Green's relations on the partition monoid and several related monoids

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Abstract

We describe Green's relations on the partition monoid, two of its submonoids, and the related Jones and Brauer monoids.

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1 The partition monoid and some submonoids

The elements of the *partition monoid* P_n [5, 8, 9] are equivalence classes of graphs on a vertex set $\mathbf{n} \cup \mathbf{n}' = \{1, 2, ...n\} \cup \{1', 2', ...n'\}$ (consisting of two copies of $\mathbf{n}$). Two such graphs are regarded as equivalent if they have the same connected components. We will thus select as a standard representative of each equivalence class a graph whose every component is a complete graph. However it facilitates visualisation to *draw* the graphs with a minimal number of edges, and we follow this convention in the diagrams below; however we should remember that each component is really a complete graph.

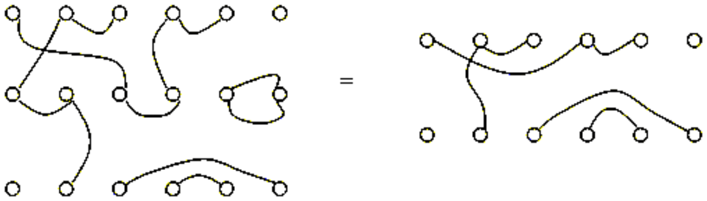

Fig. 1 Multiplication in P_6

Conventionally, we draw the graph of $a \in P_n$ so that the vertices $1, 2, \ldots n$ are in a horizontal row, with corresponding vertices $1', 2', \ldots n'$ directly below. Thus we will refer to the undashed elements as the *upper* vertices, and the dashed as the *lower* vertices. To multiply two elements of P_n, their diagrams are drawn stacked vertically, with the "interior" rows coinciding; then the connected components of the resulting graph are constructed, and finally the interior vertices are ignored. An example is seen in Fig. 1.

A component of a member a of P_n is called *transversal* if its vertices include both upper and lower elements, and *non-transversal* otherwise.

The elements of P_n also have an equivalent but more categorically-inspired description. Each element induces two partitions of $\mathbf{n}$ (ignoring for a moment the dashes), and then the transversal components set up a partial bijection between these partitions. Thus we may also describe an element a of P_n as a partial bijection between quotient sets of $\mathbf{n}$.

If every component of a is transversal, then a is called a *block bijection*; it is an isomorphism between quotient sets of $\mathbf{n}$. The block bijections form an inverse monoid I_n^* which has been studied elsewhere as the monoid of partial automorphisms of an object in the category dual to the category of sets. The monoid PT_n of partial transformations on $\mathbf{n}$ is another submonoid of P_n ($a \in PT_n$ if each component is either a singleton, or contains exactly one lower element); so also the well-studied submonoids of PT_n such as the full transformation monoid T_n and the symmetric inverse monoid I_n on n letters.

Here we shall concern ourselves with further submonoids of P_n. The *matching monoid* is the submonoid M_n of P_n consisting of matchings, that is to say, elements each of whose components has just two vertices, i.e. is an edge. The *Jones monoid* J_n consists of the matchings which may be drawn in a planar manner between the upper and lower rows. Despite an extensive literature on related semigroup algebras, there seems to have been little investigation of the semigroup properties of P_n, M_n or J_n. We sketch an initial investigation here. The results on J_n below were obtained in the second author's 2004 MAppSc project at the University of Tasmania. Those results, and others, will also appear elsewhere [7]. Here we extend those results to P_n, and apply a lemma of T. E. Hall to deduce corresponding statements about M_n and J_n, thus providing a different proof of the J_n results in [7].

