

SU(2/1) superchiral self-duality: a new quantum, algebraic and geometric paradigm to describe the electroweak interactions

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ABSTRACT: We propose an extension of the Yang-Mills paradigm from Lie algebras to internal chiral superalgebras. We replace the Lie algebra-valued connection one-form A , by a superalgebra-valued polyform \tilde{A} mixing exterior-forms of all degrees and satisfying the chiral self-duality condition $\tilde{A} = *\tilde{A}\chi$, where χ denotes the superalgebra grading operator. This superconnection contains Yang-Mills vectors valued in the even Lie subalgebra, together with scalars and self-dual tensors valued in the odd module, all coupling only to the charge parity CP-positive Fermions. The Fermion quantum loops then induce the usual Yang-Mills-scalar Lagrangian, the self-dual Avdeev-Chizhov propagator of the tensors, plus a new vector-scalar-tensor vertex and several quartic terms which match the geometric definition of the supercurvature. Applied to the SU(2/1) Lie-Kac simple superalgebra, which naturally classifies all the elementary particles, the resulting quantum field theory is anomaly-free and the interactions are governed by the super-Killing metric and by the structure constants of the superalgebra.

KEYWORDS: Anomalies in Field and String Theories, Beyond Standard Model, Gauge Symmetry

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1 Introduction

The weak interactions are chiral. Before symmetry breaking, all the Fermions of the standard model are massless, all the left states are $SU(2)$ doublets and all the right states are singlets. This fundamental asymmetry is difficult to justify in the Yang-Mills framework because Lie algebra symmetries can only connect states of a given chirality, and connecting left particles to left antiparticles as in the $SU(5)$ grand-unified theory potentially implies proton decay. However, as observed in 1979 by Ne’eman [1] and Fairlie [2], the $SU(2)U(1)$ electroweak algebra is naturally embedded in $SU(2/1)$, the smallest Lie-Kac simple superalgebra [3]. The leptons [1, 2] and quarks [4, 5], graded by their chirality, fit the smallest irreducible representations of $SU(2/1)$. Strangely, these representations are non Hermitian ([6], appendix D). But we noticed recently [6] that the resulting collection of scalar anomalies cancels, whenever the Adler-Bell-Jackiw [7, 8] vector anomaly cancels [9]. These

observations renew the interest in the construction of a generalization of the Lie algebra Yang-Mills framework to the case of a chiral superalgebra.

Merging differential geometry, superalgebra and quantum field theory concepts, we present a new paradigm. We propose to consider as fundamental a Lie superalgebra-valued superconnection polyform \tilde{A} , mixing de Rham exterior-forms of all degrees [10–13] and satisfying the new superchirality condition:

$$\tilde{A} = * \tilde{A} \chi, \tag{1.1}$$

where the $*$ denotes the Hodge duality in Minkowski 4-dimensional space-time with signature $(-+++)$ and χ is the charge-chirality of the superalgebra instrumental in the definition of the supertrace

$$STr(M) = Tr(\chi M). \tag{1.2}$$

Remarkably, this superchirality condition pairs charge conjugation with parity, naturally enforcing the Landau charge parity (CP) invariance characteristic of the weak interactions. The pairing is a consequence of the duality identities structuring the antisymmetrized products of the Pauli matrices, which imply that if \tilde{A} is a self-dual (or anti-self-dual) polyform, then the corresponding Dirac-Yukawa operator \tilde{A} only couples to left spinors ψ_L (or right spinors ψ_R):

$$\tilde{A} = * \tilde{A} \Rightarrow \tilde{A} \psi_R = 0, \quad \tilde{A} = -* \tilde{A} \Rightarrow \tilde{A} \psi_L = 0. \tag{1.3}$$

Expanding (1.1) in terms of the underlying fields, we find (2.8) that the vectors couple exactly as postulated in 1979 by Ne’eman [1] and Fairlie [2], that the scalars $\bar{\Phi}\Phi$ (2.9) couple exactly as in Thierry-Mieg [6] and that 2-form components $\bar{B}B$ of the superconnection, interpreted as self and anti-self-dual Avdeev-Chizhov fields [14], follow the same pattern (2.10). A new trilinear scalar-vector-tensor interaction $F\{\bar{B}, \Phi\}$ is induced by the Fermion loop (4.8). It must be considered as an intrinsic part of the minimal coupling of a superalgebra since the same term appears in the square of the supercurvature defined in (2.11).

Curiously, the Fermion quantum loop counterterms hesitate between a Lie algebra and a Lie superalgebra structure (3.7) (4.6) and (5.1), but when the construction is applied to the $SU(2/1)$ model of leptons and quarks [6], a generalization of the Bouchiat, Iliopoulos and Meyer (BIM) mechanism [9] lifts the ambiguity (section 7) and implies that the theory is anomaly-free and that the propagators and covariant derivatives of the scalars and the tensors are provided by the antisymmetric g_{ij} super-Killing metric of the superalgebra and by the symmetric d_{aij} structure constants.

From a geometrical perspective, our new complementary treatment of the exterior bundle (the polyform superconnection) with the spinor bundle (the chiral Fermions) reflects an element of *bona fide* internal supersymmetry. The signs generated in the quantum loops by the tensorial structures of the propagators and by the orientations of the chiral Fermion propagators match the signs generated by the grading of the superalgebra in the Clebsch-Gordan calculations and their interplay with the super-Jacobi identity. The theory is superalgebraic despite the fact that all the gauge fields are Bosons and all the matter fields are Fermions, as requested by the spin-statistics relation.

