THE MAGNETIC RAYLEIGH-TAYLOR INSTABILITY

by

Kain Chambers, BE BSc (Hons.)

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School of Mathematics and Physics
University of Tasmania
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I declare that this thesis contains no material which has been accepted for a degree or diploma by the University or any other institution, except by way of background information and duly acknowledged in the thesis, and to the best of my knowledge and belief no material previously published or written by another person except where due acknowledgement is made in the text of the thesis, nor does the thesis contain any material that infringes copyright.

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Kain Chambers

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Kain Chambers

Date: 12/11/2012
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Kain Chambers (80%), Larry Forbes (20%)

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Kain Chambers (80%), Larry Forbes (20%)

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Signed: Larry Forbes
Larry Forbes
Supervisor
School of Mathematics and Physics
University of Tasmania

Date: 12 Nov. 2012

Signed: John Dickey
John Dickey
Head of School
School of Mathematics and Physics
University of Tasmania

Date: 12 Nov. 2012
Abstract

The Rayleigh-Taylor instability (RTI) arises whenever two fluids with different densities are arranged such that the heavier fluid sits above the lighter fluid, with a sharp interface in between. The magnetic Rayleigh-Taylor instability (MRTI) has the further complication due to the presence of a magnetic field throughout both media. The two fluids in question may also have differing magnetic properties, such as the magnetic permeability. When the fluids in consideration are in fact plasmas comprised of charged particles, induced currents, magnetic fields and Lorentz forces can all act in ways that will affect the stability of the system.

The RTI has widespread applications in atmospheric physics, oceanography, meteorology, laboratory plasma physics, nuclear reactors, inertial confinement fusion as well as the field of astrophysics, where the instability plays an important role in supernova explosions, accretion discs, plasma jets and H II regions (clouds of gas in which star formation has recently taken place) amongst others. It is closely related to two other hydrodynamic instabilities, namely the Kelvin-Helmholtz instability (KHI) and the Richtmyer-Meshkov instability (RMI).

This thesis considers in detail several different flow configurations in which the RTI arises. A key feature of these configurations is that small wavelike disturbances to the flow are unstable. These configurations are studied to determine the behaviour of these unstable flows. A particular focus is the effects due to the presence of a magnetic field, and the mechanisms that alter the flow in the magnetic case.

The thesis begins by considering two dimensional planar flow in Cartesian geometry in which the amplitude of the waves is small compared with their wavelength. The flow is assumed to be incompressible and inviscid. In this scenario, linear theory is used to derive a closed form solution for the evolution of the interface between the two fluids. This solution shows quantitatively how the position of the interface depends on the ratio of the densities of the two
fluids, the wavelength of the disturbance, as well as the strength and direction of the applied magnetic field.

The unstable nature of the RTI means that after some finite time, the amplitude of the waves will grow to a size comparable with their wavelength, and in this scenario, linear theory is not appropriate. For this reason, a non-linear model is considered, again for two dimensional planar flow in Cartesian geometry. The flow in this case is considered to be weakly compressible, and viscous. Results in the non-linear case are obtained by use of a combination of streamfunction, spectral and finite difference techniques. The results show qualitatively various non-linear phenomena such as interface roll-up, fingering and bubble formation. It is shown in particular how different initial conditions give rise to outcomes that are very different in terms of the geometry of the interface between the two fluids, primarily the differences between a single mode disturbance and a multi mode disturbance to the interface at time $t = 0$.

The final problem studied in this thesis considers two dimensional flow in circularly symmetric cylindrical geometry. The configuration in this case is comprised of a heavy fluid surrounding a light fluid, and gravity is directed radially inwards. A massive object is located at the centre of the light fluid, and it behaves like a line dipole both for fluid flow and magnetic field strength. In the non-linear, weakly compressible, viscous regime, the initially circular interface between the two conducting fluids evolves into plumes, dependent on the magnetic and fluid dipole strengths and the nature of the initial disturbance to the interface. A spectral method is presented to solve the time-dependent interface shapes, and results are presented and discussed. Bipolar solutions are possible, and these are of particular relevance to astrophysics. The solutions obtained resemble structures of some HII regions and nebulae.
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Chapter 1

Introduction

The study of instabilities in fluids is a vast field of active research. Typically, an instability occurs when a stable base flow is perturbed. Various types of instabilities are possible, a few common examples being the Kelvin-Helmholtz [35, 29] instability (KHI), the Richtmyer-Meshkov [51, 41] instability (RMI) and the Rayleigh-Taylor [50, 59] instability (RTI), which are named after the pioneer researchers in each case. These instabilities are closely related, in that they all share some common features, and it is possible in some circumstances that these instabilities occur simultaneously. In each case, a number of fluids is present and of particular interest is the behaviour of an interface that separates two fluids. The configuration for an unstable RTI requires the heavier of the two fluids to be located at a region of higher gravitational potential than the lighter fluid. In general, the fluids may be subject to any acceleration, not necessarily gravitational, but in many applications the acceleration is indeed due to gravitational forces. The RTI differs from the KHI, in which the relative velocity difference at the interface is responsible for unstable flow, and the RMI which requires impulsive acceleration such as a shock wave at the interface. It is the magnetic RTI (MRTI) that is the topic of this thesis. This version of the instability includes the presence of magnetic fields, and considers various electro-magnetic effects such as conductivity, induced currents, Lorentz forces.
and magnetic permeability. The inclusion of these additional effects typically results in a flow that is less unstable than the non-magnetic equivalent, specifically in that the growth rate of the instability is suppressed. The MRTI has applications in a range of scales from laboratory plasma physics to atmospheric physics and astrophysics.

A common question that arises in the study of a fluid instability concerns the conditions for an interface between two fluids to be stable. The growth rate of an arbitrary disturbance and hence the stability of the system will depend in general on some combination of the properties of that system. Frequently, for wave-like disturbances in the linear scenario, the stability of a disturbance can be shown to be dependent on the wavelength of that disturbance by use of a common linearisation technique. This functional quantity is known as a dispersion relation, and it applies at early times during the instability where the amplitude of the waves is smaller than their wavelength. The growth rate of the interface for the linear regime in the magnetic free case is a classic result. Taylor [59] showed that this rate depends only on the densities of the two fluids, the well known Atwood number. The presence of a base magnetic field complicates matters somewhat. In 1961, Chandrasekhar [13] derived the form of the growth rate for fluids that are incompressible, inviscid, stratified and have zero resistivity. This rate depends not only on the densities of the fluids, but also on the strength and direction of the base magnetic field. Chapter two of this thesis contains a derivation of the linearised growth rate for the MRTI, under slightly different conditions than those used by Chandrasekhar.

The question of stability is naturally extended to such cases where non-linear effects contribute to the flow. These will typically be significant for modes where the wave amplitude has grown to be large compared with its wavelength and hence the approximations used in the linearised analysis become invalid. Compressibility, viscosity and Lorenz forces in the magnetic case are examples of effects that can be considered in the non-linear regime. The non-linear growth of disturbances is the topic of the third chapter of this thesis. Due to the
intractability of analytic solutions for a non-linear system of partial differential equations (PDEs), one naturally considers some type of numerical scheme to solve for the flow within some appropriate computational boundary. Such schemes are still an active area of research. A common and reasonably stable method, in particular for the two dimensional Cartesian geometry considered here, for solving such a non-linear system of PDEs is the Alternating Direction Implicit (ADI) method [31]. Such a method is unsuitable in the case of the MRTI however, given that we cannot maintain the divergence free nature of the magnetic field exactly by use of any purely numerical scheme. This is a critical fact, since it was shown by Brackbill and Barnes [9] that a small numerical error in this quantity results in an effect equivalent to a fictitious force parallel to the magnetic field. Such a force is quite undesirable, and hence it is essential to avoid such a situation. Various so called constrained transport algorithms have been designed and implemented in a range of works, for example Isobe et. al [32], Stone and Gardiner [57] and Jun et.al [33], to accomplish this task. These algorithms require some additional assumptions and approximations in order to maintain a divergent free magnetic field. Some examples are the algorithms CANS [38], FLASH [15], WP/PPM [60] and ZEUS-3D [55]. A different approach was taken in this thesis, whereby a combination of spectral and finite difference techniques is performed. The two dimensional flow is solved by use of spectral techniques in one of the dimensions, and by finite difference techniques in the other dimension. Rather than solving for the numerical values of the flow quantities directly as for a purely numerical scheme, each quantity is represented in spectral form as an analytic series, and it is the coefficients of these series that are solved for numerically by use of the Crank-Nicolson finite difference scheme. An appropriate choice for the form of each series in conjunction with a stream function approach ensures that the divergence free nature of the magnetic field is satisfied exactly. This method has two key advantages; firstly, the resulting system of equations is tri-diagonal, meaning that computationally expensive matrix inversion techniques can be avoided, and secondly,
the additional assumptions and approximations used by the above constrained transport algorithms are not required. The stream function and associated vorticity approach is particularly convenient in view of work by Forbes [24] where it was shown that fluid vorticity is the mechanism for interface roll-up, and in this present work where it is shown that magnetic vorticity is the mechanism for growth suppression.

Chapter four of this thesis is devoted to a study of the MRTI in cylindrical geometry in two dimensions with circular symmetry. The presence of curved geometry causes additional effects that have no equivalent in the flat, Cartesian geometry. The curvature of the space affects the stability of the system, and this is known as the Bell-Plesset effect, since Bell [8] was the first to discover this result for cylindrical geometry, while a short time later Plesset [49] achieved a similar result for spherical geometry. The convergent nature of cylindrical geometry requires one to also consider how to represent the center of the compressing fluid. In reality, the pressure must remain finite as the radius approaches zero. As discussed in a paper by Epstein, [20], in order to conserve mass, there must be some small central volume with appropriate properties to compensate for these constraints. A line source is such a volume in the limit that the volume tends to zero. This representation is appropriate and mathematically convenient. Due to the already discussed divergence free nature of the magnetic field however, we are forced to use a higher order singularity in this paper, a line dipole to model this small volume. As for the case in Cartesian geometry, a numerical scheme is required to solve for the flow. Ignoring for the moment the aforementioned fact that purely numerical schemes are inappropriate in the case of the MRTI, it is remarked that the use of a finite difference scheme would not be straightforward in any case given the presence of dipole type singularities at the origin. The approach taken in this thesis is to use spectral techniques in both the radial and azimuthal dimensions. The dipole nature of both the fluid flow and magnetic field is satisfied exactly using an appropriate representation. In addition, the values of all quantities on the computational boundary are
straightforward. The use of the spectral method results in a system of ordinary differential equations (ODEs) for the coefficients of the various series representations. This is integrated forwards in time using the classic Runge-Kutta RK4 method [5], which is stable for appropriately small time steps. It is observed that a single mode initial disturbance of an integer wave number gives rise to the formation of that integer number of plumes. In view of this observation, the focus of the work of this chapter is on bipolar solutions given the interest of such phenomena in astrophysics, the study of plasma jets, HII regions and nebulae being examples. A small section that considers other such possibilities, for example, tri-polar plumes is also presented. The results are compared with real observations of such structures and demonstrate how magnetic effects alter the development of the RTI in the given geometry. The resulting flows are shown to be highly dependent on a dimensionless quantity called the magnetic Froude number. Comparisons are made with the magnetic free regime, and the underlying mechanisms that alter the evolution of the flow in the magnetic case are presented and discussed.
Chapter 2

An analytic method for the time evolution of interfacial waves of a two fluid magnetic Rayleigh-Taylor configuration in the small amplitude approximation

2.1 Introduction

The classic Rayleigh-Taylor problem was first studied by Lord Rayleigh in 1883 [50], and Sir G.I. Taylor in 1950 [59]. Taylor’s analysis assumed the waves at the interface were small, and he applied linear theory to obtain an exponential growth rate for the interface. The magnetic case was studied analytically by
Chandrasekhar in 1961 [13] for fluids that are incompressible, inviscid, stratified and have zero resistivity. Chandrasekhar showed how the presence of a background magnetic field altered Taylor’s growth rate, again in the linear regime. The MRTI instability is also known as the Parker Instability, due to work by Parker in 1966 [48], in which the MRTI was suggested to be responsible for various observed astrophysical effects, such as the concentration of the interstellar plasma into discrete clouds.

The purpose of this introductory chapter is to derive the analytic form for the growth rate of the interface between two fluids in the linear regime for the MRTI. It will be shown that this quantity depends on the ratio of the densities of the two fluids and the base magnetic field strength. The result is in agreement with earlier work by Chandrasekhar [13] in which a different approach to the present work was taken. This analytic result is essential in order to understand the behaviour of the system at early times where the wave amplitude is much smaller than the wavelength. From the analytic nature of the result, it can be concluded that there are three possible qualitatively different behaviours for the flow. The main discussion concerns the role that magnetic effects have on the flow, and it is shown how the flow can be either stable, unstable or oscillatory in nature depending on the strength and direction of the base magnetic field. These behaviours are of use in conjunction with work in later chapters of this thesis that consider the MRTI in the non-linear regime. The results in the non-linear regime are of a numerical nature, and hence a comparison with the analytic results at early times gives confidence that the numeric techniques are stable and accurate.

The remainder of the chapter is structured as follows. After presenting the magnetohydrodynamical model for the system, the analytic time dependent solution for the interface is obtained for the two geometrically different cases of firstly, a horizontal base magnetic field, and secondly, for a vertical base magnetic field. The solution approaches and results have significant differences, hence the separate treatments. Finally, due to the linear nature of the assumed
solutions, it is an elementary calculation to obtain the solution for the interface in the case of an arbitrarily directed base magnetic field by use of the superposition principle. To conclude, some consequences of certain parameter combinations are discussed, along with a comparison of the obtained solution with the well known Atwood number in the case of the magnetic free RTI.

2.2 Model and Governing Equations

We begin by considering the two dimensional case of two incompressible, inviscid fluids separated by a sharp interface. Following the notation of Batchelor [6], the upper and lower fluids are denoted by layers 2 and 1 respectively. The upper fluid has density $\rho_2$ and magnetic permeability $\mu_2 = 1$, while the lower fluid has density $\rho_1$ and magnetic permeability $\mu_1 = 1$. In the case of interest, the upper fluid is heavier, hence we have $\rho_2 > \rho_1$. The upper and lower fluid meet at an interface denoted by $z = \eta(x,t)$. Flow is assumed to be periodic in the horizontal $x$ coordinate, with period $\lambda$, while the vertical $z$ direction extends indefinitely. The gravitational body force is described by $-g\hat{e}_z$.

We start with the governing MHD equations for this scenario, which in each of the two incompressible media comprise conservation of mass,

$$\nabla \cdot \mathbf{v} = 0, \quad (2.1)$$

and conservation of momentum with both gravitational and magnetic body forces,

$$\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] + \nabla p^* = \frac{1}{\mu} (\mathbf{B} \cdot \nabla) \mathbf{B}, \quad (2.2)$$

where $p^* = p + \rho g z + B^2 / (2\mu)$.

Next, we have an absence of magnetic monopoles,

$$\nabla \cdot \mathbf{B} = 0 \quad (2.3)$$
and Faraday’s Law,
\[ \frac{\partial B}{\partial t} = -\nabla \times E.\]

We substitute the Lorentz equation \( J = \sigma (E + v \times B) \) as well as Ampere’s Law \( \nabla \times B = \mu J \) into Faraday’s Law and make use of the vector identity 
\[ \nabla \times (\nabla \times B) = \nabla (\nabla \cdot B) - \nabla^2 B \]
to obtain
\[
\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \frac{1}{\mu \sigma} \nabla^2 B. \quad (2.4)
\]

Under the assumption of infinitely conducting media, \( \sigma \to \infty \), Faraday’s Law is simply
\[
\frac{\partial B}{\partial t} - \nabla \times (v \times B) = 0. \quad (2.5)
\]

The unit normal to the interface is found by taking the gradient of the level surface representing the interface. This gives
\[
\hat{n} = -\frac{(\partial \eta/\partial x) \hat{e}_x + \hat{e}_z}{\sqrt{1 + (\partial \eta/\partial x)^2}}. \quad (2.6)
\]

We use equations (2.1), (2.2), (2.3) and (2.5) in conjunction with (2.6) to derive the conditions at the interface between our two fluids. The velocity and magnetic field vectors have components \( v = (u, 0, w) \) and \( B = (B_x, 0, B_z) \) respectively, and a subscript number is used to indicate in which medium, 1 or 2, the quantity applies. From the conservation of mass (2.1), we require that the normal component of the velocity of the fluid in each medium be equal to the normal velocity of the interface, that is,
\[
-u_1 \frac{\partial \eta}{\partial x} + w_1 = -u_2 \frac{\partial \eta}{\partial x} + w_2 = \frac{\partial \eta}{\partial t} \] on \( z = \eta \). \quad (2.7)

Next, from the absence of magnetic monopoles (2.3), we require the normal component of the magnetic field in each fluid to be equal,
\[
-B_{n_1} \frac{\partial \eta}{\partial x} + B_{z_1} = -B_{n_2} \frac{\partial \eta}{\partial x} + B_{z_2} \] on \( z = \eta \). \quad (2.8)
The Faraday equation (2.5) implies that the tangential components of the electric field \( E = -v \times B \) in each fluid are equal, or \( \hat{n} \times (E_2 - E_1) = 0 \) at the interface. This is expressed in the form

\[
\hat{n} \times (E_2 - E_1) = 0 \text{ at the interface.}
\]

This is expressed in the form

\[
w_1B_{x_1} - w_2B_{x_2} = u_1B_{z_1} - u_2B_{z_2} \quad \text{on } z = \eta.
\]  

Finally, from the momentum vector equation (2.2), by Newton’s Third Law we need the forces on each side of the surface of the interface to be equal and opposite, and we obtain two conditions, one for each component of the equation,

\[
\begin{align*}
\left( \rho_1 u_1^2 + p_1^* - \frac{B_{x_1}^2}{\mu_1} \right) \left( -\frac{\partial \eta}{\partial x} \right) + \left( \rho_1 u_1 w_1 - \frac{B_{x_1} B_{z_1}}{\mu_1} \right) &= 0, \\
\left( \rho_2 u_2^2 + p_2^* - \frac{B_{x_2}^2}{\mu_2} \right) \left( -\frac{\partial \eta}{\partial x} \right) + \left( \rho_2 u_2 w_2 - \frac{B_{x_2} B_{z_2}}{\mu_2} \right) &= 0.
\end{align*}
\]  

We now introduce new dimensionless variables and coordinates, with characteristic length \( \lambda/2\pi \) and characteristic time \( \sqrt{\lambda/2\pi g} \) as in Forbes [24]. In these dimensionless variables, the flow has period \( 2\pi \) in \( x \), the characteristic speed is given by \( \sqrt{\lambda g/\pi} \), while the reference pressure, magnetic flux density and current density work out to be \( \rho_1 \lambda g/2\pi, \sqrt{\mu_1 \rho_1 g/2\pi} \) and \( \sqrt{2\pi \rho_1 g/\mu_1 \lambda} \) respectively. Finally, the gravitational body force is described by \( -\hat{e}_z \). In this dimensionless scheme, there is one parameter, \( D \), which is the ratio \( \rho_2/\rho_1 \), and we have \( \mu_1 = \mu_2 = 1 \). The system of equations in dimensionless variables for the two media, 1 and 2, require minimal adjustment from equations (2.1), (2.3) and (2.5); only the additions of the subscripts for each medium is required. The dimensionless momentum equations (2.2), however, are modified slightly,
becoming,
\[
\frac{\partial v_1}{\partial t} + (v_1 \cdot \nabla) v_1 + \nabla p_1^* = (B_1 \cdot \nabla) B_1
\]
\[
\frac{\partial v_2}{\partial t} + (v_2 \cdot \nabla) v_2 + \frac{1}{D} \nabla p_2^* = \frac{1}{D} (B_2 \cdot \nabla) B_2,
\]
(2.11)
where \( p_1^* = p_1 + z + B_1^2 / 2 \) and \( p_2^* = p_2 + Dz + B_2^2 / 2 \).

