Diffraction of a binary non-Pisot inflation tiling

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Abstract. A one-parameter family of binary inflation rules in one dimension is considered. Apart from the first member, which is the well-known Fibonacci rule, no inflation factor is a unit. We identify all cases with pure point spectrum, and discuss the diffraction spectra of other members of the family. Apart from the trivial Bragg peaks at the origin, they have purely singular continuous diffraction.

Despite various open questions on details, the theory of substitutions with pure point spectrum is fairly well developed. In particular, for any given substitution on a finite alphabet, pure pointedness of the spectrum can be decided algorithmically; see [1] and references therein. For substitutions of constant length, the situation is even better due to Dekking's result [7] and its recent extension by Bartlett [6]. In general, however, the understanding of substitutions of non-constant length with mixed spectrum is still rudimentary. Recent work has indicated that progress in this direction is easier in the geometric setting of tiling spaces with natural tile sizes; see [3] and references therein. We adopt this point of view here, too.

In this contribution, we consider the family of primitive substitution rules on the binary alphabet $\{0, 1\}$ defined by

$$\varrho_m : \quad 0 \mapsto 01^m , \; 1 \mapsto 0, \quad \text{with } m \in \mathbb{N}.$$

The substitution matrix is $M_m = \left(\begin{smallmatrix} 1 & 1 \\ m & 0 \end{smallmatrix} \right)$ with eigenvalues $\lambda_m^\pm = \frac{1}{2}\bigl(1 \pm \sqrt{4m + 1} \bigr)$, which are the roots of $\lambda^2 - \lambda - m = 0$. The frequency-normalised Perron–Frobenius (PF) eigenvector is $(1, \lambda_m^+ - 1)^t / \lambda_m^+$, whose entries are the relative frequencies of the two letters. The corresponding left eigenvector reads $(\lambda_m^+, 1)$, which is our choice of the interval lengths for the corresponding geometric inflation rule. Up to scale, this is the unique choice to obtain a self-similar *inflation tiling* of the line from ϱ_m; see [4, Ch. 4] for background.

From now on, we will mainly work with the tiling system on the real line. Note that ϱ_1 defines the ubiquitous *Fibonacci tiling*, which is well-known to have pure point spectrum, both in diffraction and in dynamical sense [8, 4, 5]. For $m = 2$, we obtain a system that is equivalent to the *period doubling chain*, as can be seen by choosing $a \mathrel{\widehat{=}} 0$ and $b \mathrel{\widehat{=}} 11$ which establishes a mutual local derivability (MLD) rule; see [4, Secs. 4.5.1 and 9.4.4] for details on the period doubling system. More generally, λ_m^+ is an integer if and only if $4m+1$ is a square. This precisely happens for $m = \ell(\ell+1)$ with $\ell \in \mathbb{N}$, giving $\lambda_m^+ = \ell+1$ and $\lambda_m^- = -\ell$. Similar to the $m = 2$ case, any of these systems can be recoded as a constant length substitution, via $a \mathrel{\widehat{=}} 0$ and $b \mathrel{\widehat{=}} 1^{\ell+1}$. The induced substitution is $a \mapsto ab^\ell$, $b \mapsto a^{\ell+1}$, which has a coincidence in the first position.

Figure 1. Illustration of the self-similar inflation rule derived from ϱ_3.

Consequently, one has pure point spectrum by Dekking's criterion [7]. In all remaining cases, the PF eigenvalue is irrational, but fails to be a Pisot–Vijayaraghavan (PV) number, which means that none of the corresponding tilings can have non-trivial point spectrum [9, 2]. In particular, the only Bragg peak is the trivial one at $k = 0$. So far, we have the following result.