The theory respects Einstein’s distinction between the force fields, which are geometrized as superalgebra-valued polyforms, and matter fields, represented by pointlike chiral Fermions. The odd couplings play with the orientation of space, which is represented by the opposite helicities of the massless left/right Fermions, and with the orientations of the p -form matter fields and their Hodge duals. This is very different from the Wess-Zumino supersymmetry which couples Bosons to Fermions, i.e. force fields to matter fields. Although the latter is much more developed, one should remember that not a single known particle is the supersymmetric partner of another known particle, for example the neutrino is not the partner of the photon, whereas on the contrary, all the known elementary particles, the leptons and the quarks, naturally fall in superchiral $SU(2/1)$ multiplets. Finally, the superconnection offers a way geometrize the Higgs fields.

In section 2, we introduce in details the new paradigm of a self superchiral superconnection. In sections 3 and 4, we show how the knowledge of the couplings of the Fermions to the scalar, vector and 2-tensor components of the superconnection induce their propagators and interactions. In section 5 to 7, we analyze the quantum anomalies. The notations are given explicitly in appendix A to E.

2 The new superchiral superconnection \tilde{A}

The de Rham complex over a 4-dimensional differentiable manifold is the space of all differential exterior-forms of all degrees from 0 to 4: $\tilde{A} = \phi + a + b + c + e$. In Yang-Mills theory, the scalars ϕ are considered as zero-forms, i.e. ordinary functions, and the Yang-Mills vector a_μ can be identified with the components of a Lie algebra-valued Cartan connection one-form $a = a_\mu^\alpha \lambda_a dx^\mu$ where the λ_a are the generators of the Lie algebra. The connection a defines the parallel transport on the manifold and specifies how to rotate the fields in internal space under an infinitesimal displacement in the base space by replacing the Cartan exterior differential d by the covariant exterior differential $D = d + a$. Since exterior-forms of even degree (ϕ, b, e) commute, and exterior-forms of odd degree (a, c) anticommute, it is natural (see Ne’eman-Thierry-Mieg [10], Quillen [11, 12]) to associate the \mathbb{Z}_2 grading of the exterior-forms to the \mathbb{Z}_2 grading of a superalgebra (appendix A) and to try to define a superconnection as a globally odd form, that is to keep only the odd exterior-forms of degree 1 and 3, $a + c = (a_\mu^\alpha dx^\mu + c_{\mu\nu\rho}^\alpha dx^\mu dx^\nu dx^\rho / 6) \lambda_a$ which are valued in the even Lie subalgebra, together with the even forms of degree 0, 2 and 4 ($\phi^i + b_{\mu\nu}^i dx^\mu dx^\nu / 2 + e_{\mu\nu\rho\sigma}^i dx^\mu dx^\nu dx^\rho dx^\sigma / 24) \lambda_i$ which are valued in the odd module of the superalgebra. In [13], we have shown that this definition is incomplete because the odd and even forms commute $\phi^i a^a = a^a \phi^i$, whereas we need (A.3) to generate the antisymmetric commutator of the even and odd matrices. The paradox is resolved by invoking the superalgebra charge chirality matrix χ (see the details in appendix A), which defines the supertrace of the superalgebra, commutes with the even matrices and anticommutes with the odd matrices:

$$STr(M) = Tr(\chi M), \quad [\chi, \lambda_a] = 0, \quad \{\chi, \lambda_i\} = 0. \tag{2.1}$$

Our final definition of the superconnection is

$$\begin{aligned} \tilde{A} &= (\phi + b + e)^i \lambda_i + \chi (a + c)^a \lambda_a, \\ \tilde{d} &= \chi d \quad \tilde{D} = \tilde{d} + \tilde{A}, \quad \tilde{F} = \tilde{d}\tilde{A} + \tilde{A}\tilde{d}. \end{aligned} \tag{2.2}$$

The presence of the superalgebra-grading-matrix χ ensures that the signs arising in the construction of the curvature polyform \tilde{F} , and in the action of \tilde{D} on all fields, are always consistent with the brackets and structure relations of the superalgebra [13]. As a result, the curvature \tilde{F} defined as the square of the covariant differential $\tilde{F} = \tilde{D}\tilde{D}$ is valued in the adjoint representation of the superalgebra, defines a linear map, and satisfies the Bianchi identity $\tilde{D}\tilde{F} = 0$, which in turn implies that the covariant differential is associative $(\tilde{D}\tilde{D})\tilde{D} = \tilde{D}(\tilde{D}\tilde{D})$. This geometric construction is satisfactory, but it does not yet explain the structure of the electroweak interactions.

The new concept presented here is, firstly, to consider, in Minkowski 4-dimensional space-time with signature $(-+++)$, a self-dual superconnection $\tilde{A} = {}^* \tilde{A}$, where $*$ denotes the Hodge duality which maps p -forms onto $(4-p)$ -forms (appendix C). In Yang-Mills theory, the connection a is a 1-form, its dual $*a$ is a 3-form, so a Yang-Mills connection cannot be self-dual and we are only familiar with the self-dual topological theories satisfying $F = {}^* F$. But because a superconnection is composed of exterior-forms of all degrees, its 1-form component a can be the dual $a = {}^* c$ of its 3-form component c and the concept of a self-dual superconnection makes sense.