At the interface between the two fluids, the conditions in dimensionless variables take the same form as equations (2.7), (2.8) and (2.9), while the momentum conditions (2.10) become
\[
(u_1^2 + p_1^* - B_{z_1}^2) \left( -\frac{\partial \eta}{\partial x} \right) + (u_1 w_1 - B_{x_1} B_{z_1}) = 0
\]
\[
(Du_2^2 + p_2^* - B_{z_2}^2) \left( -\frac{\partial \eta}{\partial x} \right) + (Du_2 w_2 - B_{x_2} B_{z_2}) \quad \text{on } z = \eta.
\]
\[
(u_1 w_1 - B_{x_1} B_{z_1}) \left( -\frac{\partial \eta}{\partial x} \right) + (w_1^2 + p_1^* - B_{z_1}^2) = 0
\]
\[
(Du_2 w_2 - B_{x_2} B_{z_2}) \left( -\frac{\partial \eta}{\partial x} \right) + (Du_2^2 + p_2^* - B_{z_2}^2) \quad \text{on } z = \eta.
\]
(2.12)
A sketch of the dimensionless flow situation is given in figure 2.1.

### 2.3 Small Amplitude Theory

As for the classical Rayleigh-Taylor problem, we perturb a stable base flow by introducing small amplitude waves of wavelength \( \lambda \) at the interface. It will be seen that these waves will grow in time due to the unstable configuration of the fluid. We consider two cases; the first case where the base magnetic field is parallel to the interface, and the second case where the base magnetic field is perpendicular to the interface. Since this small amplitude problem is linear, we may apply the super-position principle to solve for the response due to a base magnetic field at any angle relative to the interface.

A non-trivial stable 2D base flow that exists for this problem is given by \( v_1 = v_2 = 0, B_1 = B_2 = \beta (l_x \hat{e}_x + l_z \hat{e}_z), p_1 = -z, p_2 = -Dz \) and \( \eta = 0 \). The dimensionless constant \( \beta \) represents the ratio of the base magnetic flux density
Figure 2.1: Configuration of the problem. A single disturbance of period $2\pi$ in $x$ is shown.

to the reference flux density $\sqrt{\mu_1 \lambda_1 g}/2\pi$. The quantities $l_x$ and $l_z$ are direction cosines, related by the condition

$$l_x^2 + l_z^2 = 1.$$  \hspace{1cm} (2.13)

We are now ready to apply small perturbations of order $\epsilon$ to the above base flow, where a hat denotes a perturbed quantity except in the case of the unit vectors $\hat{e}_x$ and $\hat{e}_z$, but this should be clear from the context. The symbol $O(\epsilon^2)$ is used to represent additional terms of order epsilon squared. Our perturbed
quantities take the forms,

\[ v_1 = \epsilon \tilde{V}_1 + O(\epsilon^2) \]
\[ v_2 = \epsilon \tilde{V}_2 + O(\epsilon^2) \]
\[ B_1 = \beta (l_x \hat{e}_x + l_z \hat{e}_z) + \epsilon \tilde{B}_1 + O(\epsilon^2) \]
\[ B_2 = \beta (l_x \hat{e}_x + l_z \hat{e}_z) + \epsilon \tilde{B}_2 + O(\epsilon^2) \]
\[ p_1 = -z + \epsilon \tilde{P}_1 + O(\epsilon^2) \]
\[ p_2 = -Dz + \epsilon \tilde{P}_2 + O(\epsilon^2) \]
\[ \eta = \epsilon \tilde{H} + O(\epsilon^2). \]  

The model from section 2.2 is linearised about the interface \( z = 0 \) by use of equations (2.14), and we proceed to solve for \( \tilde{V} = (\tilde{u}, 0, \tilde{w}) \), \( \tilde{B} = (\tilde{b}_x, 0, \tilde{b}_z) \), \( \tilde{P} \) and \( \tilde{H} \).

It is clear that the conservation of mass and magnetic monopole equations remain unchanged in the linear regime. The linearised form of the momentum equations (2.11) becomes

\[
\frac{\partial \tilde{V}_1}{\partial t} + \nabla \tilde{P}_1 = \beta \left( \left( \nabla \times \tilde{B}_1 \right) \cdot \hat{e}_y \right) (l_z \hat{e}_x - l_x \hat{e}_z)
\]
\[
\frac{\partial \tilde{V}_2}{\partial t} + \frac{1}{D} \nabla \tilde{P}_2 = \frac{\beta}{D} \left( \left( \nabla \times \tilde{B}_2 \right) \cdot \hat{e}_y \right) (l_z \hat{e}_x - l_x \hat{e}_z),
\]

while the linearised version of the Faraday equations (2.5) is given by

\[
\frac{\partial \tilde{B}_1}{\partial t} = \beta \left( l_x \frac{\partial \tilde{V}_1}{\partial x} + l_z \frac{\partial \tilde{V}_1}{\partial z} \right)
\]
\[
\frac{\partial \tilde{B}_2}{\partial t} = \beta \left( l_x \frac{\partial \tilde{V}_2}{\partial x} + l_z \frac{\partial \tilde{V}_2}{\partial z} \right). 
\]

The same process is used to linearise the interface conditions, (2.7)-(2.9) and
(2.12), resulting in,

\[ \hat{w}_1 = \hat{w}_2 = \frac{\partial \hat{H}}{\partial t} \quad \text{on} \quad z = \eta \]

\[ \hat{B}_{z1} = \hat{B}_{z2} \quad \text{on} \quad z = \eta \]

\[ \beta l_z \hat{u}_1 = \beta l_z \hat{u}_2 \quad \text{on} \quad z = \eta \]

\[ \beta l_z \hat{B}_{x1} = \beta l_z \hat{B}_{x2} \quad \text{on} \quad z = \eta \]

\[ \hat{P}_1 + \beta l_z \hat{B}_{x1} + (D - 1) \hat{H} = \hat{P}_2 + \beta l_z \hat{B}_{x1} \quad \text{on} \quad z = \eta. \]

(2.15)

The next set of conditions is those that apply on the horizontal boundaries as \( z \to \pm \infty \). The perturbation to the flow has no effect in these regions, resulting in the base flow conditions,

\[ \hat{V} \to 0, \quad \hat{B} \to 0, \quad \hat{P} \to 0, \quad \text{as} \quad z \to \pm \infty. \]

(2.16)

Due to the already mentioned assumptions that the initial perturbation to the stable base flow is a small amplitude sinusoidal wave of wavelength \( \lambda \), the linear nature of the problem and the \( 2\pi \)-periodicity of the flow in the horizontal \( x \)-coordinate, we can represent the time-dependant perturbation to the interface in dimensionless variables by

\[ \hat{H} (x, t) = \tilde{H} (t) \cos x. \]

(2.17)

The choice of the cosine function is somewhat arbitrary, the use of a sine function or a combination of sine and cosine functions would correspond to a shift of the origin in the horizontal \( x \)-coordinate. Finally, at time \( t = 0 \) the velocity of the fluid is zero and we set

\[ \tilde{H} (0) = 1 \]

(2.18)

since this satisfies the assumption that the initial amplitude of the disturbing waves is \( \epsilon \). We now consider separate treatments for solving the equations for horizontal and vertical base magnetic fields.
2.3.1 Horizontal Base Field

The assumed nature of the perturbation to the interface (2.17) gives rise to a solution of the following form for the remaining perturbed quantities:

\[
\begin{align*}
\hat{u}(x,z,t) &= \tilde{u}(z,t) \sin x \\
\hat{w}(x,z,t) &= \tilde{w}(z,t) \cos x \\
\hat{P}(x,z,t) &= \tilde{P}(z,t) \cos x \\
\hat{b}_x(x,z,t) &= \tilde{b}_x(z,t) \cos x \\
\hat{b}_z(x,z,t) &= \tilde{b}_z(z,t) \sin x.
\end{align*}
\]

(2.19)

We seek a solution for \( \tilde{w}(z,t) \) in each medium, since this then allows calculation of \( \eta(x,t) \) by equations (2.19), (2.15), (2.18) and (2.14) respectively. The assumed forms (2.19) are substituted into the two linear systems of six first order partial differential equations (PDEs) developed in section 2.3 that describe the behaviour of the flow in each of media 1 and 2 respectively. One may eliminate all unknown quantities with the exception of \( \tilde{w}(z,t) \) from the equation governing the \( x \)-component of momentum in each medium by the use of substitution and differentiation. It is now convenient to define the intermediate quantities \( \tilde{M} \) and \( \tilde{N} \) by,

\[
\begin{align*}
\tilde{M}(z,t) &= \frac{\partial^2 \tilde{w}_1}{\partial x^2} + \beta \tilde{w}_1 \\
\tilde{N}(z,t) &= \frac{\partial^2 \tilde{w}_2}{\partial z^2} + \frac{\beta^2}{D} \tilde{w}_2.
\end{align*}
\]

(2.20)

The twelve PDEs developed in section 2.3 have thus been reduced to the following two second order PDEs for the intermediate quantities \( \tilde{M} \) and \( \tilde{N} \),

\[
\begin{align*}
\tilde{M} - \frac{\partial^2 \tilde{M}}{\partial z^2} &= 0 \\
\tilde{N} - \frac{\partial^2 \tilde{N}}{\partial z^2} &= 0.
\end{align*}
\]

(2.21)
Now we apply the separation of variables technique to equations (2.20) and (2.21) to obtain the following forms for the vertical velocities \( \tilde{w}_1(z,t) \) and \( \tilde{w}_2(z,t) \),

\[
\tilde{w}_1 = r_1 e^z \sinh s_1 t \\
\tilde{w}_2 = r_2 e^{-z} \sinh s_2 t.
\] (2.22)

We use the interface conditions (2.15), the boundary conditions (2.16) and the initial condition for the interface, (2.18) to solve for the constants \( r_1, r_2, s_1, s_2 \), and thus the velocities of the flow by equation (2.22). The pressures and magnetic flux densities then follow directly from the system of linearised PDEs. The quantity of main interest, however, is the profile of the interface,

\[
\eta(x,t) = \epsilon \cosh \left( \sqrt{\mathcal{A}_{m||}} t \right) \cos x + O(\epsilon^2),
\] (2.23)

where the magnetic Atwood number for the field parallel to the interface is given by

\[
\mathcal{A}_{m||} = \frac{D - 1 - 2\beta^2}{D + 1}.
\] (2.24)

It is trivial to see that in the absence of a magnetic field, we recover the well known Atwood number \( \mathcal{A} = (D - 1) / (D + 1) \) as for the conventional RTI; see Forbes [24].

### 2.3.2 Vertical Base Field

As for the horizontal base field case, the perturbation to the interface is described by equation (2.17). In the present section considering a vertical base field, the form of the solution variables \( \hat{u}, \hat{w} \) and \( \hat{P} \) is the same as in equation (2.19). The
form of the magnetic flux densities is replaced with

\[ \hat{b}_x (x, z, t) = \tilde{b}_x (z, t) \sin x \]
\[ \hat{b}_z (x, z, t) = \tilde{b}_z (z, t) \cos x. \]

As for the horizontal scenario, it is convenient to define two intermediate quantities, in this case, \( \tilde{K} \) and \( \tilde{L} \), by

\[ \tilde{K} (z, t) = \frac{\partial^2 \tilde{w}_1}{\partial z^2} - \tilde{w}_1 \]
\[ \tilde{L} (z, t) = \frac{\partial^2 \tilde{w}_2}{\partial z^2} - \tilde{w}_2. \]

Next, in a similar fashion as for the horizontal case, we reduce the system of twelve PDEs developed in section 2.3 to the following two second order PDEs for the intermediate quantities \( \tilde{K} \) and \( \tilde{L} \),

\[ \beta^2 \frac{\partial^2 \tilde{K}}{\partial z^2} - \frac{\partial^2 \tilde{K}}{\partial t^2} = 0 \]
\[ \beta^2 \frac{\partial^2 \tilde{L}}{\partial z^2} - \frac{\partial^2 \tilde{L}}{\partial t^2} = 0. \]  

(2.25)

To solve equations (2.25), we try solutions of the form

\[ \tilde{K} (z, t) = e^{\lambda_1 z} M (t) \]
\[ \tilde{L} (z, t) = e^{\lambda_2 z} N (t) \]

where \( \Re \{ \lambda_1 \} > 0, \Re \{ \lambda_2 \} < 0 \) are needed for stability, and \( M (t), N (t) \) are unknown functions of \( t \). One can then obtain the forms of the vertical velocities,

\[ \tilde{w}_1 (z, t) = e^{z} A_1 (t) + e^{\lambda_1 z} \left[ \frac{c_1 \cosh \beta \lambda_1 t + c_2 \sinh \beta \lambda_1 t}{\lambda_1^2 - 1} \right] \]
\[ \tilde{w}_2 (z, t) = e^{-z} A_2 (t) + e^{\lambda_2 z} \left[ \frac{d_1 \cosh \frac{\beta \lambda_2 t}{\sqrt{2}} + d_2 \sinh \frac{\beta \lambda_2 t}{\sqrt{2}}}{\lambda_2^2 - 1} \right], \]
where \( A_1(t), A_2(t) \) are unknown functions of \( t \), and \( c_1, c_2, d_1, d_2, \lambda_1 \) and \( \lambda_2 \) are unknown constants, all eight of which can be obtained by use of the boundary, interface and initial conditions. In the special case that \( \lambda_1 = 1 \) or \( \lambda_2 = -1 \), then the above are evaluated in the limit. We use the interface conditions (2.15) and the initial conditions (2.18) to solve for the required constants. Of particular interest here are the quantities \( \lambda_1 \) and \( \lambda_2 \). It may be shown that, in order to satisfy the interface and initial conditions, we require the following relationship between \( \lambda_1 \) and \( \lambda_2 \), and we thus define the new quantity \( \lambda \) by,

\[
\lambda = \sqrt{D} \lambda_1 = -\lambda_2.
\]

Again, the quantity of main interest is the profile of the interface, given by,

\[
\eta(x, t) = \epsilon \cosh \left( \sqrt{A_{m}} t \right) \cos x + O(\epsilon^2),
\]

where the magnetic Atwood number for the field perpendicular to the interface is given by,

\[
A_{m} = \beta^2 \lambda^2.
\]

All that remains is to calculate the value for growth rate \( \lambda \). It may be shown that the initial conditions (2.18) and interface conditions (2.15) lead to the following cubic equation, from which \( \lambda \) is obtained,

\[
\beta^2 (D + 1) \lambda^3 + 2 \beta^2 \left( \sqrt{D} + 1 \right) \lambda^2 + (1 - D + 2 \beta^2) \lambda + 2 \left( 1 - \sqrt{D} \right) = 0, \tag{2.28}
\]

in which we require \( \Re \{\lambda\} > 0 \) for stability. It can be shown with elementary methods as well as Descartes Rule of Signs (see [37]) that the cubic (2.28) has exactly one positive root, and hence that root must also be real. It can also be shown in the limit of no magnetic field that we recover the original Atwood number for the growth rate of the interface, as in Forbes [24].
2.3.3 Arbitrary Base Field

In this final case, our base magnetic field is directed at some angle to the interface, and has the expression \( B_1 = B_2 = \beta (l_x \hat{e}_x + l_z \hat{e}_z) \), where \( l_x \) and \( l_z \) are direction cosines as defined in equation (2.13). Again, the dimensionless constant \( \beta \) represents the ratio of the base magnetic flux density to the reference flux density \( \sqrt{\mu_0 \rho_1 g/2\pi} \). It is straightforward to use the superposition principle to obtain the solution in this case for the interface to be

\[
\eta(x, t) = \epsilon \left( l_x \cosh \sqrt{A_{m||} t} + l_z \cosh \sqrt{A_{m\perp} t} \right) \cos x + O(\epsilon^2),
\]

(2.29)

where \( A_{m||} \) and \( A_{m\perp} \) are defined in equations (2.24) and (2.27) respectively.

This result is in agreement with a similar result derived under slightly different conditions by Chandrasekhar [13].