Theorem 1 *Consider the inflation tiling defined by ϱ_m. For $m = 1$ and $m = \ell(\ell + 1)$ with $\ell \in \mathbb{N}$, the tiling has pure point diffraction, which can be calculated with the projection method. For all remaining cases, the pure point part of the diffraction consists of the trivial Bragg peak at 0, while the remainder of the diffraction is of continuous type.* $\square$

The first example with continuous component emerges for $m = 3$, where $\lambda^{\pm} = \frac{1}{2}\big(1 \pm \sqrt{13}\,\big)$. Let us discuss this case in some more detail (the other non-PV cases are fairly analogous). The concrete inflation rule for this case is illustrated in Figure 1. With $\varrho := \varrho_3$, consider the bi-infinite fixed point w of ϱ^2 with legal seed $0|0$ obtained as

$$0|0 \xrightarrow{\varrho^2} w^{(1)} = 0111000|0111000 \xrightarrow{\varrho^2} \ldots \xrightarrow{\varrho^2} w^{(i)} \xrightarrow{i \to \infty} w = \varrho^2(w);$$

see [8, 4] for background. The corresponding tiling, via the left endpoints of the two types of intervals, leads to a Delone set

$$\Lambda^w = \big\{\ldots, -1-3\lambda, -3\lambda, -2\lambda, -\lambda, 0, \lambda, 1+\lambda, 2+\lambda, 3+\lambda, 3+2\lambda, \ldots\big\} \subset \mathbb{Z}[\lambda],$$

where $\lambda = \lambda_3^+$ from now on. Defining the hull $\mathbb{Y} := \overline{\{t + \Lambda^w \mid t \in \mathbb{R}\}}^{\mathrm{LT}}$ with the closure being taken in the local topology (LT), we obtain a topological dynamical system $(\mathbb{Y}, \mathbb{R})$ under the translation action of $\mathbb{R}$. This system is *strictly ergodic*, which means it is *minimal* (because $\mathbb{Y}$ coincides with the local indistinguishability (LI) class of Λ^w) and *uniquely ergodic* (since there is a unique probability measure $\mu_{\mathbb{Y}}$, defined by the uniformly existing patch frequencies).

One important consequence is that every $\Lambda \in \mathbb{Y}$ has the same autocorrelation and the same diffraction measure as Λ^w, which are thus also called the autocorrelation and diffraction of the dynamical system. More generally, given any $\Lambda \in \mathbb{Y}$, we consider the weighted Dirac comb $\omega := \sum_{x \in \Lambda} u(x)\delta_x$ with general complex weights $u(x) \in \{u_0, u_1\}$ according to the interval type. Then, the corresponding autocorrelation γ_u is of the form

$$\gamma_u = \sum_{z \in \Lambda - \Lambda} \eta_u(z)\,\delta_z \quad \text{with} \quad \eta_u(z) = \lim_{r \to \infty} \frac{1}{2r} \sum_{y, y+z \in \Lambda_r} \overline{u(y)}\, u(y+z),$$

with $\Lambda_r := \Lambda \cap [-r, r]$.

Since we do not have any projection method at our disposal for the further analysis, other tools are needed. It has recently been shown in [3] that the *pair correlation functions* of a primitive inflation tiling satisfy a set of *exact renormalisation relations* that help to unravel the spectral type of the diffraction. Partitioning $\Lambda = \Lambda^{(0)} \dot\cup \Lambda^{(1)}$ into the two point types, one can define the pair correlation functions as

$$\nu_{ij}(z) = \lim_{r \to \infty} \frac{\mathrm{card}\big(\Lambda_r^{(i)} \cap (\Lambda_r^{(j)} - z)\big)}{\mathrm{card}(\Lambda_r)} = \frac{1}{\mathrm{dens}(\Lambda)} \lim_{r \to \infty} \frac{\mathrm{card}\big(\Lambda_r^{(i)} \cap (\Lambda_r^{(j)} - z)\big)}{2r}$$

for $i, j \in \{0, 1\}$. These functions are well-defined for any $z \in \mathbb{R}$, non-negative and satisfy the symmetry relations $\nu_{ij}(z) = \nu_{ji}(-z)$. Moreover, $\nu_{ij}(z) > 0$ if and only if $z \in \Lambda^{(j)} - \Lambda^{(i)}$, which is a consequence of strict ergodicity.