This constraint has a remarkable consequence when we consider the action of the superconnection on chiral spinors. To construct this action, we saturate the Lorentz indices of the component p -forms with Dirac γ matrices, effectively defining a map in spinor space using the Dirac-Feynman slash operator. The classic Dirac mapping $a = a_\mu dx^\mu \Rightarrow \not{a} = a_\mu(\sigma^\mu + \bar{\sigma}^\mu)$ is generalized to antisymmetric tensors of any rank, for example $b = \frac{1}{2} b_{\mu\nu} dx^\mu dx^\nu \Rightarrow \not{b} = \frac{1}{2} b_{\mu\nu}(\bar{\sigma}^\mu \sigma^\nu + \sigma^\mu \bar{\sigma}^\nu)$. As all our spinors are chiral, we use the γ_5 diagonal notation $\gamma_\mu(1 + \gamma_5)/2 + \gamma_\mu((1 - \gamma_5)/2) \rightarrow \sigma_\mu + \bar{\sigma}_\mu$ as explained in appendix B. However, the anti-symmetric product of p Pauli matrices can be rewritten as a product of $4-p$ Pauli matrices contracted with the antisymmetric Levi-Civita ϵ symbol (B.7). Therefore the Dirac operator associated to a p -form ω can be rewritten as \pm the Dirac operator associated to its Hodge dual $*\omega$, where the sign depends on the helicity of the two components Fermion on which we act (C.8). For example, if a 3-form c acts on left Fermions, this can as well be expressed in terms of the dual 1-form $*c$ (C.6):

$$\not{c} \frac{1 - \gamma_5}{2} = \frac{1}{6} c_{\mu\nu\rho} \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho = \frac{i}{6} c_{\mu\nu\rho} \epsilon^{\mu\nu\rho\sigma} \bar{\sigma}_\sigma = ({}^*c)_\mu \bar{\sigma}^\mu \quad (2.3)$$

Applying this transformation to the 2, 3 and 4 forms (b, c, e) , the Dirac operator associated to the superconnection \tilde{A} acting on the left Fermions can be rewritten as

$$\tilde{A} \frac{1 - \gamma_5}{2} = (\phi + {}^* e) \frac{1 - \gamma_5}{2} + (a + {}^* c)_\mu \bar{\sigma}^\mu + \frac{1}{2} (b + {}^* b)_{\mu\nu} \sigma^\mu \bar{\sigma}^\nu, \quad (2.4)$$

whereas the Dirac operator associated to the superconnection acting on the right Fermions can be rewritten as

$$\tilde{A} \frac{1 + \gamma_5}{2} = (\phi - {}^* e) \frac{1 + \gamma_5}{2} + (a - {}^* c)_\mu \sigma^\mu + \frac{1}{2} (b - {}^* b)_{\mu\nu} \bar{\sigma}^\mu \sigma^\nu. \quad (2.5)$$

Each parenthesized term pairs a p -form to the dual of the matching $(4-p)$ -form. As a result, see the details in appendix C, a self-dual superconnection annihilates the right Fermions

and *mutatis mutandis* an anti-self-dual superconnection annihilates the left Fermions

$$\tilde{A} = {}^* \tilde{A} \Rightarrow \tilde{A} \psi_R = 0, \quad \tilde{A} = -{}^* \tilde{A} \Rightarrow \tilde{A} \psi_L = 0. \quad (2.6)$$

To describe the electroweak interactions, we need to act both on left and on right Fermions, but with different kinds of forces. In a superalgebra framework, the charge chirality operator χ (2.1) that we have already introduced in the definition (2.2) of the superconnection provides this distinction and we postulate that our superconnection should, in addition, be superchiral

$$\tilde{A} = {}^* \tilde{A} \chi. \quad (2.7)$$

This beautiful equation (1.1) correlates the orientation of space, which is hidden in the definition of the Hodge duality, denoted by the $*$, to the charge chirality χ of the superalgebra, defined in the internal charge space, and in consequence constrains the chirality of the charged Fermions.

We illustrate the outcome of these constraints on the specific case of $SU(m/n)$ viewed as a chiral superalgebra. In the $SU(m)$ sector, the potential (in the $SU(m/n)$ fundamental representation), is accompanied by the sign $\chi = +1$, so $a = {}^* c$ (C.7). Hence in the Dirac operator only the term $(a + {}^* c)_\mu \bar{\sigma}^\mu$ survives (C.6)–(C.8) and it acts only on the left Fermion (B.3) and (C.9) and (2.6). Reciprocally, in the $SU(n)$ sector ($\chi = -1$), \tilde{A} is anti-self-dual and the Dirac operator $(a - {}^* c)_\mu \sigma^\mu$ annihilates the left Fermions and only acts on the right Fermions.

The $U(1)$ operator of $SU(m/n)$ is special in that the corresponding matrix in the fundamental representation acts at the same time on $\chi = 1$ and $\chi = -1$ states and satisfies $a = {}^* c \chi$ accordingly. In consequence we get an Abelian vector multiplying the supertraceless $U(1)$ matrix acting both on the left and right Fermions via $a_\mu((1 + \chi)\bar{\sigma}^\mu + (1 - \chi)\sigma^\mu)$.

Returning to the case of $SU(2/1)$ we get *diagonal*($\bar{\sigma}, \bar{\sigma}; 2\sigma$) exactly as postulated *ex nihilo* in 1979 by Ne’eman [1] and Fairlie [2]

$$\tilde{A} = \frac{1}{4} A_\mu^a \lambda_a (\bar{\sigma}^\mu(1 + \chi)(1 - \gamma_5) + \sigma^\mu(1 - \chi)(1 + \gamma_5)). \quad (2.8)$$

For the scalar fields, we have $\bar{\Phi} = \phi + {}^* e$ and $\Phi = \phi - {}^* e$ which act as

$$\begin{aligned} \bar{\Phi} &= \frac{1}{4} \bar{\Phi}^i \lambda_i (1 + \chi)(1 - \gamma_5), \\ \Phi &= \frac{1}{4} \Phi^i \lambda_i (1 - \chi)(1 + \gamma_5), \end{aligned} \quad (2.9)$$

exactly as we postulated *ex nihilo* in [6]. The 2-form b follows a similar pattern. Separating the self-dual and anti-self-dual parts $\bar{B} = b + {}^* b$ and $B = b - {}^* b$ the Dirac operator acts via the combinations

$$\begin{aligned} \bar{B} &= \frac{1}{8} \bar{B}_{\mu\nu}^i \lambda_i \sigma^\mu \bar{\sigma}^\nu (1 + \chi)(1 - \gamma_5), \\ B &= \frac{1}{8} B_{\mu\nu}^i \lambda_i \bar{\sigma}^\mu \sigma^\nu (1 - \chi)(1 + \gamma_5). \end{aligned} \quad (2.10)$$