2.4 Discussion and Conclusion

The main result from this chapter is equation (2.29), which governs the time evolution of interfacial waves in the case of the MRTI. This, in conjunction with the magnetic Atwood numbers for horizontal and vertical base magnetic fields given in equations (2.24) and (2.27) respectively, allows the location of the interface to be calculated at a point in time, accurate to first order in terms of the wave amplitude. These interface locations will be shown graphically in the results of the next chapter, where they will be of great use in allowing the numerical non-linear solutions to be verified at early time periods, that is, where the linear solution is valid. Several figures are presented in which a superposition of the linear and non-linear solutions is given at certain snapshots in time. A comparison is made for all three cases of horizontal, vertical and angled base magnetic fields, and the results compare favourably. This in turn gives confidence that the numerical methods of the next chapter are stable and accurate. A summary of possible behaviours for the various cases is now given.
In the case of a horizontal base magnetic field, there are three different types of behaviour depending on the sign of the magnetic Atwood number given by equation (2.24). Very weak magnetic fields give rise to an Atwood number that is positive, and thus the growth rate is exponential, and comparable with the case of no magnetic field. There is a critical value for the strength of the magnetic field in which the magnetic Atwood number is 0. The growth rate of the interface is thus fully suppressed in time in this case. Magnetic fields stronger than this critical value will thus result in a negative magnetic Atwood number, and the behaviour of the interface will be oscillatory in time.

The case of a vertical base magnetic field is somewhat more difficult to analyse given the more complicated form of the magnetic Atwood number in equation (2.27). There are, however, the same three types of behaviour as for the horizontal magnetic field case. What is more difficult is calculating in general the density ratio and magnetic field strength to give each type of behaviour. This is due in part to the requirement of finding the one positive root of a cubic equation (2.28) which has coefficients that are combinations of the density ratio and magnetic field strength.

The linear nature of the solution under consideration allows a complete description of the response due to an angled field to be described by a superposition of the behaviours in the horizontal and vertical magnetic field cases. We conclude with some analytic consequences of the calculated magnetic Atwood numbers. Firstly, in each case, it can be seen that the magnetic Atwood number depends on the square of the magnetic field strength. The result of this is that the growth rate of the interface is independent of the polarity of the magnetic field. Finally, it is important that the magnetic Atwood numbers calculated in this paper give identical growth rates to the well known Atwood number in the case of no magnetic field. This is trivial to see in the horizontal case, but more difficult to verify in the vertical case, where a complicated limit is required, but does indeed give the correct value.
Chapter 3

A numerical analysis of the effects of magnetic phenomena on the Rayleigh-Taylor instability

3.1 Introduction

The use of linear theory in the study of the MRTI relies on the underlying assumption that the amplitude of the disturbing waves located at the interface is small compared with their wavelength. Due to the exponential growth rate of these waves that was calculated in the previous chapter in this scenario, it is evident that at some finite time, the amplitude of these waves will grow to a size that is comparable, or even larger than their wavelength. At such a time, the evolution of the flow is unable to be described accurately by the use of linear theory. As for the vast majority of non-linear studies in mathematics, a non-linear treatment of the MRTI will require the use of some numerical techniques.
Research in the subject of the RTI and MRTI in the period from 1980 to the present has focussed primarily on numerical analysis of the instability in the non-linear regime. Some examples of other studies in the non-linear case, where disturbances are large, are those by Isobe et. al [32], Stone and Gardiner [57] and Jun et. al [33]. These works make use of numerical algorithms such as CANS [38], FLASH [15], WP/PPM [60] and ZEUS-3D [55] to solve the non-linear MHD equations in three dimensions, and can model both single mode and multimode disturbances. A great success of such studies in the non-linear scenario is the ability to observe interesting phenomena such as fingering and surface roll-up. Such phenomena are not realisable in the linear theory of Rayleigh, Taylor and Chandrasekhar. On a more practical side, Pacitto et. al. [47] have compared simulated results with experimental results, by using an ionic magnetic fluid made of cobalt ferrite particles dispersed in a mixture of water and glycerol. Comparing the experimental results with the linear theory was complicated by the variable magnetisation curve of the magnetic fluid. The observed growth rates for the various wavelengths of the disturbances introduced, however, were between the theoretical values calculated for the minimum and maximum magnetisation values. Later in time, fingering was indeed observed in the experiment. Another interesting work that includes Quantum effects was undertaken by Cao et. al. in 2007 [10]. Quantum effects become significant in plasma dynamics when the de Broglie wavelength of the charge carriers becomes greater than the length scale of the quantum plasma system. This case is of interest in the design of inertial confinement fusion capsules. It was found that the quantum effects had a stabilising effect on the RTI due to the quantum Bohm potential. The addition of this effect modified the equations describing the RTI stability in a similar way to how the inclusion of a magnetic field modifies the RTI equations in this present chapter; hence the similar outcomes.

The purpose of this chapter is to examine the mechanisms that alter the surface roll-up, in particular in two dimensions, in the magnetic case. The effects of viscosity and a weakly compressible fluid are included. A pseudo-
Spectral method is used to calculate the density, vorticity and current density of the fluid. This technique has the special property that in two dimensions we are able to satisfy the condition requiring an absence of magnetic monopoles at all times for all points in the fluid exactly. The results in the absence of a magnetic field are compared with those of Forbes [24], and agree almost exactly. The mechanisms for roll-up described by Forbes in the magnetic-free case in two dimensions are extended in this present chapter to include the role that magnetic effects have in the typical surface roll-ups that are observed in the simulations.

3.2 Weakly Compressible, Viscous Model

In this section, a viscous model is proposed for the MRTI problem. The flow configuration is similar to that of the previous chapter (figure 2.1), but there are now horizontal walls located at $z = \pm H$ in the vertical coordinate. This model makes use of the Boussinesq approximation as in Farrow and Hocking [23], in which the two fluids of densities $\rho_1, \rho_2$ and permeabilities $\mu_1, \mu_2$, are replaced with a single fluid of continuous density variation $\rho$. In order to mimic the two fluid situation from the previous section, we require $\rho \approx \rho_1$ over the bottom half of the finite region, and to increase rapidly yet continuously and smoothly in a thin transition region of height $2\eta$ to $\rho \approx \rho_2$ in the top half of the finite region. To be consistent with the Boussinesq approximation, the density ratio $D = \rho_2/\rho_1$ in the fluid must be close to 1, so that $D - 1$ is small. The thin transition region thus replaces the sharp interface from the previous chapter. We use a constant permeability $\mu$, since $\mu \approx 1$ in any case.

Again, the governing equations are comprised of conservation of mass and momentum, as well as Maxwell’s Equations. The density, $\rho$, takes the form $\rho = \rho_0 + \bar{\rho}$, where $\rho_0$ is a constant density component, and $\bar{\rho}$ is a small variable component. We make use of the Boussinesq Approximation obtained from Farrow and Hocking [23] to split the usual conservation of mass equation, $\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{v}) = 0$, into two separate equations; the first of these is equation
(2.1), and the second is written as

$$\rho_1 \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = \epsilon_1 \nabla^2 \rho, \quad (3.1)$$

in which a diffusive term with coefficient $\epsilon_1$ has been added to the right hand side of equation (3.1), that will aid in the stability of the numerical method later. The constant $\epsilon_1$ is an Eckmann diffusion coefficient. In the viscous regime, the conservation of momentum takes the form,

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + (\mathbf{v} \cdot \nabla) (\rho \mathbf{v}) + \nabla \left( p + \frac{B^2}{2\mu} \right) = -\rho g \hat{e}_z + \frac{1}{\mu} (\mathbf{B} \cdot \nabla) \mathbf{B} + \rho \nu \nabla^2 \mathbf{v},$$

where $\nu$ is the kinematic viscosity coefficient. The Boussinesq approximation assumes that terms proportional to $\rho$ are negligible, with the exception of the gravity force term $\rho g \hat{e}_z$. We thus obtain

$$\rho_0 \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] + \nabla p^* = -\rho g \hat{e}_z + \frac{1}{\mu} (\mathbf{B} \cdot \nabla) \mathbf{B} + \rho_0 \nu \nabla^2 \mathbf{v} \quad (3.2)$$

for conservation of momentum, where we write $p^* = p + \rho_0 g z + \frac{B^2}{(2\mu)}$.

Completing the description requires the absence of magnetic monopoles and the finite conducting form of the Faraday equation, which are given by equations (2.3) and (2.4) respectively. Non-dimensionalising with respect to the same scales and references as for the inviscid, incompressible case, with the additional reference density, $\rho_0$, does not alter the symbolic form of equations (2.1) and (2.3). Equations (3.1), (3.2) and (2.4) are replaced with Boussinesq viscous relations

$$\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla) \rho = K \nabla^2 \rho \quad (3.3)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p^* = (\mathbf{B} \cdot \nabla) \mathbf{B} + S \nabla^2 \mathbf{v} \quad (3.4)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + R \nabla^2 \mathbf{B}. \quad (3.5)$$
The constants $K, S$ and $R$ are non-dimensional diffusion coefficients defined by

$$K = \epsilon_1 \sqrt{\frac{2\pi\lambda g}{\Lambda}}, \quad S = \nu \sqrt{\frac{2\pi\lambda g}{\Lambda}}, \quad R = \frac{1}{\mu\sigma} \sqrt{\frac{2\pi\lambda g}{\Lambda}}. \quad (3.6)$$

Notice that $S$ and $R$ are respectively the inverse fluid and magnetic Reynolds numbers for the system.

Since we are now in the non-linear domain, we no longer have the superposition principle, so we proceed directly to an arbitrarily directed base magnetic field rather than the separate treatments as for the linear case in chapter 2. We write

$$v = \mathbf{0} + \mathbf{V}$$

$$\mathbf{B} = \beta (l_x \mathbf{e}_x + l_z \mathbf{e}_z) + \mathbf{B},$$

where $l_x$ and $l_z$ are the direction cosines of our base magnetic field, related by equation (2.13), and $\mathbf{V} = (U, 0, W), \mathbf{B} = (B_x, 0, B_z)$.

We define a fluid streamfunction $\Psi$ by writing

$$U = \frac{\partial \Psi}{\partial z},$$

$$W = -\frac{\partial \Psi}{\partial x}, \quad (3.7)$$

in order to satisfy $\nabla \cdot \mathbf{V} = 0$ in equation (2.1) exactly. It is convenient now to define a fluid vorticity $\xi = \partial U/\partial z - \partial W/\partial x$ for the component directed out of the $x$-$z$ plane. In view of equations (3.7), the vorticity becomes $\xi = \nabla^2 \Psi$. In a similar fashion, define a magnetic streamfunction $\mathcal{X}$ by writing

$$B_x = \frac{\partial \mathcal{X}}{\partial z},$$

$$B_z = -\frac{\partial \mathcal{X}}{\partial x}, \quad (3.8)$$

in order to satisfy $\nabla \cdot \mathbf{B} = 0$ exactly in equation (2.3). A magnetic vorticity $\Upsilon$ may similarly be defined to be $\Upsilon = \nabla^2 \mathcal{X}$. Now, take the $z$ component of
the curl of equation (3.4) to eliminate the pressure term. The same procedure is used for equation (3.5) although there is no pressure to eliminate here. The fluid and magnetic vorticity equations then become

\[
\begin{align*}
\frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x} + W \frac{\partial \xi}{\partial z} &= S \nabla^2 \xi + (\beta l_x + B_z) \frac{\partial \Upsilon}{\partial x} + (\beta l_z + B_z) \frac{\partial \Upsilon}{\partial z} + \frac{\partial \rho}{\partial x} \quad \text{(3.9)} \\
\frac{\partial \Upsilon}{\partial t} + U \frac{\partial \Upsilon}{\partial x} + W \frac{\partial \Upsilon}{\partial z} &= R \nabla^2 \Upsilon + (\beta l_x + B_z) \frac{\partial \xi}{\partial x} + (\beta l_z + B_z) \frac{\partial \xi}{\partial z} + 2 \left[ (\nabla U \cdot \nabla B_z) - (\nabla W \cdot \nabla B_z) \right] \quad \text{(3.10)}
\end{align*}
\]

respectively.

Since the solution is periodic in \(x\), we can expand in Fourier Series in the \(x\) coordinate. This also allows all of the \(x\) derivatives to be calculated exactly. In the \(z\) direction, the use of finite differences allows a rapid calculation to be performed, and also simplifies the process compared with double Fourier-Series methods. Importantly, despite the use of finite differences, we still satisfy equations (2.1) and (2.3) exactly at all points in the fluid and at all times due to the streamfunction definitions (3.7) and (3.8). The method is as follows: for the fluid, we define Fourier coefficients for the streamfunction. The vorticity-streamfunction approach then allows the coefficients for the velocities and vorticity to be calculated easily. The same process is used for the magnetic field. Finally, the density can also be written as a Fourier Series in \(x\), with the numerical coefficients in \(z\) and \(t\). A straightforward Crank-Nicolson method [2] is then used to integrate the system forward in time. The Fourier decomposition works by expressing the variables as

\[
\begin{align*}
\Psi(x, z, t) &= A_0^\Psi(z, t) + \sum_{n=1}^{N} A_n^\Psi(z, t) \cos(nx) + \sum_{n=1}^{N} A_n^\Psi(z, t) \sin(nx) \quad \text{(3.11)} \\
\mathcal{X}(x, z, t) &= D_0^\mathcal{X}(z, t) + \sum_{n=1}^{N} D_n^\mathcal{X}(z, t) \cos(nx) + \sum_{n=1}^{N} D_n^\mathcal{X}(z, t) \sin(nx) \quad \text{(3.12)} \\
\bar{\rho}(x, z, t) &= R_0^\bar{\rho}(z, t) + \sum_{n=1}^{N} R_n^\bar{\rho}(z, t) \cos(nx) + \sum_{n=1}^{N} R_n^\bar{\rho}(z, t) \sin(nx). \quad \text{(3.13)}
\end{align*}
\]

From here it is straightforward to work out the velocities and magnetic field by
use of the streamfunction definitions (3.7) and (3.8). In order to solve equations (3.3), (3.9) and (3.10) we again use the streamfunction equations (3.11), (3.12) and (3.13) to write the system for $\bar{p}$, $\xi$ and $\Upsilon$ in the Fourier domain. Write,

$$
\xi(x, z, t) = C_0^c(z, t) + \sum_{n=1}^{N} C_n^c(z, t) \cos(nx) + \sum_{n=1}^{N} C_n^s(z, t) \sin(nx)
$$

$$
\Upsilon(x, z, t) = E_0^c(z, t) + \sum_{n=1}^{N} E_n^c(z, t) \cos(nx) + \sum_{n=1}^{N} E_n^s(z, t) \sin(nx),
$$

where,

$$
C_n^c = \frac{\partial^2 A_n^c}{\partial z^2} - n^2 A_n^c; \quad C_n^s = \frac{\partial^2 A_n^s}{\partial z^2} - n^2 A_n^s
$$

$$
E_n^c = \frac{\partial^2 D_n^c}{\partial z^2} - n^2 D_n^c; \quad E_n^s = \frac{\partial^2 D_n^s}{\partial z^2} - n^2 D_n^s.
$$

Equations (3.3), (3.9) and (3.10) are multiplied by basis functions $\cos(nx)$ and $\sin(nx)$ and integrated over the interval $-\pi \leq x \leq \pi$. This results in the system of differential equations

$$
\frac{\partial R_0^c}{\partial t} = K \left[ \frac{\partial^2 R_0^c}{\partial z^2} \right] - \frac{1}{2\pi} \int_{-\pi}^{\pi} Ldx
$$

$$
\frac{\partial R_n^c}{\partial t} = K \left[ \frac{\partial^2 R_n^c}{\partial z^2} - n^2 R_n^c \right] - \frac{1}{\pi} \int_{-\pi}^{\pi} L \cos(nx)dx
$$

$$
\frac{\partial R_n^s}{\partial t} = K \left[ \frac{\partial^2 R_n^s}{\partial z^2} - n^2 R_n^s \right] - \frac{1}{\pi} \int_{-\pi}^{\pi} L \sin(nx)dx
$$

$$
\frac{\partial C_0^c}{\partial t} = S \left[ \frac{\partial^2 C_0^c}{\partial z^2} \right] - \frac{1}{2\pi} \int_{-\pi}^{\pi} Mdx + \beta_l \frac{\partial E_0^c}{\partial z}
$$

$$
\frac{\partial C_n^c}{\partial t} = S \left[ \frac{\partial^2 C_n^c}{\partial z^2} - n^2 C_n^c \right] - \frac{1}{\pi} \int_{-\pi}^{\pi} M \cos(nx)dx + \beta_l \frac{\partial E_n^c}{\partial z} + nR_n^c
$$

$$
\frac{\partial C_n^s}{\partial t} = S \left[ \frac{\partial^2 C_n^s}{\partial z^2} - n^2 C_n^s \right] - \frac{1}{\pi} \int_{-\pi}^{\pi} M \sin(nx)dx + \beta_l \frac{\partial E_n^s}{\partial z} + \beta_l \frac{\partial E_n^c}{\partial z} - nR_n^c
$$

$$
\frac{\partial E_0^c}{\partial t} = R \left[ \frac{\partial^2 E_0^c}{\partial z^2} \right] - \frac{1}{2\pi} \int_{-\pi}^{\pi} Ndx + \beta_l \frac{\partial C_0^c}{\partial z}
$$

$$
\frac{\partial E_n^c}{\partial t} = R \left[ \frac{\partial^2 E_n^c}{\partial z^2} - n^2 E_n^c \right] - \frac{1}{\pi} \int_{-\pi}^{\pi} N \cos(nx)dx + \beta_l \frac{\partial C_n^c}{\partial z} + \beta_l \frac{\partial C_n^s}{\partial z}
$$

$$
\frac{\partial E_n^s}{\partial t} = R \left[ \frac{\partial^2 E_n^s}{\partial z^2} - n^2 E_n^s \right] - \frac{1}{\pi} \int_{-\pi}^{\pi} N \sin(nx)dx - \beta_l \frac{\partial C_n^c}{\partial z} + \beta_l \frac{\partial C_n^s}{\partial z}
$$

(3.15)
for the Fourier Coefficients. Here, we have defined

\[
L = U \frac{\partial \rho}{\partial x} + W \frac{\partial \rho}{\partial z}
\]

\[
M = U \frac{\partial \xi}{\partial x} + W \frac{\partial \xi}{\partial z} - B_x \frac{\partial \Upsilon}{\partial x} - B_z \frac{\partial \Upsilon}{\partial z}
\]

\[
N = U \frac{\partial \Upsilon}{\partial x} + W \frac{\partial \Upsilon}{\partial z} - B_x \frac{\partial \xi}{\partial x} - B_z \frac{\partial \xi}{\partial z} - 2 \left[ (\nabla U \cdot \nabla B_z) - (\nabla W \cdot \nabla B_x) \right].
\]

The integrals in the equations (3.15) can be evaluated numerically to any accuracy desired using Gaussian Quadrature, with use of the MATLAB M-File written by von Winckel [61].