2 Patterns and an involution

In our descriptions of elements of the monoids, we usually find the graphical description more convenient than the purely set-theoretic. Associated with each element a of P_n are graphical structures we shall provisionally call *patterns.* These are the subgraphs of a induced on (respectively) the upper and lower vertex sets. Each pattern, upper or lower, is thus a graph on n vertices ($\mathbf{n}$ or $\mathbf{n}'$), consisting of complete connected components (possibly singletons) and having a two-tone vertex colouring, so that the vertices of the transversal and non-transversal components are given different colours. It is useful to have notation for the different kinds of components: the subgraph induced on the upper vertices by the transversal [respectively non-transversal] components of a will be denoted by $UT(a)$ [respectively $UN(a)$]. Similarly the subgraph of a induced on the lower vertices by the [non-] transversal component of a will be denoted by $LT(a)$ [resp. $LN(a)$]. Further, we write $U(a) = UT(a) \cup UN(a)$ and $L(a) = LT(a) \cup LN(a)$ and consider them as two-tone graphs, so that equality of $U(a)$ and $U(b)$ implies equality of their transversal components and also of their non-transversal components.

We refer to the number of transversal components of a pattern as its *rank.* We note that $U(a)$ and $L(a)$ have the same rank, and refer to this number as the *rank* of a, denoted $\operatorname{rank}(a)$. The following lemma is then immediate from the definitions above.

Lemma 2.1 *Given two patterns on n vertices of equal rank r, say Γ and Γ', there exist $r!$ elements a in P_n such that $U(a) = \Gamma$ and $L(a) = \Gamma'$.*

In I_n^*, M_n and J_n there are more restrictions on the patterns which may arise as upper and lower patterns. If a is a block bijection, $UN(a)$ and $LN(a)$ are empty. In a matching a, every component of $UN(a)$ and $LN(a)$ has cardinality 2, and $UT(a)$ and $LT(a)$ are discrete graphs (no edges). Moreover, for $a \in M_n$, $n - \operatorname{rank}(a)$ must be even. For J_n, in addition to the above, the non-transversal patterns must correspond to proper bracketings (using the order in $\mathbf{n}$ and $\mathbf{n}'$) where no transversal vertex occurs within an "open" bracket. We refer to these patterns as *admissible* for each submonoid.

Lemma 2.2 *(a) Given two patterns on n vertices of equal rank r, say Γ and Γ', both admissible for M_n, there exist $r!$ elements a in M_n such that $U(a) = \Gamma$ and $L(a) = \Gamma'$.*
(b) Given two patterns on n vertices of equal rank, say Γ and Γ', both admissible for J_n, there exists a unique element a in J_n such that $U(a) = \Gamma$ and $L(a) = \Gamma'$.

Define, for $a \in P_n$, a diagram $a^* \in P_n$ obtained by 'turning a upside-down.' Together with the definition of multiplication, this gives the following.

Lemma 2.3 *For $a, b \in P_n$,*
(i) $U(a) = L(a)$;
(ii) $a^{**} = a$;
(iii) $(ab)^* = b^* a^*$;
(iv) $aa^*a = a$.

By the definitions of I_n^*, I_n, M_n, and J_n, each is closed under the unary operation $a \mapsto a^*$. So parts (ii) to (iv) assert that each of P_n, M_n, and J_n is a *regular *-semigroup* as introduced by [10] (and so in particular, regular). Of course, for I_n^* and I_n, the operation $*$ is the inversion which makes them inverse semigroups.

3 Green's relations

Theorem 3.1 *For $a, b \in P_n$,*
(i) $a \mathcal{R} b$ if and only if $U(a) = U(b)$, i.e., $UT(a) = UT(b)$ and $UN(a) = UN(b)$;
(ii) $a \mathcal{L} b$ if and only if $LT(a) = LT(b)$ and $LN(a) = LN(b)$;
(iii) $a \mathcal{D} b$ if and only if a and b have equal rank;
(iv) $\mathcal{D} = \mathcal{J}$;
(v) $a \in P_n b P_n$ if and only if $\operatorname{rank}(a) \leq \operatorname{rank}(b)$, and consequently all ideals of P_n are principal and form a chain of length n.