Φ and B absorb right Fermions and emit left Fermions, and their antiparticles $\bar{\Phi}$ and \bar{B} absorb left Fermions and emit right Fermions, as illustrated below in the Feynman diagrams

presented in sections 3 and 4. Our point is that the superchiral constraint allows us to derive from first principles the same interactions that had to be imposed in the previous SU(2/1) literature to force the gauge superalgebra to look like the standard model. The price we pay is the appearance of a new scalar sector represented by the $\overline{B}B$ fields.

The reader should notice that the 2-form component of the curvature polyform \tilde{F} (2.2) reads in these new notations

$$\tilde{F} = \left(dA^a + \frac{1}{2} \left(f_{bc}^a A^b A^c + d_{ij}^a \left(\overline{\Phi}^i B^j + \Phi^i \overline{B}^j \right) \right) \right) \lambda_a, \quad (2.11)$$

generating inside the Lagrangian \tilde{F}^2 a new scalar-vector-tensor interaction $F(\{\overline{B}, \Phi\} + \{\overline{\Phi}, B\})$. As shown below, this term plays a crucial role in the self consistency of the theory.

Given these algebraic and geometrical definitions, let us now study how the Dirac action of the superconnection on the chiral Fermions gets promoted in the quantum field theory into the definition of the propagators and interactions of its components, the complex scalar field Φ , the vector A , and the complex self-dual anti-self-dual antisymmetric tensor $\overline{B}B$ all correctly satisfying the spin-statistics relation.

3 The Avdeev-Chizhov propagator is induced by the Fermion loop

In their seminal study [14], Avdeev and Chizhov have introduced a new type of quantum field: a self-dual and anti-self-dual antisymmetric tensor \overline{B} and B satisfying in Minkowski space the conditions

$$\overline{B} = * \overline{B}, \quad B = -* B, \quad (3.1)$$

where $*$ denotes the Hodge dual (C.1) in 4-dimensional Minkowski space-time with signature $(-+++)$:

$$B = \frac{1}{2} B_{\mu\nu} dx^\mu dx^\nu, \quad *B = -\frac{i}{2} \epsilon_{\mu\nu\rho\sigma} B^{\mu\nu} dx^\rho dx^\sigma \quad (3.2)$$

and ϵ is the fully antisymmetric symbol with $\epsilon_{0123} = 1$. These fields coincide with the antisymmetric tensor fields identified in (2.10) as part of the superchiral superconnection \tilde{A} : compare (3.2) with (C.1), (C.5) and the definition of the Hodge dual of the field components in (C.6) and (C.8).

Until their discovery, the existence of a Lagrangian compatible with the self-duality condition seemed unlikely and its structure appeared at first complicated. Some efforts were needed to demonstrate that the Avdeev-Chizhov tensors describe a complex scalar field with one real degree of freedom for B and one for \overline{B} and to delineate their possible interactions [15, 16]. With hindsight, we can reconstruct the model just from the rules of quantum field theory. The possible couplings of a 2-tensor to a chiral Fermion are strongly constrained by Lorentz invariance. The $\mu\nu$ indices must act on the Fermions via the antisymmetrized product of two Pauli matrices (see appendix B for our precise notations) and this product is by itself self-dual:

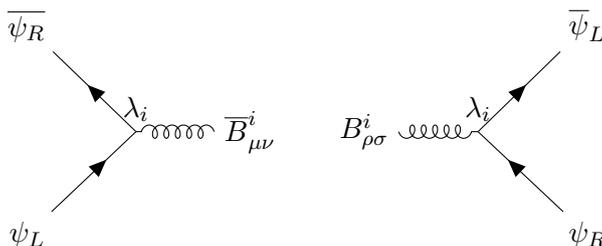
$$\sigma\overline{\sigma} = P^+ \sigma\overline{\sigma}, \quad \overline{\sigma}\sigma = P^- \overline{\sigma}\sigma, \quad (3.3)$$

where P^\pm are the self-duality projectors

$$P_{\mu\nu\rho\sigma}^\pm = \frac{1}{4} (g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} \mp i \epsilon_{\mu\nu\rho\sigma}) , \tag{3.4}$$

$$(P^+)(P^+) = P^+, \quad (P^-)(P^-) = P^-, \quad (P^+)(P^-) = (P^-)(P^+) = 0 .$$

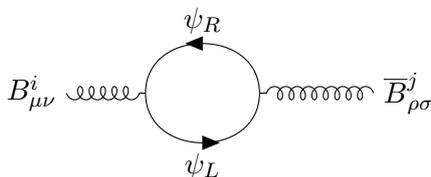
Therefore, the only antisymmetric tensors which can couple to chiral Fermions are self or anti-self-dual. The anti-self-dual field B absorbs right states and emits left states, and the self-dual field \bar{B} absorbs left states and emits right states according to the Feynman diagrams:



Assuming the standard propagator for the chiral Fermions defined by the Lagrangian

$$\mathcal{L} = i(\overline{\psi_R}) \sigma^\mu \partial_\mu \psi_R + i(\overline{\psi_L}) \bar{\sigma}^\mu \partial_\mu \psi_L , \tag{3.5}$$

the knowledge of these 2 vertices is sufficient to compute the pole part of the propagator of the $\bar{B}B$ field by closing the Fermion loop:



Carefully computing this Feynman diagram (appendix E), we recover the tensorial structure of Avdeev-Chizhov propagator [14]

$$\mathcal{L}_B = -\kappa_{ij} g^{\mu\nu} \partial^\alpha \bar{B}_{\alpha\mu}^i \partial^\beta B_{\beta\nu}^j , \tag{3.6}$$

however [6], an unexpected consequence of the chiral couplings of the $\bar{B}B$ fields (2.10) is that the κ_{ij} metric is calculated as a chiral trace:

$$\kappa_{ij} = \frac{1}{2} Tr((1 + \chi)\lambda_i\lambda_j) = \frac{1}{2} Tr(\lambda_i\lambda_j) + \frac{1}{2} STr(\lambda_i\lambda_j) . \tag{3.7}$$

The theory hesitates between a Lie algebra like metric: $Tr(\lambda_i\lambda_j)$, and a Lie-Kac superalgebra supermetric: $STr(\lambda_i\lambda_j)$. The resolution of this dilemma depends on the number and types of chiral Fermions described by the model and is discussed below in section 6.

4 The Bosonic interaction terms are induced by the Fermion loops

Following our above discussion of the Avdeev-Chizhov fields, we now extend the method to determine the propagators and self interactions of the remaining components of the superchiral superconnection. We postulate the generalized Dirac Lagrangian

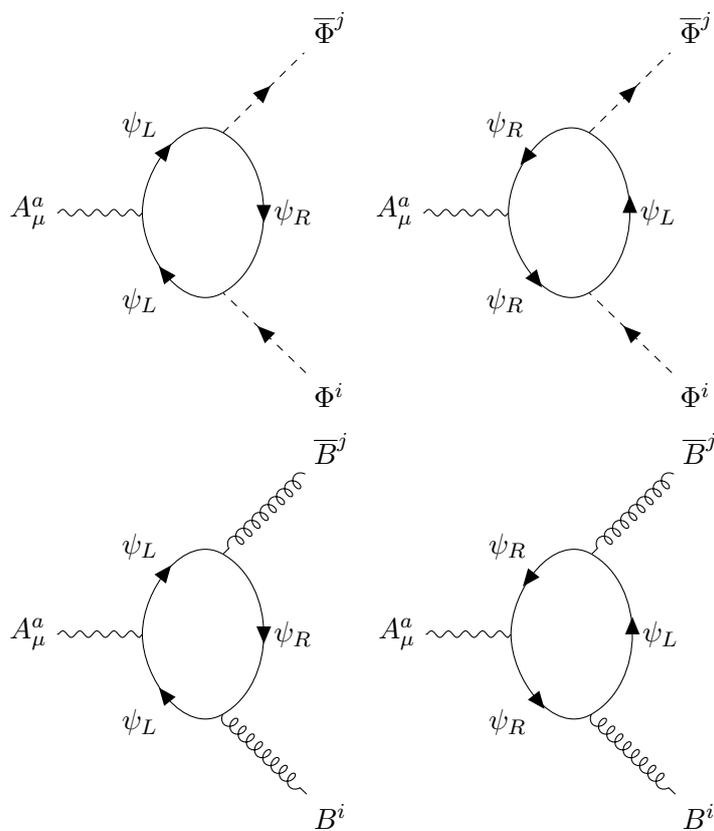
$$\mathcal{L} = i \overline{(\psi)} \tilde{D} \psi, \tag{4.1}$$

where $\tilde{D} = \chi d + \tilde{A}$, and \tilde{A} is our new superchiral superconnection (2.7). The renormalization of the wave function upon inclusion of a Fermion loop as above gives the well known propagator of the scalars and the vectors, as well as the Avdeev-Chizhov propagator [14] as derived in (3.6):

$$\begin{aligned} \mathcal{L}_\Phi &= -\kappa_{ij} g^{\mu\nu} \partial_\mu \bar{\Phi}^i \partial_\nu \Phi^j, \\ \mathcal{L}_A &= -\frac{1}{4} \kappa_{ab} g^{\mu\rho} g^{\nu\sigma} \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a \right) \left(\partial_\rho A_\sigma^b - \partial_\sigma A_\rho^b \right), \\ \mathcal{L}_B &= -\kappa_{ij} g^{\mu\nu} \partial^\alpha \bar{B}_{\alpha\mu}^i \partial^\beta B_{\beta\nu}^j, \end{aligned} \tag{4.2}$$

where the same κ_{ij} metrics controls the scalar (4.2) and tensor propagator (3.7). The vector metric $\kappa_{ab} = g_{ab} = \frac{1}{2} Tr(\lambda_a \lambda_b)$ is the only term that is purely algebraic and does not hesitate.

The interaction terms are given by the pole part of the Fermion loops with 3 external fields. The Feynman diagrams



induce the expected covariant derivative minimal coupling

$$\mathcal{L} = -D_\mu \bar{\Phi} D_\mu \Phi - D^\alpha \bar{B}_{\alpha\mu} D^\beta B_{\beta\mu} \tag{4.3}$$

with a caveat [6]: since the orientation of the loop is correlated with the chirality of the looping Fermions, the interaction term hidden in the definition of the covariant derivative