Since the flow starts from rest, the initial velocity components are zero. The initial conditions for density are chosen to mimic the density profile of the linear two-fluid case as closely as possible, although using a continuous, smooth profile. For the magnetic field, we choose this to start from 0, despite the fact that the linear solution had the magnetic field starting from some small value proportional to the perturbed wave magnitude \( \epsilon \). This is done to avoid discontinuities that would produce Gibbs’ phenomenon oscillations in the Fourier series representations (3.11), (3.12) and (3.13) of the solution (see Kreyszig [36]). Furthermore, the zero initial perturbed magnetic field satisfies the divergence-free condition (2.3) identically; this avoids non-zero divergence being propagated by the Faraday equation (3.5). We thus use

\[
U(x, z, 0) = 0 ; \quad W(x, z, 0) = 0
\]

\[
\bar{\rho}(x, z, 0) = \begin{cases} 
0, & z < \epsilon \cos x - \eta \\
\left( \frac{D-1}{2} \right) \left( \sin \left[ \frac{\pi}{2\eta} (z - \epsilon \cos x) \right] + 1 \right), & \epsilon \cos x - \eta < z < \epsilon \cos x + \eta \\
D - 1, & z > \epsilon \cos x + \eta 
\end{cases}
\]

\[
B_x (x, z, 0) = 0 ; \quad B_z (x, z, 0) = 0
\]

(3.16)

as the initial conditions. As well as the initial condition for the density in equation (3.16), we also model the response to a smooth partial cosine interface.
In the paper by Forbes [24], in the non-magnetic case, this resulted in a bubble of lighter fluid forming that actually broke apart from the remainder of the light fluid. It will be seen that the presence of the magnetic field can stop this break-off from occurring. In this case, the initial condition for density is given in two parts; firstly, for \( |x| < q\pi \),

\[
\varrho(x, z, 0) = \begin{cases} 
0, & z < \epsilon \cos \frac{x}{\pi} - \eta \\
\left(\frac{D-1}{2\eta}\right)
\left(z - \epsilon \cos \frac{x}{\pi} + \eta\right), & \epsilon \cos \frac{x}{\pi} - \eta < z < \epsilon \cos \frac{x}{\pi} + \eta \\
D - 1, & z > \epsilon \cos \frac{x}{\pi} + \eta,
\end{cases}
\]  

(3.17a)

and secondly, for \( q\pi \leq |x| \leq \pi \),

\[
\varrho(x, z, 0) = \begin{cases} 
0, & z < -\epsilon - \eta \\
\left(\frac{D-1}{2\eta}\right)(z + \epsilon + \eta), & -\epsilon - \eta < z < -\epsilon + \eta \\
D - 1, & z > -\epsilon + \eta,
\end{cases}
\]  

(3.17b)

where \( 0 < q \leq 1 \) is required. We now need to work out the Fourier Coefficients in the expressions (3.11) and (3.12) for the fluid and magnetic streamfunctions, plus the density (3.13). These can be done accurately using Gaussian Quadrature since the integrals in the first case (3.16) can not be evaluated in closed form; however the integrals in the second case (3.17) are evaluated exactly in closed form. The initial values for the Fourier Coefficients are calculated from,

\[
\begin{align*}
A^c_0(z, 0) &= 0, \quad A^c_n(z, 0) = 0, \quad A^n_0(z, 0) = 0 \\
D^c_0(z, 0) &= 0, \quad D^c_n(z, 0) = 0, \quad D^n_0(z, 0) = 0 \\
R^c_0(z, 0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \varrho(x, z, 0) \, dx \\
R^c_n(z, 0) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \varrho(x, z, 0) \cos(nx) \, dx \\
R^n_n(z, 0) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \varrho(x, z, 0) \sin(nx) \, dx.
\end{align*}
\]
The initial values for the $C_n$ and $E_n$ are calculated using equations (3.14), and are all zero.

It is clear that at the top ($z = +H$) and bottom ($z = -H$) boundaries we want to simulate the stable conditions that will occur far from the centre of the medium, resulting in the following boundary conditions for the Fourier coefficients for density:

\[
R_0^c (-H, 0) = 0 \\
R_0^c (+H, 0) = D - 1,
\]

and all others zero at $z = \pm H$. These conditions result in zero initial velocity on the walls at $z = \pm H$. At later times, the fluid is allowed to slip on these boundaries, in spite of the fact that the fluid is viscous. This has been done to simplify the calculations, and is of minor consequence, since the interest here is focussed on the region of the interface, rather than the wall boundary layers. This is consistent with the Boussinesq approximation as used by Farrow & Hocking [23] and Forbes [24].

It is now a lengthy but somewhat straightforward matter to discretise and integrate equations (3.15) forward in time starting from the initial conditions given in equations (3.16). The first set of the coefficients evaluated at each iteration is the $R_n$, followed by the $C_n, E_n, A_n$ and finally the $D_n$. Then the integrals are evaluated by Gaussian Quadrature. The finite-difference approach that takes half the values at the current step and half at the next time step, as for the Crank-Nicolson method [2], is found to give excellent numerical stability, as expected. It is important that the step size in the vertical $z$-coordinate is very small in order to resolve the very thin density transition region of height $2\eta$.

The use of a thin transition region in the initial density profile gives virtually identical results in the non-magnetic case with those of Forbes [24], in which a strict two fluid initial density profile is used. Since the growth rate of modes is suppressed in the magnetic case, it is assumed that the transition region
approximation is still valid for this scenario.

3.3 Results

In this section, we present a selection of the flows simulated by the pseudo-spectral method in the compressible, viscous case. These are compared with results from Forbes [24] in the non-magnetic case, and the discussion is focussed on these comparisons. In order to compare with Forbes, we set the density ratio to $D = 1.05$, the diffusion parameters are taken as $K = S = R = 10^{-4}$ and we use $N = 25$ Fourier Coefficients in our $x$ variable. A trade off exists for the choice for the value of $N$. This value needs to be large enough to resolve any fine scale features of the plots, yet small enough to avoid significant numerical error. This trade off exists because it can be shown by stability analysis that high frequency modes become numerically unstable for this problem. The resolution of the parameter $N$ was tested by comparing the relative amplitudes of the Fourier modes at various intervals during the simulation. A situation in which the amplitudes of the higher modes grows with increasing frequency is a clear indication of numerical error. We use $\epsilon = 0.03$ and $\eta = 0.01$ in equations (3.16) and (3.17) for both initial density profiles. The top and bottom walls of the fluid are set to $H = \pm 1$, with the exception of the final section, where the partial cosine initial profile is simulated with the walls further from the interface at $H = \pm 2$. Regarding the numerical grid, a very small spacing of $10^{-3}$ was required in the vertical $z$-coordinate in order to resolve the very thin density transition region. A total of 201 grid points are used in the horizontal $x$-coordinate over the range $0 < x < \pi$, with symmetry arguments used to calculate values in the range $-\pi < x < 0$ with the exception of the case of an angled base magnetic field where 401 points are used over the full period $-\pi < x < \pi$. These points had a non-uniform spacing due to the use of Gaussian Quadrature in the method to calculate various integrals with respect to variable $x$. The values of these integrals were found to be consistent with those obtained if double or quadruple
the number of grid points was used, but varied slightly if half the number of grid points used, indicating that the 201 points chosen is numerically accurate. A time step of $10^{-3}$ was required in most cases, although some of the longer runs with an angled field required a time step of half this value. The shorter runs ($t = 24$) could use a much larger time step, even as large as 0.1, and still be numerically stable. In general, each run was performed at various decreasing time steps until the result was consistent. The resolution study performed has eliminated several possible sources of numerical error, which gives some confidence as to the accuracy of the numerics.

3.3.1 No magnetic field

The results of Forbes [24] are reproduced using the Pseudo-Spectral method described in this current paper, but setting the background magnetic field parameter $\beta$ to zero. This gives confidence that the method used is correct. Figure 3.1 shows contours of vorticity in color at time $t = 24$. Superimposed in black is the location of the interface, which is simulated by the $\rho = 1.025$ contour of density. A solution at later time $t = 40$ is presented in figure 3.2. The plots in figures 3.1 and 3.2 clearly show the evolution of the classic rolled up, mushroom type shape for the interface that results in the absence of magnetic fields. Also visible is the fact that there are two regions of high vorticity, centred on the interface, and that the vorticity is almost zero at the majority of points in the fluid, with only the locations near the interface having significant vorticity. This is in agreement with Forbes [24], where it was shown that small concentrated regions of vorticity develop in the viscous problem at the same points and times that curvature singularities form in the inviscid problem, and that these are thus the trigger for the roll up.

3.3.2 Horizontal Base Field

In this case, we have even, odd and even symmetry in $x$ for density, vorticity and current density respectively. From equation (2.23) it can be seen that there
Figure 3.1: Simulation of the RTI at time $t = 24$ in the absence of a magnetic field. The plot shows contours of vorticity in color, where the red end of the scale represents clockwise rotation, and the blue end counter-clockwise rotation. The black line is the location of the interface, taken to be the contour for which $\eta = 1.025$. 

Parameters:
- $D = 1.05$
- $\varepsilon = 0.03$
- $H = 1$
- $\beta = 0$
- $t = 24$
Figure 3.2: Simulation of the RTI at time $t = 40$ in the absence of a magnetic field. The plot shows contours of vorticity in color, where the red end of the scale represents clockwise rotation, and the blue end counter-clockwise rotation. The black line is the location of the interface, taken to be the contour for which $\rho = 0.025$. 

\[ D = 1.05, \, \epsilon = 0.03, \, H = 1, \, \beta = 0, \, t = 40 \]
are three different types of behaviour possible for the evolution of the interface in the linear regime, occurring when the magnetic Atwood number (2.24) is either positive, zero or negative. A negative base magnetic field produces the same evolution of the interface as for a corresponding base magnetic field of the same magnitude but positive in sign, since the interface profile depends on $\beta^2$. The following figures explore the above scenarios in turn for the case of the non-linear viscous model with a horizontal base field described in section 3.2.

Figure 3.3 shows contours of density as solid lines at time $t = 24$ where the base magnetic field is of strength $\beta = 0.1$ in dimensionless units. The location of the interface is in the region where the contours are closely spaced, since the density of the fluid changes rapidly near the interface. The dashed lines show the corresponding linear, incompressible, inviscid, infinite wall solution (2.23) for the interface under the same parameters. At this time there is quite good agreement between the two different solutions, thus giving confidence in the non-linear method which should of course give a similar interface profile to the linear method at early times while the amplitude of the waves is still small.

The coloured lines in figures 3.4 and 3.5 show the corresponding contours of vorticity and current density respectively. In a similar fashion to figures 3.1 and 3.2, the $\rho = 1.025$ contour is superimposed in black, in order to show the location of the interface, and its proximity to the regions of high vorticity and current density, for the latter two plots. As in the previous case with no base magnetic field, the majority of points in the fluid have no vorticity nor current density; only points around the interface have significantly non-zero values of these quantities. In the horizontal $x$-coordinate, the odd symmetry for vorticity and even symmetry for current density is clearly evident.

The location of the regions of high vorticity in figure 3.4 has been explained by Forbes [24]; the centres of these regions occur in the viscous problem in precisely the same locations that curvature singularities form in the inviscid problem. In order to explain the location of the regions of high induced current
density, we approximate the non-dimensional form of the Lorentz equation by

\[ \mathbf{J} = \mathbf{v} \times \mathbf{B} / R, \]  

(3.18)

where \( R \) is the inverse Magnetic Reynolds number defined in equation (3.6). Notice that the electric field, \( \mathbf{E} \) does not appear in the Lorentz equation (3.18). Under the present circumstances in which particles travel at non-relativistic speeds, \( |\mathbf{E}| \ll |\mathbf{v} \times \mathbf{B}| \), as explained in detail in [19]. In view of equation (3.18), it is evident that the induced current is largest when the direction of the velocity of the flow is perpendicular to the direction of the total magnetic field.

At the early time shown in figure 3.5, the induced magnetic field is much smaller in magnitude than the base magnetic field, resulting in a total magnetic field that is directed horizontally. The induced currents will thus have maximum magnitude in regions in which the velocity of the flow is directed vertically. The largest velocities occur in the regions of high vorticity, with the largest vertical components occurring on the interface at \( x = \pm \pi \), in which the velocity is directed downwards, and at \( x = 0 \), where the direction is upwards. This explains the induced current flowing out of the page at \( x = \pm \pi \), and flowing into the page at \( x = 0 \) in figure 3.5.

If we compare figure 3.1 with figure 3.4, it can be seen that the magnitude of the vorticity at corresponding points in the fluid is much smaller when the base magnetic field is present, despite all the other parameters being identical. We conclude at this stage that the presence of the magnetic field has in some way dampened the generation of vorticity. We thus expect less roll up in the MRTI compared with the magnetic-free version of the instability.

The mechanism for how the magnetic field reduces the vorticity is the induced current density, contours of which are shown in figure 3.5. A simplified version of events is that firstly, the curvature singularity that would develop in the inviscid problem is suppressed by viscosity, which instead creates regions of high vorticity. Then secondly, since the fluid is comprised of charged particles,
this vorticity causes a change in the magnetic field, and by Lenz’ Law, the ensuing currents are directed such as to oppose the change that caused them, that is, such as to suppress the vorticity. From the Lorentz equation (3.18), we thus expect that when the total magnetic field is horizontal, that it will be the vertical components of velocity that are suppressed, with the horizontal components to remain unaffected.

We now consider a simulation at the later time of $t = 44$. Figures 3.6, 3.7 and 3.8 show contours of density, vorticity and current density respectively, in exactly the same fashion as figures 3.3, 3.4 and 3.5. In figure 3.6, it is immediately clear that the linear and non-linear solutions have now diverged, meaning that our growing wave amplitude is no longer considered small at $t = 44$, so that non-linearity is genuinely important at this time. A comparison of figure 3.7 with figure 3.2, which showed the growth of the interface in the magnetic free case at time $t = 40$, supports the notion that a horizontal magnetic field will suppress vertical components of velocity. One might argue that this result is due to the walls at $H = \pm 1$ having an effect on the flow, but in fact these are two separate effects as will be further demonstrated in section 3.3.5 when the walls are further away at $H = \pm 2$.

The contours of vorticity are given in figure 3.7. Whilst the vorticity is still concentrated in regions near the interface as was the case at the earlier time $t = 24$, at this later time $t = 44$ we have a much higher magnitude of vorticity, over 10 dimensionless units in magnitude compared with only 0.2 units at $t = 24$. The same is true of the current density, shown in figure 3.8. Now the difference is even greater, with maximum values of over 20 dimensionless units at $t = 44$ compared with 0.2 units at $t = 24$.

The important conclusion is that for each point on the interface, there will be a given direction where the velocity of the interface is maximally suppressed, which will be perpendicular to the direction of the total magnetic field. By the same token, there will be no suppression in the direction of the total magnetic field, and thus it is not possible to suppress fully the RTI in all directions through
the use of a background magnetic field alone, as noted by Stone and Gardiner [57].

We discuss briefly the case of zero magnetic Atwood number (2.24), and whilst the linear solution (2.23) predicts oscillations with zero growth for the interface, it is evident that non-linear effects will still generate some growth in this case, albeit much slower than that for flows with a positive magnetic Atwood number. Obviously the Eckmann diffusion $\epsilon_1$ in equation (3.1) takes part as well, causing the density to spread. No oscillatory behaviour is observed in the non-linear simulation, although a slowing of the growth rate compared with the case of positive magnetic Atwood number occurs.

The third case analysed was that in which the magnetic Atwood number (2.24) from the linear model is negative. Figures 3.9 to 3.11 show the simulation at time $t = 16$ with the base magnetic field strength taking the value $\beta = 0.2$. This value gives oscillating growth behaviour in the linear regime, a snapshot of which is given by the dashed line representing the location of the interface in the linear problem in figure 3.9. If we compare this with figure 3.12 which shows the same fluid at the later time $t = 24$, we see that while there is no direct visual confirmation that the interface oscillates in the non-linear simulation, it can be seen from figures 3.10 and 3.13 that the high base magnetic field has managed to reverse the polarity of the vorticity generated by the curvature of the interface, suggesting that oscillations could in fact be present. There is no such reversal for the current density at these particular times, shown in figures 3.11 and 3.14. The growth rate of the interface is virtually static here, suppressed by the magnetic field at these time scales. The far slower growth rate for the $\beta = 0.2$ case compared with the $\beta = 0.1$ case is evident from the comparison between figures 3.12 and 3.3, which have identical parameters with the exception of the base magnetic field strength $\beta$.

We conclude this section by considering the effects of a negative base magnetic field on the MRTI. This does not change the fact that regions of high vorticity are generated in order to prevent curvature singularities from forming.
Figure 3.3: Horizontal Base Field with positive magnetic Atwood Number at time $t = 24$. Contours are shown for density. The dashed line represents the linear solution (2.23) from section 2.2.
Figure 3.4: Horizontal Base Field with positive magnetic Atwood Number at time $t = 24$. Contours are shown for vorticity. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modelled by the contour for $\theta = 1.025$. 
Figure 3.5: Horizontal Base Field with positive magnetic Atwood Number at time $t = 24$. Contours are shown for current density. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line is the non-linear interface, modelled by the contour for $\overline{\rho} = 1.025$. 
Figure 3.6: Horizontal Base Field with positive magnetic Atwood Number at time $t = 44$. Contours are shown for density. The dashed line represents the linear solution (2.23) from section 2.2.
Figure 3.7: Horizontal Base Field with positive magnetic Atwood Number at time $t = 44$. Contours are shown for vorticity. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modelled by the contour for $\rho = 1.025$. 
Figure 3.8: Horizontal Base Field with positive magnetic Atwood Number at time $t = 44$. Contours are shown for current density. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line is the non-linear interface, modelled by the contour for $\rho = 1.025$. 
Figure 3.9: Horizontal Base Field with positive magnetic Atwood Number at time \(t = 16\). Contours are shown for density. The dashed line represents the linear solution (2.23) from section 2.2.
Figure 3.10: Horizontal Base Field with positive magnetic Atwood Number at time $t = 16$. Contours are shown for vorticity. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modeled by the contour for $\rho = 1.025$. 
Figure 3.11: Horizontal Base Field with positive magnetic Atwood Number at time $t = 16$. Contours are shown for current density. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line is the non-linear interface, modelled by the contour for $\bar{\rho} = 1.025$. 
Figure 3.12: Horizontal Base Field with positive magnetic Atwood Number at time $t = 24$. Contours are shown for density. The dashed line represents the linear solution (2.23) from section 2.2.
Figure 3.13: Horizontal Base Field with positive magnetic Atwood Number at time $t = 24$. Contours are shown for vorticity. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modelled by the contour for $\rho = 1.025$. 
Figure 3.14: Horizontal Base Field with positive magnetic Atwood Number at time $t = 24$. Contours are shown for current density. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line is the non-linear interface, modelled by the contour for $\bar{\rho} = 1.025$. 
on the interface. What changes is that whilst the charged particles caught in the regions of high vorticity will be directed in the same direction as before, this will now cause the opposite change to the magnetic field, since the base magnetic field has reversed. This subsequently causes the opposite induced current response, and the two cancel out to give the same response in the density equation for the simulated interface. The simulation for a negative magnetic field does indeed look identical to that for a magnetic field of the same magnitude, but positive. The result is that the non-linear case agrees with the result from the linear case, in which the interface profile given by equation (2.23) on page 34 was shown to be independent of the sign of the base magnetic field strength $\beta$.