The autocorrelation coefficients $\eta_u(z)$ can now be expressed as

$$\eta_u(z) \;=\; \mathrm{dens}(\Lambda) \sum_{i,j \in \{0,1\}} \overline{u_i}\, \nu_{ij}(z)\, u_j$$

and similar expressions will later emerge for the diffraction measure $\widehat{\gamma_u}$. As follows from [3, 2], one has the following result.

Theorem 2 *The pair correlations functions ν_{ij} satisfy the linear renormalisation equations*

$$\nu_{00}(z) \;=\; \frac{1}{\lambda}\left(\nu_{00}\!\left(\tfrac{z}{\lambda}\right) + \nu_{01}\!\left(\tfrac{z}{\lambda}\right) + \nu_{10}\!\left(\tfrac{z}{\lambda}\right) + \nu_{11}\!\left(\tfrac{z}{\lambda}\right)\right),$$

$$\nu_{01}(z) \;=\; \frac{1}{\lambda}\left(\nu_{00}\!\left(\tfrac{z-\lambda}{\lambda}\right) + \nu_{00}\!\left(\tfrac{z-1-\lambda}{\lambda}\right) + \nu_{00}\!\left(\tfrac{z-2-\lambda}{\lambda}\right) + \nu_{10}\!\left(\tfrac{z-\lambda}{\lambda}\right) + \nu_{10}\!\left(\tfrac{z-1-\lambda}{\lambda}\right) + \nu_{10}\!\left(\tfrac{z-2-\lambda}{\lambda}\right)\right),$$

$$\nu_{10}(z) \;=\; \frac{1}{\lambda}\left(\nu_{00}\!\left(\tfrac{z+\lambda}{\lambda}\right) + \nu_{00}\!\left(\tfrac{z+1+\lambda}{\lambda}\right) + \nu_{00}\!\left(\tfrac{z+2+\lambda}{\lambda}\right) + \nu_{01}\!\left(\tfrac{z+\lambda}{\lambda}\right) + \nu_{01}\!\left(\tfrac{z+1+\lambda}{\lambda}\right) + \nu_{01}\!\left(\tfrac{z+2+\lambda}{\lambda}\right)\right),$$

$$\nu_{11}(z) \;=\; \frac{1}{\lambda}\left(3\,\nu_{00}\!\left(\tfrac{z}{\lambda}\right) + 2\,\nu_{00}\!\left(\tfrac{z+1}{\lambda}\right) + 2\,\nu_{00}\!\left(\tfrac{z-1}{\lambda}\right) + \nu_{00}\!\left(\tfrac{z+2}{\lambda}\right) + \nu_{00}\!\left(\tfrac{z-2}{\lambda}\right)\right).$$

Subject to the condition that the support of each ν_{ij} is $\Lambda^{(j)} - \Lambda^{(i)}$, the solution space of this infinite system of linear equations is one-dimensional. $\square$

Via $\Upsilon_{ij} := \sum_{z \in S_{ij}} \nu_{ij}(z)\, \delta_z$, the pair correlation functions are turned into pure point measures. The autocorrelation measure γ_u can now be written as $\gamma_u(\mathcal{E}) = \mathrm{dens}(\Lambda) \sum_{i,j \in \{0,1\}} \overline{u_i}\, \Upsilon_{ij}(\mathcal{E})\, u_j$, where $\mathcal{E} \subset \mathbb{R}$ is any bounded Borel set. Taking the Fourier transform, one obtains the diffraction measure as

$$\widehat{\gamma_u}(\mathcal{E}) \;=\; \mathrm{dens}(\Lambda) \sum_{i,j \in \{0,1\}} \overline{u_i}\, \widehat{\Upsilon}_{ij}(\mathcal{E})\, u_j,$$

where the Fourier transform of each term can be shown to exist [3]. Now, the renormalisation relations from Theorem 2 induce measure-valued relations for the $\widehat{\Upsilon}_{ij}$, which have to be satisfied for each part of their Lebesgue decompositions separately. An analysis of the asymptotic behaviour of the absolutely continuous components shows that the only contribution compatible with local integrability of the Radon–Nikodym densities and the translation boundedness of $\widehat{\gamma_u}$ is the trivial one, which means that no absolutely continuous component is possible [2]. Consequently, one has the following result.