$$D_\mu \Phi_i = \partial_\mu \Phi_i + t_{aij} A_\mu^a \Phi^j, \tag{4.4}$$

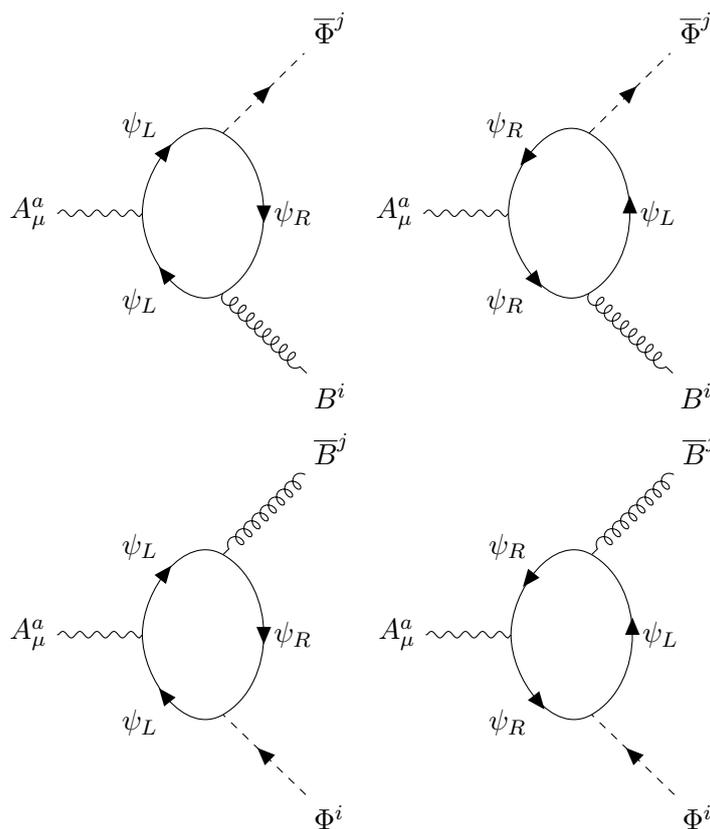
$$D^\alpha B_{i\alpha\mu} = \partial^\alpha B_{i\alpha\mu} + t_{aij} A^{a\alpha} B_{\alpha\mu}^j, \tag{4.5}$$

is given by the chiral trace

$$t_{aij} = Tr((1 + \chi) \lambda_a \lambda_i \lambda_j - (1 - \chi) \lambda_a \lambda_j \lambda_i) = Tr(\lambda_a [\lambda_i, \lambda_j]) + STR(\lambda_a \{\lambda_i, \lambda_j\}). \tag{4.6}$$

As found for the tensor propagators (3.7) and (4.2), the t_{aij} interaction terms (4.6) are neither fish nor meat. They hesitate between a Lie algebra trace and a Lie-Kac supertrace. They are not universal. They depend on the Fermion content of the model.

Another novelty is the apparition of a new mixed $AB\bar{\Phi}$ coupling, which must be considered as a genuine component of the superchiral minimal coupling, and is induced by the Feynman diagrams:



The tensorial structure of these counterterms is unusual because the propagator (3.6) of the $\bar{B}B$ field has a rather complex structure

$$P_{\mu\nu\alpha\beta}^+ k^\alpha g^{\beta\gamma} k^\delta P_{\gamma\delta\rho\sigma}^- / (k^2)^2. \tag{4.7}$$

When we perform the calculation, we get with the same strength as in (4.3) the interaction:

$$\mathcal{L}_{AB\Phi} = \frac{1}{4} t_{aij} F_{\mu\nu}^a \left(\bar{B}_{\mu\nu}^i \Phi^j + B_{\mu\nu}^i \bar{\Phi}^j \right). \tag{4.8}$$

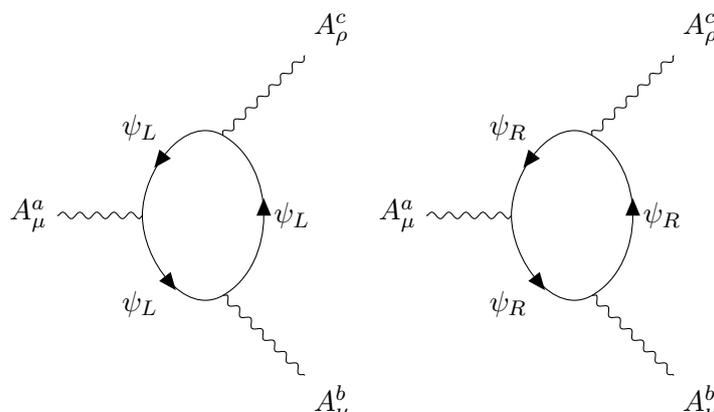
This is the only term which is Lorentz invariant and invariant under the Lie subalgebra. The coupling matrix t_{aij} is the same mixture (4.6) of trace and supertrace which appeared above in $D\Phi$ and DB , and is common to $A\bar{\Phi}\Phi$, $A\bar{B}B$ and $AB\bar{\Phi}$ because the Φ and the B fields have the same chiral interactions to the Fermions (2.9), (2.10). Regrouping all terms we get

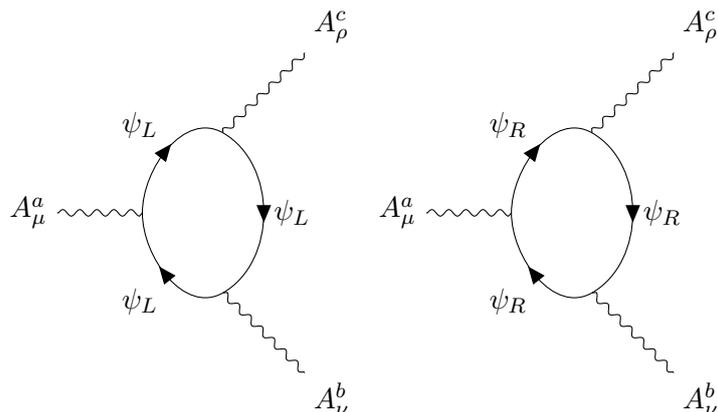
$$\mathcal{L}_{B\Phi} = -\kappa_{ij} D^\alpha \bar{B}_{\alpha\mu}^i D^\beta B_{\beta\mu}^j - \kappa_{ij} D^\alpha \bar{\Phi}^i D_\alpha \Phi^j - \frac{1}{4} t_{aij} F^{\alpha\mu\nu} \left(\bar{B}_{\mu\nu}^i \Phi^j + B_{\mu\nu}^i \bar{\Phi}^j \right). \tag{4.9}$$