### 3.3.3 Vertical Base Field

In this case, we still have even symmetry in $x$ for density, odd symmetry in $x$ for vorticity, but we now have odd symmetry in $x$ for current density as well. Since the linear theory allowed exponential growth as the only solution for the profile of the interface (2.26) in the case of a vertical magnetic field, we need only consider this one case in the non-linear model, as opposed to the horizontal field where multiple behaviours were possible.

We present precisely the same format of figures as for the horizontal magnetic field case. Figure 3.15 shows that while the agreement between the non-linear and linear solutions for the interface at the early time $t = 24$ is not as close as that for the case of the horizontal base field, it is still reasonable nonetheless. It can also be seen that the growth rate is somewhat slower than in the case of zero base magnetic field shown in figure 3.1 as expected. Figure 3.16 shows that there are two regions of peak vorticity located on the interface either side of centre with respect to the horizontal $x$-coordinate, the position of which is again explained by Forbes [24]. Note that the peak magnitude of vorticity is significantly lower than in both the case of a horizontal base magnetic field and no magnetic field.
The change in symmetry for the current density is clearly evident from figure 3.17, as is the fact that rather than being located on the interface as in the horizontal magnetic case, the centres of the regions of high current density are located just above and just below the interface. We analyse the Lorentz equation (3.18) to explain these positions, in the same fashion used to explain the corresponding positions in the case of a horizontal base field. A similar analysis yields the conclusion that the induced current will be high in regions where the direction of the velocity is horizontal. Again, the velocity is maximum in the regions of high vorticity. The horizontal velocity maxima thus occur just above and just below the interface at $x \approx \pm \pi/2$, exactly where the induced current density magnitude maxima are located in figure 3.17.

At the later time $t = 48$, the profile of the interface has become slightly rolled up, but not as much as in the case of no base magnetic field. The lower observed vorticity in the present case is the explanation for this result. The profile is flat at the top, due to the presence of the wall at $H = 1$, as can be seen in figure 3.18. Finally, figures 3.19 and 3.20 show that the regions of high vorticity and current density remain close to the interface, with both the vorticity and current density virtually zero everywhere else in the fluid. In summary, the results are very similar to the horizontal base magnetic field case; the main differences being the rather different shape for the profile of the interface, the lower magnitude of vorticity and current density, plus the change in symmetry and position of the current density.

As for the horizontal magnetic field case, we analyse the Lorentz equation (3.18) to explain the resulting shape of the interface. Now that the base magnetic field is directed vertically, it will be the horizontal components of velocity that are suppressed, whereas vertical components are largely unaffected. This horizontal suppression is particularly effective at early times in dampening the vorticity. The regions of high vorticity that develop in the case of no magnetic field are long in the horizontal direction, and narrow in the vertical direction. The fluid particles in this region thus spend the majority of the time travel-
ling perpendicular to the magnetic field, thus restricting the vorticity. Far from
the interface, matters are complicated by the presence of the vertical walls at
\( H = \pm 1 \). It is evident that the top wall suppresses the interface from rising
upwards. Later plots in section 3.3.5 show plots with the walls further out at
\( H = \pm 2 \), and it is clearer from these that the vertical components of velocity
are not being suppressed so much as the horizontal components.

In view of the above explanation for growth suppression, one would expect
that, for very strong vertical magnetic fields, horizontal motion and hence the
MRTI will be almost totally suppressed. Indeed, figure 3.15 shows very minimal
growth of the MRTI at time \( t = 24 \) where the magnetic field is of strength
\( \beta = 0.1 \). At the later time of \( t = 48 \), a weaker magnetic field of strength
\( \beta = 0.05 \) was used in order to show the distinguishing features of a slightly
suppressed growth rate in figure 3.18, since the growth rate in the former case
was insufficient to show such features.

### 3.3.4 Arbitrary Base Field

For an arbitrarily directed base magnetic field, we lose the bilateral symmetry
that both the horizontal and vertical base field cases exhibited, because the cur-
crent density is now a combination of the even and odd parities of sections 3.3.2
and 3.3.3 respectively. This results in the vorticity and density profiles requiring
non-zero values for both the even and odd components of their respective
Fourier Series representations as well. As expected, we get growth suppres-
sion for the interface in both the vertical and horizontal directions due to the
\( \mathbf{v} \times \mathbf{B}/R \) term in the Lorentz equation (3.18) being significant for both vertical
and horizontal velocities. Each point in the fluid still has directions in which
the magnetic field is ineffective at dampening the growth of the interface, being
precisely the direction of the total magnetic field at that point.

Figures 3.21 to 3.26 show the results in this case, where the magnetic field
is at the angle \( \pi/3 \) up from horizontal, and we again retain the same formats
for the graphs as for the previous cases.
$D = 1.05 ; \varepsilon = 0.03 ; H = 1 ; \beta = 0.1 ; \theta = \pi / 2 ; t = 24$

Figure 3.15: Horizontal Base Field with positive magnetic Atwood Number at time $t = 24$. Contours are shown for density. The dashed line represents the linear solution (2.26) from section 2.2.
Figure 3.16: Horizontal Base Field with positive magnetic Atwood Number at time $t = 24$. Contours are shown for vorticity. Red regions show clockwise rotation, and blue regions show counterclockwise rotation. The solid black line is the non-linear interface, modeled by the contour for $\rho = 1.025$. 
Figure 3.17: Horizontal Base Field with positive magnetic Atwood Number at time $t = 24$. Contours are shown for current density. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line is the non-linear interface, modelled by the contour for $\bar{\rho} = 1.025$. 
Figure 3.18: Horizontal Base Field with positive magnetic Atwood Number at time $t = 48$. Contours are shown for density. The dashed line represents the linear solution (2.26) from section 2.2.
Figure 3.19: Horizontal Base Field with positive magnetic Atwood Number at time $t = 48$. Contours are shown for vorticity. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modelled by the contour for $\bar{\rho} = 1.025$. 
Figure 3.20: Horizontal Base Field with positive magnetic Atwood Number at time $t = 48$. Contours are shown for current density. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line is the non-linear interface, modelled by the contour for $\bar{\rho} = 1.025$. 

The figure shows the current density distribution with two sets of contours. One set of contours is for red regions indicating current flowing into the page, and the other set for blue regions indicating current flowing out of the page. The solid black line represents the non-linear interface, which is modelled by the contour for the density ratio $\bar{\rho} = 1.025$. The color scale indicates the range of current density values.
Figure 3.21: Horizontal Base Field with positive magnetic Atwood Number at time $t = 36$. Contours are shown for density. The dashed line represents the linear solution (2.29) from section 2.2.
Figure 3.22: Horizontal Base Field with positive magnetic Atwood Number at time $t = 36$. Contours are shown for vorticity. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modelled by the contour for $\rho = 1.025$. 
Figure 3.23: Horizontal Base Field with positive magnetic Atwood Number at time $t = 36$. Contours are shown for current density. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line is the non-linear interface, modelled by the contour for $\bar{\rho} = 1.025$. 
Figure 3.24: Horizontal Base Field with positive magnetic Atwood Number at time $t = 48$. Contours are shown for density. The dashed line represents the linear solution (2.29) from section 2.2.
Figure 3.25: Horizontal Base Field with positive magnetic Atwood Number at time $t = 48$. Contours are shown for vorticity. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modelled by the contour for $\rho = 1.025$. 
Figure 3.26: Horizontal Base Field with positive magnetic Atwood Number at time $t = 48$. Contours are shown for current density. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line is the non-linear interface, modelled by the contour for $\rho = 1.025$. 
In figure 3.21, contours of density at $t = 36$ are presented. At this time, the lack of bilateral symmetry in the interface profile has just become apparent. The linear solution for the profile of the interface (2.29) is shown as a dashed line, although it is clear that by time $t = 36$ that the linear model is invalid. The vorticity plot given by figure 3.22 shows that whilst there are two regions of high vorticity as for the previous cases, points in the region centred around $x \approx -\pi/2$ have a higher magnitude than the corresponding points centred around $x \approx \pi/2$. This is due to the angle of the base magnetic field. At early times ($t < 36$), the velocity of particles in the region of high vorticity centred about $x \approx -\pi/2$ is approximately parallel to the base magnetic field over most of the profile. For particles in the region of high vorticity centred about $x \approx \pi/2$, the opposite is true; the velocity is approximately perpendicular to the base magnetic field over most of the profile. Hence, by equation (3.18), the vorticity around $x \approx \pi/2$ is suppressed, whereas the vorticity around $x \approx -\pi/2$ is not suppressed. This then results in the very asymmetric profile for the interface as shown in figure 3.24, where more roll up exists in the region $x < 0$ than for the region $x > 0$. Whilst the symmetry of the previous cases is not retained in the present example, the fact that the regions of high vorticity and high current density are located near the interface as has been observed in all the previous cases thus far is in fact still true.

3.3.5 Partial Cosine Initial Profile

The results in this final section show that the evolution of the interface is significantly influenced by the initial density profile of the fluid. In the absence of a magnetic field, a similar conclusion was also reached by Forbes [24]. We now extend the results from Forbes to include the case where a magnetic field is present in the fluid. The initial density profile is the periodic partial cosine profile given by equation (3.17), with parameter $q = 0.3$. This is a multi-mode disturbance, as opposed to the single mode disturbance of all the previous cases considered. The top and bottom walls of the fluid are extended out to $H = \pm 2$
in order to compare the shape of the resulting interface with that of Forbes. First, however, the results of Forbes in the non magnetic case are reproduced with the pseudo spectral method of this paper. Figures 3.27 and 3.28 show the profile of the interface at two different times, \( t = 15 \) in figure 3.27 and \( t = 25 \) in figure 3.28. The plots show contours of vorticity in color, with the center density contour shown in black to represent the location of the simulated interface. By \( t = 25 \), the bottom figure shows that a bubble of lighter fluid has fully pinched off and sits in the top half of the fluid, unconnected from the remaining lighter fluid in the bottom half. This situation is in agreement with Forbes [24].

Figures 3.29 to 3.32 show the resulting shape of the interface when the partial cosine initial density profile is used in combination with a horizontal base magnetic field (\( \theta = 0 \)) of strength \( \beta = 0.05 \), again at two different times. In the magnetic regime, we show separately contours of vorticity and current density, both superimposed by the center density contour in black to represent the location of the simulated interface. It can be seen that the presence of a horizontal base magnetic field seems to have prevented the pinch off as observed in the non-magnetic scenario (figure 3.28) from occurring.

The results with the partial cosine initial density profile for the case of a vertical base magnetic field (\( \theta = \pi/2 \)) are shown for time \( t = 26 \) in figures 3.33 and 3.34, and time \( t = 32 \) in figures 3.35 and 3.36. The same type of plots as for the horizontal base magnetic field case are shown; that is, contours of vorticity and current density in color, superimposed by the center contour of density in black. As for the case of a horizontal base magnetic field, there is no pinch off in this simulation. The resulting profile is completely different however.

The results are consistent with the earlier reasoning based on the strength of the induced currents as determined by the Lorentz equation (3.18), and the use of Lenz’ law. When the magnetic field is directed horizontally, we see retardation of the growth in the vertical direction, and when the magnetic field is directed vertically, we see retardation of the growth in the horizontal direction. Whilst it can not be proven that the magnetic field prevents the pinch off from occurring
Figure 3.27: Contours of vorticity for the partial cosine profile with no magnetic field at time $t = 15$. The red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line in plots (a) and (b) is the non-linear interface, modelled by the contour for $\bar{\rho} = 1.025$. 

$D = 1.05; \varepsilon = 0.03; q = 0.3; H = 2; t = 15$
Figure 3.28: Contours of vorticity for the partial cosine profile with no magnetic field at time $t = 25$. The red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line in plots (a) and (b) is the non-linear interface, modelled by the contour for $\bar{\rho} = 1.025$. 
Figure 3.29: Contours of vorticity for the partial cosine profile with $\theta = 0$ at time $t = 25$. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modelled by the contour for $\rho = 1.025$. 
Current Density

Figure 3.30: Contours of current density for the partial cosine profile with $\theta = 0$ at time $t = 25$. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line in plots (a) and (b) is the non-linear interface, modeled by the contour for $p = 1025$. The parameters are $D = 0.03$, $\varepsilon = 0.03$, $q = 0.3$, $\beta = 0.05$, $\theta = 0$, $H = 2$, and $t = 25$. The current density is normalized for clarity.
Figure 3.31: Contours of vorticity for the partial cosine profile with $\theta = 0$ at time $t = 30$. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modelled by the contour for $\rho = 1.025$. 

$D = 1.05; \gamma = 0.03; q = 0.3; \beta = 0.05; \theta = 0; H = 2; t = 30$
Figure 3.32: Contours of current density for the partial cosine profile with $\theta = 0$ at time $t = 30$. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line in plots (a) and (b) is the non-linear interface, modelled by the contour for $\bar{\rho} = 1.025$. 
Figure 3.33: Contours of vorticity for the partial cosine profile with $\theta = \pi/2$ at time $t = 26$. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modelled by the contour for $\rho = 1.025$. 
Figure 3.34: Contours of current density for the partial cosine profile with $\theta = \pi/2$ at time $t = 26$. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line in plots (a) and (b) is the non-linear interface, modelled by the contour for $\rho = 1.025$. 

D = 1.05 ; $\varepsilon = 0.03 ; q = 0.3 ; \beta = 0.05 ; \theta = \pi/2 ; H = 2 ; t = 26$
Figure 3.35: Contours of vorticity for the partial cosine profile with $\theta = \pi/2$ at time $t = 32$. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modelled by the contour for $\rho = 1.025$. 

$D = 1.05, r = 0.03, q = 0.3, \beta = 0.05, \phi = \pi/2, H = 2; \gamma = 32$
Figure 3.36: Contours of current density for the partial cosine profile with $\theta = \pi/2$ at time $t = 32$. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line in plots (a) and (b) is the non-linear interface, modelled by the contour for $\rho = 1.025$. 

$D = 1.05$; $\varepsilon = 0.03$; $q = 0.3$; $\beta = 0.05$; $\theta = \pi/2$; $H = 2$; $t = 32$
in general, the results from figures 3.31 to 3.36 suggest that the presence of a magnetic field will at least delay the time that pinch off occurs, and maybe even prevent it from occurring all together.

Finally, we present the results for the case of an angled magnetic field ($\theta = \pi/3$). Figures 3.37 and 3.38 show the results at time $t = 20$ while figures 3.39 and 3.40 show the results for time $t = 25$. The figures retain the same format as for the previous cases of horizontal and vertical magnetic fields. The multimode nature of the initial density profile results in four distinct regions of high vorticity near the interface at early times (see figure 3.27 in the non-magnetic case), as opposed to just two such regions in the single mode case. This in combination with an angled base magnetic field allows the more complicated behaviour of the present case to ensue, whereby the mushroom type shape for the interface is tilted to one side by time $t = 25$. Significantly, no bubbles of fluid have been observed to form, unlike the case with no background magnetic field.

3.4 Conclusion

This chapter describes a detailed study of the MRTI in two dimensions for viscous, weakly compressible flow. A pseudo spectral method has been used to follow the evolution of the interface profile in time. A significant benefit of this method, indeed its very motivation, is the fact that it ensures that the total magnetic field is divergence-free at all locations in the fluid at all times. The value of this should not be under estimated. Brackbill and Barnes [9] show that the effect of artificial magnetic monopoles due to numerical errors from using a finite difference approximation is to produce a fictitious force parallel to the magnetic field. The results in this thesis are thus immune to effects from such fictitious forces automatically. The additional assumptions and approximations used in typical constrained transport methods (as used in the numerical algorithms mentioned in the introduction) to satisfy the divergence-free nature of the magnetic field are thus not required in the method presented in this chap-
Figure 3.37: Contours of vorticity for the partial cosine profile with $\theta = \pi/3$ at time $t = 20$. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modelled by the contour for $\mathcal{P} = 1.025$. 

\[
\begin{align*}
D &= 1.05 ; 
\varepsilon &= 0.03 ; 
q &= 0.3 ; 
\beta &= 0.05 ; 
\theta &= \pi/3 ; 
H &= 2 ; 
t &= 20
\end{align*}
\]
Figure 3.38: Contours of current density for the partial cosine profile with $\theta = \pi/3$ at time $t = 20$. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line in plots (a) and (b) is the non-linear interface, modelled by the contour for $\rho = 1 / 0.25$. 

\[ \begin{align*} 
D &= 1.05 \\
\varepsilon &= 0.03 \\
q &= 0.3 \\
\beta &= 0.05 \\
\theta &= \pi / 3 \\
H &= 2 \\
t &= 20 
\end{align*} \]
Figure 3.39: Contours of vorticity for the partial cosine profile with $\theta = \pi/3$ at time $t = 25$. Red regions show clockwise rotation, and blue regions show counter-clockwise rotation. The solid black line is the non-linear interface, modelled by the contour for $\rho = 1.025$. 
Figure 3.40: Contours of current density for the partial cosine profile with $\theta = \pi/3$ at time $t = 25$. Red regions show current flowing into the page, and blue regions show current flowing out of the page. The solid black line in plots (a) and (b) is the non-linear interface, modelled by the contour for $\rho = 1.025$. 
ter. This method also allows the regions of high vorticity and induced current density to be located easily, which is significant since it is the induced current density that is responsible for stabilising the interface roll up that results from high vorticity. This has been confirmed by running the cases from sections 3.3.2 and 3.3.3 but setting the inverse magnetic Reynolds number to $R = 10^4$. The parameter $R$ controls the relative proportion of induced current effects to diffusion effects, as can be seen from equation (3.5). Choosing $R = 10^4$ has the effect of negating induced currents, so that diffusion is the dominant process in this regime. The results with $R = 10^4$ look identical to the results from the case of no magnetic field in section 3.3.1, and are therefore not shown in the interests of space.