Proof. (i) Suppose $a = bx$. Let X be a component of $UT(a)$, and $i \in X$. Then there is an edge $\{i, j\}$ with $j \in LT(b)$, so i is a vertex of some component Y of $UT(b)$. If i' is another vertex in Y, then there is an edge $\{i', j\}$ and hence $\{i, i'\} \in UT(a)$. So X is a union of components of $UT(b)$. Similarly if $b = ay$ then every component of $UT(b)$ is a union of components of $UT(a)$. So $a \mathcal{R} b$ implies $UT(a) = UT(b)$. Again if $a = bx$, $UN(b) \subseteq UN(a)$, so $a \mathcal{R} b$ implies $UN(a) = UN(b)$.

For the converse, suppose $UT(a) = UT(b)$ and $UN(a) = UN(b)$. There is a one-to-one correspondence between the components of $LT(a)$ and the components of $LT(b)$, induced by their respective bijections with the components of $UT(a) = UT(b)$. This defines $x \in P_n$ with $U(x) = L(b)$ and $L(x) = L(a)$ such that $bx = a$ and $ax^* = b$ (in fact, $x = b^*a$); thus $a \mathcal{R} b$.

(ii) This is dual to part (i).

(iii) If there is $c \in P_n$ such that $a \mathcal{R} c$ and $c \mathcal{L} b$, then the ranks of a and b are equal to that of c. Conversely, given $a, b \in P_n$ of equal rank, we use Lemma 1 to construct $c \in P_n$ such that $U(c) = U(a)$ and $L(c) = L(b)$, whence $a \mathcal{R} c$ and $c \mathcal{L} b$. Thus $a \mathcal{D} b$.

(iv) follows from the finiteness of P_n and Theorem 3 of [3] (or see Proposition 2.1.4 in [6]).

(v) By (iii), for given $a \in P_n$ of rank r, we have $a \mathcal{D} e_r$, where e_r has r transversal edges $\{i, i'\}$ with $1 \leq i \leq r$ and all other components singletons. Similarly $b \mathcal{D} e_s$. Clearly $e_r e_s = e_s e_r = e_{\min(r,s)}$ and the result follows from this and part (iv). ■

Now the submonoids $PT_n, T_n, I_n, I_n^*, M_n$, and J_n are regular, and so their $\mathcal{L}$ and $\mathcal{R}$ relations are the restrictions of those on P_n, by a result of T. E. Hall in [4] (see also [6], Proposition 2.4.2). Thus the well-known characterizations of Green's relations $\mathcal{L}$ and $\mathcal{R}$ on the first three monoids in the list above are corollaries of the Theorem—the UN and LN graphs are discrete, so part (i) of the theorem simplifies to equality of the UT graphs, which in this case can be recognised as kernels of mappings, and part (ii) reduces to equality of ranges. For I_n^*, the UN and LN graphs are empty and the conditions reduce to equality of set partitions.

For M_n and J_n we have

Corollary 3.2 *Let $a, b \in M_n$ [respectively, J_n]. Then*
(i) $a \mathcal{R} b$ if and only if $UN(a) = UN(b)$;
(ii) $a \mathcal{L} b$ if and only if $LN(a) = LN(b)$; and
(iii) $a \in M_n b M_n$ [respectively, $a \in J_n b J_n$] if and only if $\text{rank}\,(a) \leq \text{rank}\,(b)$.