The interesting point is that the F coupling cannot be freely adjusted. It comes as a consequence of the \tilde{D} coupling of all the connection fields to the Fermion and should be considered as an indispensable part of the minimal coupling of the Avdeev Chizhov fields. The same coupling appears in (2.11) as part of the classic Lagrangian \check{F}^2

5 The Adler-Bell-Jackiw vector anomaly viewed as superalgebraic

The main surprise of the previous calculations is that the theory seems to hesitate between a Lie algebra and a Lie superalgebra structure. The scalar propagator κ_{ij} (3.7) and the vector-scalar or vector-tensor vertex t_{aij} (4.6) contain a Lie algebra and a Lie superalgebra tensor, which cannot both be well defined at the same time. But *a posteriori*, this is not so surprising; this situation is actually very well known in physics. If we compute just as before the chiral Fermion loop contributions to the triple vector interaction:





we also obtain two types of terms:

$$\begin{aligned} Z_f &= Tr(\lambda_a [\lambda_b, \lambda_c]) A^{a\mu} A^{b\nu} \partial_\mu A_\nu^c, \\ d_{abc} &= STR(\lambda_a \{\lambda_b, \lambda_c\}). \end{aligned} \tag{5.1}$$

The $f_{abc} = Tr(\lambda_a [\lambda_b, \lambda_c])$ term is the expected counterterm to the Lie algebra triple vector vertex contained in the classic Yang-Mills Lagrangian $Tr(F^2)$. The $d_{abc} = STR(\lambda_a \{\lambda_b, \lambda_c\})$ term is the surprise Adler-Bell-Jackiw [7, 8] anomaly coming from the measure of the chiral Fermions (see for example chapter 5 of Bilal lectures [20]), where the supertrace is defined as the trace over the left Fermions minus the trace over the right Fermions. Using the superchirality condition (1.1), we can reinterpret this helicity supertrace in the sense of Hermann Weyl (B.1), as the internal superalgebra supertrace in the sense of Kac (1.2) and (A.1), and identify the vector anomaly with the even part of the rank-3 super-Casimir operator (A.7) of the superalgebra. The role of the Hodge dual in the Adler anomaly is also consistent with our superchiral condition (1.1) which correlates the chirality of the spinor bundle with the Hodge duality of the exterior bundle (1.3).

To conclude, the triple-vector vertex (5.1) also hesitates between a Lie algebra and a Lie superalgebra structure. The Adler-Bell-Jackiw anomalous term (5.1) is superalgebraic in nature and cancels out if the supertrace of the Casimir of rank 3 (A.7) of the Lie subalgebra vanishes.

The Fermion loop counterterms to the quartic vertices A^4 , $A^2\bar{\Phi}\Phi$, $A^2\bar{B}B$, $A^2\bar{\Phi}B$, $A^2\bar{B}\Phi$ also contain anomalies, but they automatically follow the structure of the cubic terms because of the Lie algebra Ward identities. For example the classic A^4 vertices are the complements of the A^3 vector terms in the classic Yang-Mills Lagrangian $Tr(F^2)$. The $(A^2\bar{B}\Phi)$ counterterm is the complement of the $(A\bar{B}\Phi)$ term in the $(F\{\bar{B}, \Phi\})$ Lagrangian. The quartic potentials $\bar{\Phi}^2\Phi^2$, $\bar{\Phi}\Phi\bar{B}B$ and \bar{B}^2B^2 remain to be studied.

6 Classification of the anomaly-free superchiral supeconnections

We have identified three obstructions to the construction of the quantum field theory: (3.7), (4.6) and (5.1). We wish to show here that these hesitations between trace and supertrace are resolved in many superchiral models.

Consider first the scalar anomalies. Since the trace operator is invariant under circular permutation, we can use the closure relation (A.3) of the superalgebra to rewrite the trace term in (3.7) as

$$Tr(\lambda_i \lambda_j) = Tr(\lambda_j \lambda_i) = \frac{1}{2} Tr(\{\lambda_i, \lambda_j\}) = \frac{1}{2} d_{ij}^a Tr(\lambda_a). \quad (6.1)$$

In the same way, we can rewrite (4.6) as

$$\begin{aligned} Tr(\lambda_a [\lambda_i, \lambda_j]) &= Tr(\lambda_a \lambda_i \lambda_j - \lambda_a \lambda_j \lambda_i) \\ &= Tr(\lambda_i \lambda_j \lambda_a - \lambda_i \lambda_a \lambda_j) = -Tr(\lambda_i [\lambda_a, \lambda_j]) \\ &= -f_{aj}^k Tr(\lambda_i \lambda_k) = -\frac{1}{2} d_{ik}^b f_{aj}^k Tr(\lambda_b). \end{aligned} \quad (6.2)$$

Hence if all the even generators satisfy the constraint

$$Tr(\lambda_a) = 0, \quad (6.3)$$

the theory is superalgebraic: the propagators of the scalars and of the Avdeev Chizhov tensors are controlled by the odd part of the super-Killing metric $\kappa_{ij} = \frac{1}{2} STr(\lambda_i \lambda_j)$ and their interactions with the vectors are governed by the symmetric structure constants of the superalgebra $t_{aij} = d_{aij}$. For U(1) factors, this constraint is non trivial.