The significant result that the RTI can not be fully suppressed in all directions by the introduction of a magnetic field alone is obtained from analysis of the mechanism in which the presence of a magnetic field reduces the growth rate of the interface compared with the case where no magnetic field is present. This result is in agreement with a similar result obtained by Stone and Gardiner [57]. Highly conducting plasmas will suppress the growth of the interface much more than weakly conducting plasmas however.

The development of the interface profile was shown to be strongly influenced by its initial condition. This is not surprising, since Forbes [24] has shown that it is the curvature of the interface that determines the location of regions of high vorticity. The present work extends this result to the MRTI by showing that, in turn, the location of the regions of high vorticity in conjunction with the direction of the total magnetic field determines the location of the regions of high current density, which are responsible for stabilising the roll up that is caused by the vorticity. An interesting result is obtained in the classical RTI whereby certain initial interface profiles develop in such a way as to form a bubble of the lighter fluid that breaks apart from the remainder of the light fluid. These same profiles were shown to behave very differently when a magnetic field is present. In this case, the bubble of fluid did not seem to form, and the fluids
remained separated into the two connected regions of light fluid and heavy fluid respectively.

We conclude with some brief remarks on the role of the Kelvin-Helmholtz instability (KHI) in this problem. Whilst the MRTI is the primary instability of this problem, the triggering of this instability results in relative motion between the two fluids and secondary KHI ensue. The presence of a background magnetic field will also suppress the KHI, since by primarily suppressing the regions of high vorticity, there will be reduced relative motion between the two fluids. The results of this thesis showed that both the growth rate and the amount of roll up are reduced in the presence of a magnetic field, which is consistent with Jun et. al. [33], where it was noted that the KHI results in a significant roll-up on the mushroom cap. More detail on the specifics of the KHI can be found in the literature, for example, Chen and Forbes [14].
Chapter 4

An accurate method for computing the viscous, weakly compressible magnetic Rayleigh-Taylor instability in cylindrical geometry

4.1 Introduction

While the original Rayleigh-Taylor instability (RTI), pioneered by Lord Rayleigh in 1883 [50], and Sir G.I. Taylor in 1950 [59], considered horizontal planar flow, generalisations to this basic RTI problem have since been devised. The RTI in cylindrical coordinates was evidently first considered by Bell [8] in 1951.
Bell discovered that when considering curved geometry, additional effects can occur, which have no equivalent in the basic Cartesian geometry. In the linear model derived by Bell in cylindrical coordinates, the curvature of the interface contributes to the evolution of the interface in time. It is, in fact, possible for an otherwise stable system to in fact become unstable due to the curvature of the interface. In 1954, Plesset [49] derived the analogous result for spherical coordinates, and this effect is now collectively known as the Bell-Plesset effect.

The combination of cylindrical coordinates together with magnetic effects first appeared in a study by Harris [28] in 1962.

As for the case of the MRTI in Cartesian geometry, study in the period from 1980 to the present in the case of cylindrical geometry has focussed primarily on numerical analysis of both the RTI and MRTI in the non-linear regime. A much smaller volume of work in this period considers the RTI in the linear regime. Papers by Mikaelian [43] and Yu & Livescu [63] in 2005 and 2008 respectively are two examples from the present era that consider linear effects of the RTI in cylindrical coordinates. Mikaelian considered the three dimensional RTI with multiple interfaces where the fluids are arranged in a system of $N$ cylindrical coaxial shells. In the case where $N = 3$, Mikaelian showed that, for high frequency perturbations, the system of differential equations decouples, so that Bell’s equation can be applied at each interface independently for such a perturbation. Yu & Livescu defined a geometry parameter and considered how the qualitative nature of the flow changed as the flow varied from 2D axisymmetric $(r, z)$ to 3D $(r, \theta, z)$ to 2D circular $(r, \theta)$. Specifically, a comparison between two configurations is made. The convergent configuration is similar to the configuration of this thesis, whereby a heavy fluid surrounds a light fluid, and gravity is directed radially inwards. The divergent configuration is vice-versa, that is, a light fluid surrounds a heavy fluid, and gravity is directed radially outwards. The main result is that the growth rate of the interface in the convergent case is faster than for the divergent case, with the exception of the 2D circular regime, where the reverse occurs. For this reason, it is concluded
that the 3D results are qualitatively the same as for the 2D axisymmetric case, but qualitatively different to the 2D circular case.

The MRTI is the subject of frontier research in the field of astrophysics, with the study of HII regions and nebulae being the focus of applications in this chapter. These similar topics differ in that the ionization is higher in the case of nebulae compared with HII regions [53]. The basic theory behind the formation of such structures is well known, beginning with the classic 1939 Strömgren paper [58] in which the problem of the ionization and excitation of interstellar hydrogen was modelled by what is now known as a Strömgren sphere. Later, in 1954, Kahn considered in detail the acceleration of interstellar clouds by ionizing radiation [34]. The first to consider magnetic effects was Spitzer in 1956 [42], in which a lower limit to the mass that can be gravitationally bound was calculated. The basic theory of HII regions has been a textbook result from as early as 1978 with the subsequent work by Spitzer [53]. Modern numerical treatments consider the ideal MHD equations [26], and even photoionization and recombination in some cases [4, 27, 44]. The numerical component of such works is performed by various algorithms, such as Athena [56] in the case of [27], SPH/N [7] for works by Dale and Bonnell [17, 18] and the PHAB-C² code [30] used by Arthur et. al [4]. Mizuta found that when recombination is included, the growth rates were observed to be similar to the isothermal model of Williams [62].

The contents of this chapter contain a spectral-numerical method used to calculate the density, vorticity and current density of the fluid in cylindrical geometry. As for the model presented in chapter three, the methodology used in this current chapter is able to satisfy the condition requiring zero divergence of the magnetic field at all points in the space at all times in an exact, analytic fashion. The evolution of initially concentric fluids is shown to be highly dependent on a parameter called the magnetic Froude number, which represents a ratio of the strength of magnetic forces to gravitational forces. A variety of flow configurations is possible, with striking features of various simulated flows
compared with real flows as observed in HII regions and nebulae. The ability for the model of this chapter to predict a variety of such features in unstable flows gives some confidence as to the accuracy of the calculations.

4.2 Weakly Compressible, Viscous Model

This section presents a viscous model for the MRTI of two fluids arranged initially in two concentric discs, with the lighter fluid of density $\rho_1$ and magnetic permeability $\mu_1$ interior to the heavier fluid of density $\rho_2$ and permeability $\mu_2$. The initial interface of the two fluids is the circle located at radius $r = a$ from the origin, and the boundary of the computational domain is taken to be the circle $r = b$, for convenience. An opportune representation of the geometry is the cylindrical polar coordinate system $(r, \theta)$, which is defined from the rudimentary Cartesian $(x, y)$ coordinate system by the conventional relations $x = r \cos \theta$ and $y = r \sin \theta$. We employ the Boussinesq approximation from Farrow and Hocking [23], replacing the two fluids of densities $\rho_1$ and $\rho_2$ with a single fluid of continuous density variation $\rho$. For this approximation to be valid, we require $\rho \approx \rho_1$ in the region $0 < r < a$ and $\rho \approx \rho_2$ in the region $a < r < b$. We also require that the change in $\rho$ in the neighbourhood of $r = a$ is to be both rapid and smooth. The form of the density is thus $\rho = \rho_0 + \mathcal{P}$, where $\rho_0$ is a constant density component, and $\mathcal{P}$ is a small variable component. A gravitational field is directed radially inwards and is described by $-g \hat{e}_r$. Finally, we set $\mu_1 = \mu_2 = 1$ for simplicity, since $\mu \approx 1$ for many applications.

The governing equations of this model are a combination of Navier-Stokes’ and Maxwell’s equations. These equations take a similar form to the corresponding equations developed in the previous chapter. For convenience, we give a self contained development in this chapter due to a small number of differences. Under the Boussinesq approximation, the conventional equation representing the conservation of mass $\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{v}) = 0$ is split into the following two
equations,

\[ \nabla \cdot v = 0, \quad (4.1) \]
\[ \frac{\partial \rho}{\partial t} + v \cdot \nabla \rho = \epsilon_1 \nabla^2 \rho, \quad (4.2) \]

where the small Eckmann diffusion term on the right hand side of equation (4.2) has been added for the purpose of numerical stability. The conservation of momentum with both gravitational and Lorentz forces as well as viscous effects,

\[ \rho \left[ \frac{\partial v}{\partial t} + (v \cdot \nabla) v \right] + \nabla p^* = -\rho g \hat{e}_r + \frac{1}{\mu} (B \cdot \nabla) B + \rho \nu \nabla^2 v, \quad (4.3) \]

simplifies somewhat under the Boussinesq approximation, since terms proportional to \( \bar{\rho} \) are deemed to be negligible, with the exception of the gravity force term \(-\rho g \hat{e}_r\). Under this regime, the conservation of momentum takes the form

\[ \rho_0 \left[ \frac{\partial v}{\partial t} + (v \cdot \nabla) v \right] + \nabla p^* = -\rho_0 g \hat{e}_r + \frac{1}{\mu} (B \cdot \nabla) B + \rho_0 \nu \nabla^2 v, \quad (4.3) \]

where \( p^* = p + \rho_0 gr + B^2 / (2\mu) \).

Next, the absence of magnetic monopoles requires

\[ \nabla \cdot B = 0 \quad (4.4) \]

and, finally, Faraday’s Law takes the form

\[ \frac{\partial B}{\partial t} = -\nabla \times E. \]

The electric field \( E \) in Faraday’s Law can be eliminated by substitution of the Lorentz equation \( J = \sigma (E + v \times B) \) as well as Ampere’s Law \( \nabla \times B = \mu J \) and making use of the vector identity \( \nabla \times (\nabla \times B) = \nabla (\nabla \cdot B) - \nabla^2 B \) to obtain

\[ \frac{\partial B}{\partial t} = \nabla \times (v \times B) + \frac{1}{\mu \sigma} \nabla^2 B. \quad (4.5) \]
We remark that thermal considerations have been ignored due to results by Crutcher [16] in which a study of the properties of 27 interstellar clouds showed that magnetic effects dominate thermal effects in such regions, as well as repeating the comment from the introduction in which Mizuta [44] found that when recombination is included, the growth rates were observed to be similar to the isothermal model of Williams [62].

At time $t = 0$, a fluid dipole of strength $p_D$ located at the origin and directed along the $x$ axis, and a magnetic dipole of strength $q_D$ also located at the origin and directed along the $x$ axis are switched on impulsively. At this instant, the flow is irrotational, and can be described by a fluid potential $\Phi$ and a magnetic potential $X$, given by

$$\Phi = \frac{p_D \cos \theta}{2\pi r}; \quad X = \frac{q_D \cos \theta}{2\pi r}. \quad (4.6)$$

Non-dimensional variables are now introduced; the scales for length and density are given naturally by the quantities $a$ and $\rho_0$ respectively. One may use the fluid dipole strength $p_D$ together with the length scale $a$ to derive scales for both time, $a^3/p_D$, and speed, $p_D/a^2$. The scale for magnetic field strength unsurprisingly requires the quantity $q_D$, and is given by $q_D/a^2$. A reference pressure of $p_D^2 \rho_0 / a^4$ ensues. Several dimensionless parameters control the behaviour of the system. The density ratio $D = \rho_2 / \rho_1$ is required to be greater than 1 for the case of interest in this thesis which is unstable growth, but close to 1 so that $D - 1$ is small in order for the Boussinesq approximation to be valid.

We define the fluid Froude number $F$ and magnetic Froude number $G$ to be

$$F^2 = \frac{p_D^2}{g a^5}; \quad G^2 = \frac{q_D^2}{\rho_0 \mu g a^5}.$$ 

The parameter $F$ is essentially a ratio of inertia to gravitational forces, while $G$ represents the relative strength of magnetic and gravitational forces. The fluid
Reynolds number $R_e$ and the magnetic Reynolds number $R_m$, defined by

\[ R_e = \frac{\rho D}{\nu a}; \quad R_m = \frac{\rho D \mu \sigma}{a}, \]

are assumed to be large in this chapter, that is, inertia forces dominate shear forces and advection of the magnetic field $B$ is far more significant than diffusion of $B$. This is comparable with the models of other works that study HII regions, such as those by Fukuda & Hanawa [26] and Gendelev & Krumholz [27] in which the ideal MHD equations are used, that is $R_e \to \infty$ and $R_m \to \infty$. Finally, the dimensionless Eckmann diffusion coefficient takes the form $E = \epsilon_1 a / p_D$, and is considered to be very small in this chapter for the purpose of numerical stability.

For time $t > 0$, the flow will become rotational in nature, hence the use of the potential functions (4.6) is not appropriate. However, under the Boussinesq approximation, there is still an incompressible component to the velocity $\mathbf{v}$ in view of equation (4.1). In the magnetic case, the absence of monopoles guarantees that $\mathbf{B}$ is incompressible, regardless of any approximation. We use a stream function approach to satisfy these two relations exactly. It is convenient to replace the conventional radial and azimuthal velocity components $(u, v)$ with the modified components $(\tilde{U}, \tilde{v})$, where $\tilde{U} = ru$. In these variables, we write the modified velocity components in terms of the fluid stream function $\Psi$ as

\[ \tilde{U} = \frac{\partial \Psi}{\partial \theta}; \quad \tilde{v} = -\frac{\partial \Psi}{\partial r} \]  

(4.7)

A similar argument results in the modified magnetic field components being written in terms of the magnetic stream function $\mathcal{X}$ as

\[ \tilde{B}_r = \frac{\partial \mathcal{X}}{\partial \theta}; \quad B_\theta = -\frac{\partial \mathcal{X}}{\partial r}, \]  

(4.8)

where $\tilde{B}_r = rB_r$. Conjoint with each stream function is an associated vorticity
function, and we define

$$\xi = -\nabla^2 \Psi ; \quad \Upsilon = -\nabla^2 \chi$$

to be the fluid and magnetic vorticity functions respectively. The scalar functions $\xi$ and $\Upsilon$ are the $\hat{e}_z$ components of the corresponding vectors $\nabla \times v$ and $\nabla \times B$, and represent twice the angular speed and twice the magnetic field magnitude in each case. It is also convenient to define a modified density function by $\tilde{R} = \rho/r$, similar to Forbes [25]. In view of the stream function formulation, the entire system of equations is reduced to the following three transport equations for density, vorticity and magnetic vorticity (current),

$$r \frac{\partial \tilde{R}}{\partial t} + \tilde{U} \left( \frac{\partial \tilde{R}}{\partial r} + \frac{\tilde{R}}{r} \right) + v \frac{\partial \tilde{R}}{\partial \theta} = E \left( \nabla^2 \tilde{R} + 2 \frac{\partial \tilde{R}}{\partial r} + \frac{\tilde{R}}{r} \right) \quad (4.9)$$

$$\frac{\partial \xi}{\partial t} + \tilde{U} \frac{\partial \xi}{\partial r} + \frac{v}{r} \frac{\partial \xi}{\partial \theta} = \frac{G^2}{F^2} \left( \frac{\tilde{B}_r}{r} \frac{\partial \Upsilon}{\partial r} + \frac{\tilde{B}_\theta}{r} \frac{\partial \Upsilon}{\partial \theta} \right) - \frac{1}{F^2} \frac{\partial \tilde{R}}{\partial \theta} + \frac{1}{R_e} \nabla^2 \xi \quad (4.10)$$

$$G^2 \frac{\partial \Upsilon}{\partial t} = G^2 \nabla^2 L + \frac{G^2}{R_m} \nabla^2 \Upsilon \quad (4.11)$$

where $L = (v \times B) \cdot \hat{e}_z$.