Proof. $UT(a)$ and $UT(b)$ consist of singleton components, so part (i) is immediate; likewise the LN graphs and part (ii). For part (iii), "only if" is clear, so suppose $\text{rank}\,(a) = r \leq \text{rank}\,(b) = s$. Since $a, b \in M_n$, $n - r$ and $n - s$ are even. The rank-r pattern Γ_r having singletons $\{i\}$ $(i = 1, \ldots r)$ for its transversal components, and edges $\{j, j+1\}$ $(j = r+1, r+3, \ldots n-1)$ for its non-transversal components, is admissible for J_n. Thus there is, by Lemma 2.2, $c \in J_n$ with $U(c) = U(a)$ and $L(c) = \Gamma_r$, and hence $a \mathcal{R} c$. Similarly there is $d \in J_n$ with $U(d) = \Gamma_r$ and $L(d) = L(b)$, whence $d \mathcal{L} b$. Now let f_r have $U(f_r) = \Gamma_r = L(f_r)$; it follows that $a \mathcal{D} f_r$. Similarly $b \mathcal{D} f_s$, but (as in Theorem 3.1, part (v)) $f_r f_s = f_s f_r = f_r$ and it follows that $a \in M_n b M_n$ $[a \in J_n b J_n]$. ■

In the case of J_n, this is one of our results in [7], with different notation.

4 Related monoids

Recall from section 1 that the interior vertices were ignored in forming the product of two elements of P_n. We can construct new semigroups from those above in the following manner. (The construction is formally the same as the *alteration* product discussed by Sweedler [11] in the context of algebras.) First denote by $\gamma(a,b)$ the number of components of the graph $LN(a) \cup UN(b)$ induced on the interior vertices during the juxtaposition of $a, b \in P_n$, in the manner shown in Fig. 1 (which is an example where $\gamma(a,b) = 1$.) Let S be a subsemigroup of P_n, and define a product on $\mathbb{N} \times S$ by the rule

$$(k,a) \circ (l,b) = (k + l + \gamma(a,b)\,,\, ab)\,;$$

it is easily checked to be associative and have the identity element $(0,1)$ if S is a monoid. The elements of the new monoid are represented by graphs as in P_n, but augmented by *circles* (without vertices); (a,k) is represented by a P_n-diagram a with k circles adjoined. In multiplying two such diagrams, cycles formed in the middle row are retained in the product (as circles without vertices).

When S is the Jones monoid J_n, this construction gives the *Kauffman monoid* K_n investigated by Borisavljević, Došen and Petrić [1]. This is the monoid generated by the generators of the Temperley-Lieb algebra TL_n. When S is the matching monoid M_n, the construction gives the *Brauer monoid* B_n which has been well studied (beginning with [2]) because of the significance of the Brauer algebra which it generates. When $S = P_n$ we shall denote the new monoid by Q_n.

In the above cases, all components in the interior row are cycles; but when S is I_n^*, PT_n, or any of their subsemigroups, $\gamma(a,b) = 0$ and we obtain simply the direct product.

In [7] we made a more detailed study of the range of the function $\gamma(a,b)$ and used the results to determine the ideal structure of the Kauffman monoid and two of its quotient monoids. We rephrase here some of these findings:

Theorem 4.1 *Let $(k,a)\,,(l,b) \in K_n$ with $k,l \in \mathbb{N}$ and $a,b \in J_n$. Then*
(i) $(k,a)\ \mathcal{R}\ (l,b)$ if and only if $a\,\mathcal{R}\,b$ and $k = l$;
(ii) $(k,a)\ \mathcal{L}\ (l,b)$ if and only if $a\,\mathcal{L}\,b$ and $k = l$;
(iii) $(k,a)\ \mathcal{D}\ (l,b)$ if and only if $a\,\mathcal{D}\,b$ and $k = l$, and $\mathcal{D} = \mathcal{J}$;
(iv) the lattice of principal ideals of K_n is the product of a chain of length $\lfloor \frac{n}{2} \rfloor$ with a chain isomorphic to $\mathbb{N}$ (with the order $0 > 1 > \dots$);
(v) all ideals of K_n are finitely generated.

The analogous theorem holds in each of B_n and Q_n, with the only difference being that the chain length in part (iv) is n rather than $\lfloor \frac{n}{2} \rfloor$. The proof is simpler than for K_n, since there is no need to construct planar elements.

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