Theorem 3 *The diffraction measure $\widehat{\gamma_u}$, which is the same for all $\Lambda \in \mathbb{Y}$, has the pure point part $\left|\frac{2\lambda-1}{13}u_0 + \frac{7-\lambda}{13}u_1\right|^2 \delta_0$. The remainder of $\widehat{\gamma_u}$ is singular continuous.* $\square$

To give an impression of the singular continuous part, let us choose $u_0 = 1 - \lambda$ and $u_1 = 1$. With this choice, the Bragg peak at 0 is extinct, so $\widehat{\gamma_u}$ is purely singular continuous, with $\eta_u(0) = (6\lambda - 3)/13 \approx 0.832$. Consequently, the *distribution function F* defined by $F(x) := \widehat{\gamma_u}\big([0, x]\big)$ is continuous, with average slope given by $\eta_u(0)$; see Figure 2 for an illustration. Note that F is strictly increasing, despite the appearance of 'flat' regions which resemble plateaux.

Let us close by commenting on the other non-Pisot members of our inflation tiling family. So, let m be any integer such that the pure point part of the diffraction, according to Theorem 1,
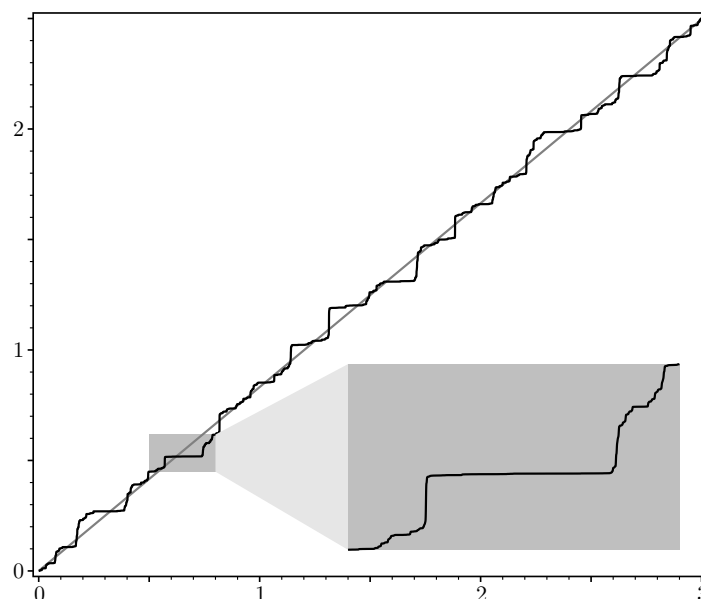
Figure 2. Sketch of the distribution function F for the singular continuous diffraction measure discussed in the text. The average slope of F is $\eta_u(0)$, which is indicated by a straight line.

is trivial. Then, with our choice of interval lengths from the beginning, the Bragg peak at the origin has intensity

$$
I_0 \;=\; \left|\mathrm{dens}(\varLambda)\,(u \cdot v_{\mathrm{PF}})\right|^2 \;=\; \frac{\left|u_0 + (\lambda_m^+ - 1)u_1\right|^2}{4m+1},
$$

where $\mathrm{dens}(\varLambda) = \frac{\lambda_m^+}{2\lambda_m^+ - 1} = \frac{\lambda_m^+ + 2m}{4m+1}$ and v_{PF} is the PF frequency vector from above.

For each such inflation tiling, the pair correlation functions are again well-defined, and satisfy a set of exact linear renormalisation relations in analogy to Theorem 2. Completing the corresponding analysis on the Fourier side, our analysis indicates that we can never have an absolutely continuous component. A numerical calculation of the distribution function analogous to F above produces graphs that are very similar to the one shown in Figure 2.

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