We now seek a spectral solution to the non-linear system of equations (4.9)-(4.11) that satisfies given boundary and initial conditions exactly. This chapter imposes the condition that as $r \to 0$, the flow converges to the irrotational solution (4.6), that is,

$$\tilde{U} \to \frac{\cos \theta}{2\pi r} ; \quad v \to -\frac{\sin \theta}{2\pi r} ; \quad \Psi \to -\frac{\sin \theta}{2\pi r} ; \quad \xi \to 0$$

$$\tilde{B}_r \to \frac{\cos \theta}{2\pi r} ; \quad \tilde{B}_\theta \to -\frac{\sin \theta}{2\pi r} ; \quad \chi \to -\frac{\sin \theta}{\pi \nabla} ; \quad \Upsilon \to \frac{\pi}{t}$$

$$\tilde{R} \to 0 ; \quad \frac{\partial \tilde{R}}{\partial \theta} \to 0 \quad (4.12)$$

These conditions (4.12) represent the exact singular behaviour near the line dipoles. Along the outer dimensionless boundary $r = \beta = b/a$, convenient
conditions are given by
\[ \tilde{U} = -\frac{\cos \theta}{2\pi \beta}, \quad v = -\frac{\sin \theta}{2\pi \beta^2}, \quad \Psi = -\frac{\sin \theta}{2\pi \beta}, \quad \xi = 0 \]
\[ \tilde{B}_r = -\frac{\cos \theta}{2\pi \beta}, \quad B_\theta = -\frac{\sin \theta}{2\pi \beta^2}, \quad X = -\frac{\sin \theta}{e \pi \beta}; \quad \mp = \tau \]
\[ \tilde{R} = \frac{D - 1}{\beta}. \quad (4.13) \]

The initial condition for the density perturbation needs to correspond to an effective interface at \( r = 1 \). For this reason, we impose

\[ \tilde{R} (r, \theta, 0) = \begin{cases} 0, & 0 < r < 1 \\ (D - 1)/r, & 1 < r < \beta \end{cases} \quad (4.14) \]

for the density function \( \tilde{R} (r, \theta, t) \). In addition to the fluid and magnetic dipoles that are switched on impulsively at time \( t = 0 \), we include a small impulsive sinusoidal disturbance of magnitude \( \epsilon \) to the K-th Fourier mode of the stream functions, which results in the initial conditions

\[ \Psi (r, \theta, 0) = X (\nabla, \theta, t) = -\frac{\sin \theta}{2\pi r} + \left\{ \begin{array}{ll} \epsilon r^K \sin (K \theta), & 0 < r < 1 \\ \epsilon r^{-K} \sin (K \theta), & 1 < r < \beta. \end{array} \right. \quad (4.15) \]

We are now ready for the full time dependent spectral representation of the quantities \( \Psi, X \) and \( \tilde{R} \). The stream functions take a similar form,

\[ \Psi (r, \theta, t) = \frac{\sin \theta}{2\pi r} + \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} (t) J_{2m} \left( \frac{j_{2m,n}}{\beta r} \right) \sin (2m \theta) \quad (4.16) \]
\[ X (\nabla, \theta, \mp) = \frac{\sin \theta}{2\pi r} + \sum_{m=1}^{M} \sum_{n=1}^{N} D_{mn} (t) J_{2m} \left( \frac{j_{2m,n}}{\beta r} \right) \sin (2m \theta) \quad (4.17) \]

where the notation \( J_\nu (z) \) denotes the \( \nu \)-th order Bessel function of the first kind, and \( j_{\nu,n} \) denotes the n-th zero of \( J_\nu (z) \). One may obtain the spectral forms of both the velocity and magnetic field strength components by performing the appropriate differentiation given in equations (4.7) and (4.8) respectively. Only
even order trigonometric terms are included in the double series term, due to the interest in bipolar solutions in this section.

The spectral form of the density function is chosen with the auxiliary constraint that certain integrals with respect to variable \( r \) in equations (4.19)-(4.22) are required to be non singular in the limit that \( r \to 0 \), in the spectral method described below. We therefore represent the density by

\[
\tilde{R}(r, \theta, t) = \left( \frac{D - 1}{\beta^3} \right) r^2 \beta^3 \sin \left( \frac{n \pi r^2}{2 \beta^2} \right) + \sum_{m=1}^{M} \sum_{n=1}^{N} B_{mn}(t) J_{2m} \left( \frac{j_{2m,n} r}{\beta} \right) \cos (2m\theta).
\] (4.18)

It is a trivial matter to see that the conditions (4.12) and (4.13) are satisfied exactly by the given representations (4.16) - (4.18). The ability to deal exactly with the singular behaviour of the line dipoles is a strength of the present spectral approach. The use of the sine function in the single summation term in the density representation is particularly convenient, allowing a greater portion of the upcoming decomposition to be performed analytically rather than numerically, compared with various other choices for this representation. To decompose the vorticity equation (4.10), we multiply it by the basis functions \( r J_{2k} (j_{2k,l} r / \beta) \sin (2k\theta) \) and integrate over the cylindrical region \( 0 < r < \beta, -\pi < \theta < \pi \). This is then simplified considerably, primarily by application of the rudimentary trigonometric orthogonality relations [1, p.77,78], as well as the following orthogonality relation of the Bessel functions,

\[
\int_{0}^{\beta} r J_{\nu} (j_{\nu,n} r / \beta) J_{\nu} (j_{\nu,l} r / \beta) = \begin{cases} 
0, & n \neq l \\
\beta^2 J_{\nu+1}^2 (j_{\nu,l}) / 2, & n = l
\end{cases}
\]

which has been scaled appropriately for this problem from the standard form found in [1, p.485]. After a little algebra, the following system of ordinary
differential equations (ODEs) for the coefficients $A_{kl}$ ensues:

$$\frac{dA_{kl}}{dt} = -\frac{2}{\pi j_{2k,l} J_{2k+1}(j_{2k,l})^2} \int_{-\pi}^{\pi} \int_{0}^{\beta} \left[ \frac{\partial \xi}{\partial r} + v \frac{\partial \xi}{\partial \theta} - \frac{C^2}{F^2} \left( \tilde{B}_r \frac{\partial \xi}{\partial r} + B_{\theta} \frac{\partial \xi}{\partial \theta} \right) \right] \times$$

$$J_{2k} \left( \frac{j_{2k,l}}{\beta} r \right) \sin (2k\theta) dr d\theta - 2k \left( \frac{\beta}{F j_{2k,l}} \right)^2 B_{kl}(t) - \frac{1}{Re} \left( \frac{j_{2k,l}}{\beta} \right)^2 A_{kl}(t).$$

(4.19)

The current equation (4.11) is decomposed in an identical fashion as for the vorticity equation, resulting in the following system of ODEs for the coefficients $D_{kl}$:

$$\frac{dD_{kl}}{dt} = -\frac{2}{\pi j_{2k,l} J_{2k+1}(j_{2k,l})^2} \int_{-\pi}^{\pi} \int_{0}^{\beta} G(r, \theta, t) r J_{2k} \left( \frac{j_{2k,l}}{\beta} r \right) \sin (2k\theta) dr d\theta$$

$$- \frac{1}{Rm} \left( \frac{j_{2k,l}}{\beta} \right)^2 A_{kl}(t),$$

(4.20)

where the intermediate function

$$G(r, \theta, t) = U \nabla^2 B_{\theta} + 2 \left( \frac{\partial U}{\partial r} - \frac{U}{r} \right) \frac{\partial B_{\theta}}{\partial r} + \frac{1}{r^2} \frac{\partial U}{\partial \theta} \frac{\partial B_{\theta}}{\partial \theta}$$

$$+ B_{\theta} \left( \nabla^2 U - \frac{2 \partial U}{r} \frac{\partial r}{\partial r} + \frac{2U}{r^2} \right) - v \left( \nabla^2 \tilde{B}_r - \frac{2 \partial \tilde{B}_r}{r} \frac{\partial r}{\partial r} + \frac{2\tilde{B}_r}{r^2} \right)$$

$$- 2 \left( \frac{\partial \tilde{B}_r}{\partial r} - \frac{\tilde{B}_r}{r} \right) \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial v}{\partial \theta} \frac{\partial \tilde{B}_r}{\partial \theta} \right) - \tilde{B}_r \nabla^2 v$$

has been defined for convenience. We now consider the density equation (4.9) in order to obtain the ODEs for the coefficients $B_{kl}$ and $C_1$. Firstly, the $B_{kl}$ are obtained by multiplying equation (4.9) by basis functions $J_{2k}(j_{2k,l}r/\beta) \cos (2k\theta)$ and integrating over the cylindrical region $0 < r < \beta, -\pi < \theta < \pi$. Again, orthogonality conditions are used to simplify the expression, resulting in a system
of ODEs for the $B_{kl}$ in the form

$$\frac{dB_{kl}}{dt} = -\frac{2}{\pi [\beta J_{2k+1}(j_{2k,l})]} \int_{-\pi}^{\pi} \int_{0}^{\beta} \left[ \widehat{U} \frac{\partial \widehat{R}}{\partial r} + v \frac{\partial \widehat{R}}{\partial \theta} + \frac{\widehat{U}}{r} \right] \times$$

$$J_{2k} \left( \frac{j_{2k,l}}{\beta} \right) \cos (2k\theta) d\theta dr - E \left( \frac{j_{2k,l}}{\beta} \right)^2 B_{kl}(t) + \frac{2E}{[\beta J_{2k+1}(j_{2k,l})]^2} \times$$

$$\sum_{n=1}^{N} B_{kn}(t) \int_{0}^{\beta} \left[ 2 \left( \frac{j_{2k,n}}{\beta} \right) J_{2k} \left( \frac{j_{2k,n}}{\beta} r \right) + \frac{1}{r} J_{2k} \left( \frac{j_{2k,n}}{\beta} r \right) \right] J_{2k} \left( \frac{j_{2k,l}}{\beta} r \right) dr.$$ (4.21)

To obtain the final system of ODEs for the $C_n$, we return to the density equation (4.9), but this time multiply by basis functions $\sin \left( \frac{n\pi r}{2\beta} \right)$. The orthogonality relation,

$$\int_{0}^{\beta} r \sin \left( \frac{n\pi r}{2\beta} \right) \sin \left( \frac{l\pi r}{2\beta} \right) dr = \begin{cases} 0, & n \neq l \\ \beta^2/4, & n = l \end{cases}$$

is used to simplify the expression, resulting in

$$\frac{dC_l}{dt} = -\frac{2}{\pi^2} \int_{-\pi}^{\pi} \int_{0}^{\beta} \left[ \widehat{U} \frac{\partial \widehat{R}}{\partial r} + \frac{\widehat{R}}{r} \right] \sin \left( \frac{l\pi r}{2\beta} \right) dr d\theta$$

$$+ E \left[ \frac{36 (D-1)}{\pi \beta^2} \right] \sin^2 \left( \frac{l\pi}{2} \right) + \mathcal{F}(r),$$ (4.22)

where

$$\mathcal{F}(r) = \frac{32\pi}{\beta^2} \sum_{n=1}^{N} n C_n(t) \int_{0}^{\beta} r \cos \left( \frac{n\pi r}{2\beta} \right) \sin \left( \frac{l\pi r}{2\beta} \right) dr -$$

$$\frac{16\pi^2}{\beta^6} \sum_{n=1}^{N} n^2 C_n(t) \int_{0}^{\beta} r^3 \sin \left( \frac{n\pi r}{2\beta} \right) \sin \left( \frac{l\pi r}{2\beta} \right) dr +$$

$$\frac{4\pi^6}{\beta^2} \sum_{n=1}^{N} C_n(t) \int_{0}^{\beta} \frac{1}{r} \sin \left( \frac{n\pi r}{2\beta} \right) \sin \left( \frac{l\pi r}{2\beta} \right) dr.$$ (4.23)

The integrals appearing in equation (4.23) can all be evaluated in closed form, making use of the cosine integral function [1, p.231],

$$Ci(z) = \gamma + \log z + \int_{0}^{z} \frac{\cos t - 1}{t} dt,$$
in which \( \gamma \approx 0.577 \) is the Euler constant [1, p.255]. The equations (4.19) to (4.22) form a system of ODEs for the coefficients \( A_{kl} \) to \( D_{kl} \). We invoke the classic fourth order Runge-Kutta method [5, p.371] to integrate the system forwards in time from an initial state at time \( t = 0 \). There is a small number of integrals that cannot be evaluated in closed form; these are calculated approximately by use of Gauss-Legendre quadrature in both the \( r \) and \( \theta \) coordinates.

The abscissae and weights for the Gauss-Legendre scheme are calculated by a MATLAB program written by von Winckel [61]. The initial conditions for the coefficients \( A_{kl} \) to \( D_{kl} \) are calculated by decomposition of equations (4.14) and (4.15). After simplification, including the use of various identities for the Bessel functions [1, p.361], one obtains in the case \( K = 2 \) (bipolar solutions),

\[
A_{1l}(0) = \frac{2e}{J_3^2(j_{2,l})} \left[ \frac{4}{(j_{2,l}/\beta)^2} J_{2} \left( \frac{j_{2,l}}{\beta} \right) - \frac{J_3 \left( j_{2,l} \right)}{\beta^2 j_{2,l}} \right] \\
C_{1l}(0) = \frac{2(D - 1)}{\beta} \left[ \sqrt{2} \left\{ S \left( \sqrt{2l} \right) - S \left( \sqrt{2l}/\beta \right) \right\} + \cos \left( \frac{l\pi}{\beta} \right) \right] \\
D_{1l}(0) = \frac{2e}{J_3^2(j_{2,l})} \left[ \frac{4}{(j_{2,l}/\beta)^2} J_{2} \left( \frac{j_{2,l}}{\beta} \right) - \frac{J_3 \left( j_{2,l} \right)}{\beta^2 j_{2,l}} \right] \tag{4.24}
\]

where \( S(z) \) is the Fresnel sine integral \( \int_0^z \sin \left( \frac{\pi t^2}{2} \right) dt \) [1, p.300]. The initial value of those coefficients not listed above is zero.
4.3 Results

In this section, we present results of the spectral-numerical method of the previous section for a range of Magnetic Froude number values. This will enable the effect of a magnetic field on the growth of the MRTI to be observed. In order for the Boussinesq approximation to be valid, we require $D \approx 1$, and in view of this fact, we set the density ratio to $D = 1.05$. The initial disturbance mode is set to $K = 2$ since bipolar solutions are of primary interest in this chapter. A total of 31 Bessel modes, and thus 15 non-zero Fourier modes is found to be an appropriate balance between a representation that sufficiently spans the frequency domain, yet is numerically stable. For the latter consideration, it is known that the higher order modes are the most numerically unstable. The numerical value of both the fluid and magnetic Reynolds numbers is chosen to be $10^4$; this results in inertia forces that are large in comparison with shear forces, and means that diffusion of the magnetic field is weak compared with advection. The reason that we consider such a low shear regime is in view of results by Zhang et al. [64] where it was shown that the RTI becomes more unstable in the high shear flow regime. With regards to the magnetic advection and diffusion, Chambers and Forbes [11] demonstrated that the RTI was largely unaffected by a magnetic field in the strong advection regime due to the weakness of induced currents, as discussed in chapter 3. This choice is comparable to other models in the literature that use the ideal MHD equations [26, 44, 27]. We also require the diffusion of the density function to be weak in some sense so that the single continuous, smooth fluid density function does indeed approximate the constant densities of the two fluid case. For this reason, a small diffusion coefficient of $\epsilon_1 = 10^{-4}$ is used. The outer numerical boundary is located at $\beta = 2$. This value is chosen in order for the observed features of the flow to be easily visible from the density plots, whilst being located far enough from the origin that it does not significantly affect the flow. The initial amplitude of the disturbance is set to $\epsilon = 0.03$. Finally, the fluid Froude number $F$ takes
the value 2, since this is appropriate for a typical bipolar flow in astrophysics, as opposed to say a laboratory application whereby a light fluid is injected into a heavier one which requires $F \to \infty$. The numerical grid has a non-uniform spacing in both the $r$ and $\theta$ coordinates due to the use of Gaussian quadrature. A total of 161 points in the $r$-coordinate over the range $0 < r < 2$ and 81 points in the $\theta$-coordinate over the range $0 < \theta < \pi$ were used. Symmetry arguments were used to calculate the value of quantities in the range $-\pi < \theta < 0$. A time step of $1/200$ was found to give stable results in all cases. The accuracy of the numerical component of the method is deemed to be reliable by use of similar techniques as described in the results section of the previous chapter.

4.3.1 No magnetic field

We begin by considering the case in which there is no magnetic field. This is equivalent to setting the magnetic Froude number to $G = 0$ and all of the $D_{kl}$ coefficients to 0. Figure 4.1 is a density plot that shows the configuration of the two fluids at the four times $t = 6, 9, 12$ and 15. The lighter of the two fluids is shown in purple (dark) with blue (soft) representing the heavier fluid. By time $t = 6$, the initially concentric configuration from time $t = 0$ has evolved into a dumbbell type shape for the lighter fluid. Motion of the lighter fluid in the $x$-direction is favoured over motion in the $y$-direction since the fluid dipole at the origin is oriented along the $x$-axis. This becomes even more apparent by the time $t = 15$. At this time, we also observe the classic mushroom roll-up of the interface for the two plumes. Much after this time, the single continuous, smooth density function fails to be able to distinguish between the heavy and light fluids near the interface due to diffusion and numerical error. These results are comparable with those of Forbes [25] in which a line source replaces the line dipole of this thesis. This comparison gives confidence in the method of this chapter, since a linearised solution is unavailable for comparison at early values of time. The main difference between the two configurations is that the lighter fluid penetrates further into the heavier fluid in the case of the line source, since
the effect of the line source is stronger than that of the line dipole for radius 
$r > 1$.

![Density plot](image1)

Figure 4.1: Density plot at the four times (a) $t = 6$, (b) $t = 9$, (c) $t = 12$ and 
(d) $t = 15$ in the absence of a magnetic field, showing the lighter fluid in purple 
(dark), and the heavier fluid in blue (soft). The magnetic Froude number is 
$G = 0$.

The results are also comparable with real physical observations such as those 
of HII regions and nebulae in the field of astrophysics. Figure 4.2 shows a rotated 
image of MyCn18, known more commonly as the hourglass nebula, taken by the 
Hubble Space Telescope in 1996 [52]. This nebula lies about 8000 light years 
from Earth in the southern constellation Musca. The three different colours in 
the image represent three different gases; nitrogen in red, hydrogen in green and 
oxygen in blue. Contemporary thinking is that the hourglass shape is produced
by the expansion of a fast stellar wind within a slowly expanding cloud which is denser near its equator than its poles [52]. The Rayleigh-Taylor instability is thus applicable to this situation. The fact that this configuration is reproduced with a magnetic Froude number of $G = 0$ gives rise to the possibility that the effects of gravitational forces completely dominate the effects of magnetic forces for this nebula. Verification of this is not directly possible due to very little information being available about this nebula. The hourglass shape of the nebula closely resembles the simulated density profile from figure 4.1(c).

![Hourglass Nebula](image)

Figure 4.2: The "hourglass nebula", MyCn18, taken by the Hubble Space Telescope [52]. Credits: Raghvendra Sahai and John Trauger (JPL), the WFPC2 science team, and NASA.

When considering the vorticity of the fluids in the case of Cartesian geometry with a single mode disturbance, it is observed [24] that the vorticity is zero at almost all points in the fluid. There are just two small regions where the vorticity is non-zero; these positions are centred about the points where the curvature of the interface becomes infinite after a finite time in the non-viscous case. The case is very different in cylindrical geometry as there is non-zero vorticity at almost all points of the fluid. Notwithstanding this, it is observed that the magnitude of the vorticity is largest around the interface, but the more complicated distribution of vorticity compared with the case in Cartesian geometry makes vorticity a somewhat less useful quantity with which to explain the behaviour of the flow with time in the case of cylindrical geometry. We show how the vorticity evolves in figures 4.3. The same points in time are sampled as...
for the case of the density above, so that it can be seen clearly that the vorticity is largest near the interface.

