalar ${}^{l}m$ is real. Now, wherever ${}^{l}m = 0$ and ${}^{l}\lambda$ is finite our interpretation of the conditions is that the velocity of light has been attained. The simple view is that this condition holds at the internal boundary of every electron and that in every proton ${}^{l}m$ attains a very large, or perhaps indefinitely large, value.

The work of earlier writers suggests a first form for ${}^{l}W^{*}$ namely ${}^{l}m - \frac{1}{2} l V_0 {}^{l}\omega {}^{d}\psi^{-1} {}^{l}\omega$ where the "extensive" meaning understood for ${}^{d}\psi$ is that which makes ${}^{d}\psi^{-1} V_2 \alpha\beta = V_2 {}^{d}\psi^{-1}\alpha \, {}^{d}\psi^{-1}\beta$. The considerations advanced in the last paragraph suggest a first modification of this form by the addition thereto of $-l$. The general nature of the Bohr theories suggests a further change by which the invariant coefficient of l is replaced by a corresponding exponential thus

$$\left.\begin{aligned} {}^{l}W^{*} &= {}^{l}m - l\cosh\sqrt{(V_0 {}^{l}\omega {}^{d}\psi^{-1} {}^{l}\omega)} \\ &= {}^{l}m - l\cos\sqrt{(-V_0 {}^{l}\omega {}^{d}\psi^{-1} {}^{l}\omega)} \end{aligned}\right\} \quad (16)$$

Let us enquire whether (16) should lead us to expect the automatic-formation of those intense concentrations and the Bohr orbits whose existence has hitherto proved so baffling. Such an enquiry may perhaps suggest further modifications of (16) before we seriously face the labour of exact mathematical analysis. The argument will be easier to follow if we write (16) in the following invariant form

$$\left.\begin{aligned} W^* &= m - \cosh\sqrt{(D^2 - H^2)} \\ &= m - \cos\sqrt{(H^2 - D^2)} \end{aligned}\right\} \quad (17)$$

W^* stands for $l^{-1l}W^*$ and m for $l^{-1l}m$. D (displacement) and H (magnetic force) are the invariant magnitudes of the two vector densities ${}^l\delta$, ${}^d\zeta$, given by

$$\left.\begin{aligned} {}^l\delta &= V_1{}^d\psi^{-1l}\lambda^l\omega, \\ {}^d\zeta &= -\psi V_3{}^l\lambda^l\omega \end{aligned}\right\} \quad (18)$$

(17) suggests that perhaps the proper form for the exponential is $\cosh D + \cos H - 1$ in place of $\cosh\sqrt{(D^2 - H^2)}$. Our quarternion notation suggests several alternative forms.

Think now of (17) in connection with the problem of the atomic concentrations, and first consider the great (mainly stagnant) interstellar spaces. We may suppose the normal condition here to be that D, H and $\theta - 1$ are all very small or zero according as radiation is present or absent. Further we may plausibly endow these great physically empty spaces with the negative property of contributing zero to the action integral. Thus the characteristic of empty space is that $m = 1$. (Perhaps a more plausible criterion for the value of m in empty space should be sought in equipartition of energy between the whole of ether and the whole of gross matter; but I believe the search will always fail from the want of a natural boundary between the two.)

When just now we said $\theta - 1$ is nearly zero in the ether we tacitly assumed that in a practical but no absolute sense it is possible to choose a system of coordinates which is natural and simple. From this point of our argument let us use such a system and permit ourselves freely to contemplate an evolution of the physical world as time progresses.

At some remote epoch in the past all the energy of the universe existed as a chaos or welter of radiation.

Gravitation at once began to make such fortuitous congestions of energy as existed still more congested and to make the emptier places still more empty. Each congestion had a high electric potential and the descending potentials in the emptier surroundings had but one limit namely zero, corresponding to a zero value of energy density. Incipient atoms had evolved from primeval chaos. Each atom consisted of a pair of singular points, at one of which was a concentration of energy and at the other a sink of potential and a boundary of the physical world. Equipartition of energy necessarily ensued, and the incipient atoms had become the hydrogen atoms with which we are familiar to-day. - Needless to say the details of this brief sketch of the growth of physical law are not to be insisted upon. Rather are they given to indicate in what direction exact analysis is called for.

Similarly (17) while possessing several instructive features bearing on the possible mode of origin of atoms is not very likely to prove the exact form required. If (18) is to be of use in this problem we should expect a general explanation to run somewhat as follows. (1) For a proton $D^2 - H^2$ is positive, and D and m assume large values. (2) For an electron $D^2 - H^2$ is negative and m is between unity and zero. (3) The apparent mass of an electron is practicly entirely of electric origin. (4) The energy levels of Bohr's Theory no doubt depend on the periodic cosine term in W^*, but the working out of the mathematical details will probably prove difficult.

Near an isolated proton when the electron has been removed it is not improbable that, between limits of distance from the centre of the proton about 10^{-5} cm. to 10^{-10} cm., m varies approximately inversely as the distance, rising from value unity. When the electron is present it probably pushes a sort of pit or crater of unit density into these denser previously spherical layers, the crater forming a kind of cometary tail, the electron itself being the head or nucleus. On the other hand it is possible that the critical Rydberg length, about 10^{-5} cm., is olosely connected with the linear dimension of an isolated electron rather than an isolated proton, and in that case we should expect to attain the value 2 near an isolated proton, at some distance between 10^{-10} and 10^{-13} cm.

Why, it may be asked, do not the pairs of concentrations we have pictured run to the extreme of forming one great single pair instead of a vast number of atoms? Four alternative general answers seem reasonable. First, analysis may show that the pairs when once formed will be highly stable. Secondly, the large number of atoms may depend on a constant of intergration, perhaps in association with the invariant y of our first paper. Thirdly, a very large mathematical number (such as $e^{16\pi^2}$) may be involved in the ratio of the linear dimensions of the universe to those of an atom. Fourthly, the bounding vacuities inside electrons may be original unchangeable features of the universe, and form necessary nuclei for the atoms to gather round.