Figure 4.3: Vorticity plot at the four times (a) $t = 6$, (b) $t = 9$, (c) $t = 12$ and (d) $t = 15$, in the absence of a magnetic field, showing clockwise rotation in dark, and counter clockwise rotation in soft. The magnetic Froude number is $G = 0$.

4.3.2 Magnetic Field

The base magnetic field is due to the presence of a magnetic dipole located at the origin, directed along the $x$-axis. Of course, there is also an induced magnetic field due to the motion of the fluid across the magnetic field lines. The effect of the total magnetic field can be progressively made stronger by choosing larger values of the magnetic Froude number $G$. We begin with the case $G = 1$, in
which the magnetic effects are considered weaker than the fluid effects since the fluid Froude number is \( F = 2 \) and thus we have \( G < F \). Figure 4.4 shows the configuration of the fluids at the same times as in the non-magnetic case so that we can see the effect that the presence of this field together with \( G = 1 \) has on the flow. At this value of \( G \), the flow is qualitatively similar to the non-magnetic case, but suppressed in time. This is in agreement with Lenz’ law, whereby the induced current flows in such a direction to oppose the change that caused it. The induced current flows when the fluid crosses the magnetic field lines; hence this motion and thus the growth of the instability is suppressed. We do not show contours of vorticity nor current density due to the reasons given in the previous case and in the interests of space.

The next consideration is the case where the fluid and magnetic effects may be considered to have equal weight due to the dimensionless Froude numbers taking the values \( F = G = 2 \). The flow configuration is shown in figures 4.5, again at identical times to the previous cases. In this regime, it is apparent that the presence of the magnetic field is not merely just suppressing the flow; the lighter fluid is penetrating deeper into certain regions of the heavier fluid compared with the non-magnetic case. This feature was not observed in Cartesian geometry, where the magnetic field acted only to suppress the flow. The configuration still features the classic mushroom shape of the previous cases, but the mushroom cap now appears thinner and more elongated.

This change is made even clearer by doubling the magnetic Froude number to \( G = 4 \). Figure 4.6 shows this case. The previous roundness of the mushroom cap now looks distorted, with the tips of the mushroom “snaking” into the lighter fluid. Even the stem of each mushroom is behaving differently now. Rather than a more-or-less uniform width stem, each stem appears more bulbous in regions near the origin, due to growth near the mushroom cap being suppressed.

We remark in particular on the likeness of the features from figure 4.6(c) and figure 4.7, which shows an image of NGC 6302 [45], otherwise known as the butterfly nebula. This picture was taken by the Hubble Space Telescope in
Figure 4.4: Density plot at the four times (a) $t = 6$, (b) $t = 9$, (c) $t = 12$ and (d) $t = 15$ in the absence of a magnetic field, showing the lighter fluid in purple (dark), and the heavier fluid in blue (soft). The magnetic Froude number is $G = 1$.

2009 in ultra-violet and visible light. This nebula is located in the constellation Scorpius, about 3800 light years from Earth within the Milky Way galaxy. The primary features of the nebula are the butterfly wings and the numerous finger like projections. Both the butterfly wings and finger projections are also present in the results of the numerical simulation shown in figure 4.6(c), which under rotation is of a similar form to the image of the butterfly nebula 4.7.

Finally, we re-double the magnetic Froude number to $G = 8$. In this regime, magnetic effects are very dominant over the fluid effects on the flow. The configurations given for this case in figure 4.8 are qualitatively very different to the
Figure 4.5: Density plot at the four times (a) \( t = 6 \), (b) \( t = 9 \), (c) \( t = 12 \) and (d) \( t = 15 \) in the absence of a magnetic field, showing the lighter fluid in purple (dark), and the heavier fluid in blue (soft). The magnetic Froude number is \( G = 2 \).

Configurations that resulted from much lower magnetic Froude numbers. This is contrast to the case of Cartesian geometry, where it was observed \([11]\) that the presence of a magnetic field merely suppressed the flow in time; it did not alter the regions that the lighter fluid could penetrate.

The “snaking” or “tentacular” nature of the flow from figure 4.8 is reminiscent of behaviour observed in the tarantula nebula, an image of which is available from either of the public websites \([21, 22]\). Unfortunately, due to copyright restrictions, this image can not be printed in this paper. The curled over light blue tentacle featured in the top right of the nebula image is especially
Figure 4.6: Density plot at the four times (a) \( t = 6 \), (b) \( t = 9 \), (c) \( t = 12 \) and (d) \( t = 15 \) in the absence of a magnetic field, showing the lighter fluid in purple (dark), and the heavier fluid in blue (soft). The magnetic Froude number is \( G = 4 \).

striking in similarity to the same region in the simulated image. This nebula is also known as the 30 Doradus nebula in the NGC 2070 star cluster. This cluster is one of the largest known star formation regions [22]. It is very bright, in fact bright enough to be observed optically despite it being located outside our Milky Way galaxy in the large Magellanic Cloud [3].

The effect of the polarity of the magnetic dipole was also considered, but was found to have no effect on the flow. In the interests of space, the results are thus not presented here. This result was as expected, since an earlier analysis of similar nature performed in Cartesian geometry [11] produced the same result.
This can be explained with the already mentioned Lenz’ law argument as in chapter 3. The induced current always flows in the direction so as to oppose the change that caused it, hence the growth of the MRTI is suppressed by the same amount regardless of the polarity of the magnetic dipole located at the origin.

4.3.3 Higher mode disturbances

In this section, we present some results in which tripolar plumes evolve. The initial disturbance to generate such a configuration requires $K = 3$ in equation (4.15). This alteration to the quantity $K$ in turn affects the spectral representations for both the streamfunctions and the density given previously by equations (4.16) and (4.18). In the present case of tripolar plumes, only the coefficients of trigonometric terms of an order that is a multiple of three are non-zero, and hence we may write the spectral forms of the streamfunctions as

$$
\Psi (r, \theta, t) = \frac{\sin \theta}{2\pi r} + \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} (t) J_{3m} \left( \frac{j_{3m,n}}{b} r \right) \sin (3m\theta)
$$

$$
\mathcal{X} (\nabla, \theta, \underline{1}) = \frac{\sin \theta}{2\pi r} + \sum_{m=1}^{M} \sum_{n=1}^{N} D_{mn} (t) J_{3m} \left( \frac{j_{3m,n}}{b} r \right) \sin (3m\theta) \quad (4.25)
$$
Figure 4.8: Density plot at the four times (a) $t = 6$, (b) $t = 9$, (c) $t = 12$ and (d) $t = 15$ in the absence of a magnetic field, showing the lighter fluid in purple (dark), and the heavier fluid in blue (soft). The magnetic Froude number is $G = 8$.

and the spectral form of the density function as

$$
\tilde{R}(r, \theta, t) = \frac{(D - 1) r^2}{b^3} + \sum_{n=1}^{N} C_n(t) \sin \left( \frac{n\pi r^2}{b^2} \right) + \sum_{m=1}^{M} \sum_{n=1}^{N} B_{mn}(t) J_{3m} \left( \frac{j_{3m,n} r}{b} \right) \cos (3m\theta). \quad (4.26)
$$

Whilst the main applications in the field of astrophysics require bipolar solutions, the study of the general case is valuable for a deeper understanding of the instability. A study by Matsuoka and Nishihara [40] on the topic of the associated RMI presents results depicting single, bipolar, tripolar and octopolar
plumes. The growth rates of bubbles (where light fluid penetrates a heavy fluid) and spikes (where heavy fluid penetrates a light fluid) are quantities of interest in the study of the RMI. A later paper by the same authors [39] argues that the results in which the disturbing mode is greater than or equal to three are fundamentally different to the case in the disturbing mode is less than three. Under the parameters used by the authors in that paper, the curvature of the spikes is positive for modes less than three, but negative for modes greater than three.

We now present a series of density plots showing the configuration of the two fluids at different points in time for a range of magnetic Froude numbers. The first case considers a magnetic Froude number of $G = 0$, which corresponds with the case of the non-magnetic RTI, in figure 4.9. The format of the figures from the previous section is retained, that is, the lighter of the two fluids is shown in purple (dark) with blue (soft) representing the heavier fluid. In order to facilitate a meaningful comparison, the same instants in time are sampled as for previous cases.

A comparison of figures 4.9 with figures 4.1 from section 4.3.1 shows that there is a greater degree of roll up in the case of the tripolar plume than for the bipolar plume under the same parameters. Matsuoka and Nishihara [40] argue that, due to the existence of two independent spatial scales when using cylindrical geometry, radius and wavelength, the growth rate is highly dependent on mode number. This same argument is used by the author of this thesis to justify the higher degree of roll up just observed in view of the fact that both radius and wavelength are used as spatial scales in the model of this chapter.

As was the case of the previous section, we alter the magnetic Froude number, and examine the effect of a magnetic field on the instability. The first case to include magnetism for the mode three disturbance has a magnetic Froude number of $G = 1$, and results for this case are shown in figure 4.10. In the case of the mode two disturbance, the classic mushroom cap shapes of the non-magnetic case gradually became more distorted as the magnetic Froude number
increased. The tips of the caps were seen as snakes of heavy fluid penetrating the light fluid. The same scenario is observed in the case of mode three disturbances, albeit at seemingly faster growth rates due to the fact that the time scale is dependent on the wave number. These snake like structures are clearly visible in the case of \( G = 2 \), in figure 4.11, with longer snakes appearing in the cases of magnetic Froude number \( G = 4 \) and \( G = 8 \) shown in figures 4.12 and 4.13.

As a further extension, we present a comparison of a configuration that evolves into five plumes with a famous nebula called the Rosette nebula. This
Figure 4.10: Density plot at the four times (a) $t = 6$, (b) $t = 9$, (c) $t = 12$ and (d) $t = 15$ in the absence of a magnetic field, showing the lighter fluid in purple (dark), and the heavier fluid in blue (soft). The magnetic Froude number is $G = 1$ and the disturbance mode is $K = 3$.

result requires an initial disturbance wave number of $K = 5$ in equation (4.15).

The Rosette nebula, or NGC 2237 as it is more formally known, is a well studied HII region generated by a compact group of five stars. It is located at a distance of about 5000 light years from the Earth, in the Monoceros region inside the Milky Way band [54]. Figure 4.14(a) shows an image of the Rosette nebula [46] and figure 4.14(b) is the result of a numerical simulation using the method of this chapter. The main feature of the Rosette nebula is the five petal structure, which is able to be reproduced using the techniques of this chapter.
Figure 4.11: Density plot at the four times (a) $t = 6$, (b) $t = 9$, (c) $t = 12$ and (d) $t = 15$ in the absence of a magnetic field, showing the lighter fluid in purple (dark), and the heavier fluid in blue (soft). The magnetic Froude number is $G = 2$ and the disturbance mode is $K = 3$.

### 4.4 Conclusion

A spectral-numerical analysis of the MRTI has been performed to study the effect of a magnetic field on the RTI in cylindrical coordinates for viscous, weakly compressible flow. The mechanism for instability in cylindrical geometry is quite different from that in Cartesian geometry, due to the Bell-Plesset effect [49]. A key feature of the method from the current chapter was the use of a magnetic stream function which enabled the divergence-free nature of the magnetic field to be satisfied identically. This rules out the possibility that the results of this chapter are due to either fictitious magnetic monopoles or possible errors con-
sequent of other assumptions, for example those used in constrained transport methods typical of the numerical schemes mentioned in the introduction. The presence of numerically generated magnetic monopoles was shown by Brackbill and Barnes [9] to produce a fictitious force parallel to the magnetic field, while the constrained transport methods require an artificial alteration of the magnetic field in order for it to remain divergence free at each time step.

By varying the magnetic Froude number it is possible to observe the effect that a base magnetic field due to a line dipole located at the origin and directed along the $x$-axis has on the flow. A qualitative change in the nature of the
configuration of the two fluids was observed as the magnetic Reynolds number varied. The main observation is that, in contrast to Cartesian geometry where the magnetic field had a universal stabilising effect on the flow, in cylindrical geometry, the magnetic field stabilised some parts of the flow, but de-stabilised others. This non-universal behaviour has been observed many times in the literature [33], but is normally associated with certain modes being stable whilst others are unstable. It is conjectured that it is the injection of fluid that is responsible for the behaviour observed in the present chapter.

This chapter considered the MRTI in cylindrical coordinates, which whilst
being a well known problem, has not been studied in the case where there are dipole type singularities at the origin. The dipole type singularities were chosen over, say, a source type singularity due to the impossibility of such a singularity in the magnetic case in view of the divergence-free nature of any magnetic field. The disturbances considered focussed mainly on those that were expected to produce bi-polar plumes, such as those observed by astrophysicists in jet structures, pictures of which may be found in [54]. The flows observed in this chapter did not form the typical thin jet structures even in the case of large magnetic fields. An obvious extension of the bipolar results is to consider higher modes, with different numbers of outflow plumes. This has been illustrated in this chapter for the tripolar case $K = 3$. In addition, more complex multi-mode initial disturbances might also be investigated. Other geometries are also possible, such as cylindrical coordinates with radial or azimuthal symmetry, or various 2D spherical coordinate systems, or indeed some more general 3D geometry. In the latter case, the stream function technique would not be so straightforward,
and it is expected that some sort of constrained transport method would be required to keep the magnetic field divergence free.

The results from this chapter are able to be partially verified by real observations in the field of astrophysics, in particular HII regions and nebulae. Many examples were found that exhibited the same striking features as obtained by the simulated results using the methodology developed in this chapter. In particular, the hourglass, butterfly, tarantula and Rosette nebulae are examples that exhibit a wide range of behaviours over a wide range of parameters, such as the magnetic Froude number.
Chapter 5

Conclusion

This thesis considered three different scenarios of the magnetic Rayleigh-Taylor instability. The consideration of accuracy, efficiency and stability of the solution ultimately dictated that different solution techniques were required to be implemented for each scenario. This is particularly evident in the difference between the model of chapter two with that of chapters three and four. The linear model of chapter two allows a very life like representation of the system as two distinct fluids with an infinitely thin interface located in between. This simpler model is still accurate provided that the assumptions under which it is valid are understood. The models of chapters three and four are valid under more general conditions, and as such are more complicated, for example by the inclusion of weak compressibility and viscosity. The efficiency and stability of the solution in these cases dictates that such a life like representation of the fluids be replaced with an approximation of just a single fluid in which the density varies very rapidly yet smoothly in the region simulating an interface. The models themselves were chosen with a view to being applicable to situations such as HII regions and nebulae from the field of astrophysics. As such, a full treatment modelling every quantity is unnecessary; for example, arguments are presented justifying the fact that temperature considerations are ignored.

The linear model presented in the introductory second chapter considers the
MRTI in Cartesian geometry under the assumption that the amplitude of the disturbing waves is small compared to the wavelength. The advantage of the simpler linear model is that exact analytic solutions are obtained. The model is valid at early time periods, but due to the exponential growth rate that is calculated for the amplitude of such waves at the interface, there will inevitably become such a time in which the assumptions of the model are violated and the results thereafter are meaningless. The growth rate of the interface is obtained for the cases in which the base magnetic field is horizontal, vertical and indeed at any angle by use of superposition. These rates agree with earlier work by Chandrasekhar [13] despite the different approach taken by that author. The rates also agree in the limit of no magnetic field with the well known Atwood number from the pioneering works of Rayleigh [50] and Taylor [59]. The purpose of calculating the growth rates analytically in the linear regime is so that the growth rates for non-linear models obtained by numerical methods in chapter three can be verified to be accurate at least for early time periods where the approximations used in the linear model are valid.

Chapter three builds on the work from chapter two by adding additional non-linear effects to the model, such as convective acceleration, weak compressibility, viscosity and finite conductivity. The more complicated non-linear model requires some type of numerical method since analytic solutions are not available. Common finite difference numerical methods not only require a great amount of computational resources, but also suffer from instability. In addition, such methods can not solve a key equation of the model exactly, the equation specifying that the magnetic field must have zero divergence. We can improve the accuracy, efficiency and stability of the solution by performing a combination of analytic and numerical techniques with a so called pseudo spectral method. The numerical component of this technique uses a tri-diagonal matrix method which is very fast compared with a full finite difference scheme. This has the added bonus that the analytic component of the technique that uses vorticity and streamfunction methods is crucial to be being able to understand and ex-
plain the behaviour of the flow. Various mushroom type shapes are obtained for the interface between the two fluids. Of particular interest is the mechanism for which the magnetic field suppresses the growth of such shapes, for both single mode and multi mode disturbances, and for base magnetic fields directed at various angles. It is also shown that the presence of a magnetic field can prevent bubbles of fluid breaking apart as is observed in the non-magnetic case.

The final scenario is considered in chapter four and adds the complication of circularly symmetric cylindrical geometry. The convergent nature of such geometry is dealt with by the use of dipole type singularities at the origin for both fluid flow and magnetic field. This is typically expected to apply to the cases of HII regions, nebulae and astrophysical jets. In keeping with the considerations of accuracy, efficiency and stability, the numerical method chosen for this chapter differs slightly from that used in the previous chapter. Double series techniques are used in the present case, not only to, satisfy the dipole nature of the flows at the origin exactly, but also to maintain the divergence-free nature of the magnetic field exactly while allowing the very efficient Runge-Kutta RK4 method to be used to integrate the system of PDEs forwards in time. Of interest is how the flow evolves into an integer number of plumes dependent on the integer wave number of the disturbing wave. The shapes obtained bear striking similarity to those observed in real HII regions and nebulae, which are topics of frontier research in astrophysics.

The work of this thesis has scope for various extensions. Additional effects such as surface tension would be compatible with the methods of either chapters three or four. It would also be straightforward to apply the methods to cylindrical geometry with radial or azimuthal symmetry or two dimensional spherical geometry, each of which would have applications in astrophysics. Less straightforward but still possible would be to consider three dimensional versions of the instability. The greatest difficulty for the three dimensional version would be the maintenance of the divergence free nature of the magnetic field. It is expected that some type of constrained transport algorithm would be required in
such a case. It is not known whether the inclusion of relativistic effects would be compatible with the methods used in this thesis, and would require further research. Even within the current model of chapter four, there is scope for further work by considering different types of disturbances, in particular multi-mode disturbances. In chapter three it was shown that there is a vast difference in behaviour between the single mode and multi-mode disturbances, and the same may be possible for the conditions of chapter four.
Bibliography