Consider now the vector anomaly (5.1). The simple Lie algebras are of type A, B, C, D, E and F. Among those, only $A_m = SU(m+1)$, $m \geq 2$, including $A_3 = D_3 = SO(6) = SU(4)$, admit a Casimir of rank 3. In addition, we can have a U(1) algebra, denoted Y, which generates two supplementary Casimirs of rank 3: Y^3 and YC_2 where C_2 is a rank 2 Casimir of any other Lie algebra present in the model. A superchiral model associated to the simple superalgebras $G(3)$ or $OSp(m/n)$ with $m \geq 7$ has no Casimir of rank 3 and no U(1) factor, so it cannot have a vector anomaly. As all its generators are traceless, it has no scalar anomaly either. Therefore the model is superalgebraic and anomaly-free. It is nevertheless chiral whenever non-Abelian charges of the left and right Fermions differ.

7 Anomaly cancellation in the superchiral SU(2/1) model of leptons and quarks

It is also possible to cancel the scalar anomalies by combining several irreducible representations. For example, in the SU(2/1) model of the electroweak interactions, the superchirality condition (1.1) implies that SU(2) only acts on the left doublets (2.8). The cumulated hypercharge of the {right-electron/(left-electron, left-neutrino)} triplet $Tr(Y) = -4$ ([6] appendix B) is compensated by the cumulated hypercharge of the 3 colored (up, down) quarks quadruplets $Tr(Y) = 3 * 4/3$ ([6] appendix D). This is equivalent to the observation that the electric charge of the hydrogen atom (1 electron plus three *uud* quarks) vanishes. Using (6.1) and (6.2), the scalar anomalies (3.7) and (4.6) cancel out. As found in 1972 by Bouchiat, Iliopoulos and Meyer (BIM [9]), the four vector anomalies Y^3 , $Y SU(2)^2$, $Y SU(3)^2$ and $SU(3)^3$ (5.1) also cancel out in the standard model, separately for each family. Indeed, the complete rank 3 super-Casimir tensor cancels for each family, removing any potential measure anomaly $STr(\lambda_a [\lambda_i, \lambda_j])$ in the $A\bar{B}B$ and $A\bar{B}\Phi$ triangle diagrams.

8 Discussion

The concept of a superconnection defined as an odd polyform, a linear combination of exterior-forms of all degrees valued in a Lie superalgebra (2.2), was first introduced by Thierry-Mieg and Ne’eman in 1982 [10] in terms of the primitive forms (ϕ, a, b, c, e) as

$$\tilde{A} = (\phi + b + e)^i \lambda_i + (a + c)^a \lambda_a, \tag{8.1}$$

and then by Quillen and Mathai in their seminal papers [11, 12] and recently modified in [13]. In Quillen [11, 12], the covariant differential is defined as $\tilde{D} = d + A + L$, where $L = L^i \lambda_i$ is as for us a mixed exterior-form of even degree valued in the odd module of the superalgebra. But because L must be odd relative to the differential calculus to ensure that the curvature $\tilde{F} = \tilde{D}\tilde{D}$ defines a linear map, Quillen assumes that the components L^i of L are valued in another graded algebra which anticommute with the exterior-forms. The difficulty is that these partially anticommuting L^i cannot be represented in quantum field theory by commuting scalar fields. This is probably why the works of Ne’eman and Sternberg [17] or of the Marseille-Mainz group (see for example [18, 19]) who have all adopted the Quillen formalism, stop short of the quantum theory. In our construction [13], the components ϕ^i of $\phi = \phi^i \lambda_i$ are just ordinary commuting functions. Nevertheless ϕ is odd with respect to our differential calculus as requested by Quillen [11], because the λ_i matrices anticommute with the chirality χ (2.1) which decorates our exterior differential $\tilde{d} = \chi d$ (2.2). As a result, the commuting ϕ^i can be represented by Bose scalars and we can develop a quantum field theory formalism as here. This modifies the calculation of the superconnection cohomology [12] which should be reexamined and we conjecture that the Adler-Bell-Jackiw quantum anomalies play a role as obstructions in this purely geometrical context. However, an inconvenience of the superconnection formalism is the excessive number of component fields.

The new self-dual superchiral constraints $\tilde{A} = * \tilde{A} \chi$ introduced in the present work as equation (1.1) provides a subtle way to eliminate the higher forms. The Hodge duality conspires with the chiral properties of the Pauli matrices such that the action of the Dirac operator is focused on a single chirality (2.6). If \tilde{A} is self-dual, it only absorbs left Fermions, if it is anti-self-dual, it only absorbs right Fermions. Coupling the Hodge duality signs (C.1) with the superalgebra supertrace operator χ , defined as $Str(\dots) = Tr(\chi \dots)$ ((1.2) and appendix A), links the spinor bundle (appendix B) to the exterior bundle (appendix C) and focuses the action of the superconnection (2.8)–(2.10) on the CP positive Fermions: $\tilde{A} = \tilde{A} (1 \pm \chi)(1 \mp \gamma_5)/4$. The constraint also eliminates the primitive fields (2.2) in terms of the self-chiral fields (Φ, A, B) (2.8)–(2.10). Applied to the $SU(2/1)$ superalgebraic model of the elementary particles, we recover exactly the vector interactions postulated *ex nihilo* by Ne’eman [1] and Fairlie [2] and the scalar interactions postulated in [6]. The superchirality focuses, as observed in Nature, the action of the $SU(2)$ vector Bosons on the left leptons and quarks (section 7).

Because the left and right leptons and quarks do not carry the same $SU(2)$ or $U(1)$ charges, the Yang-Mills sector is subject to the Adler-Bell-Jackiw anomaly [7, 8] which curiously (section 5) involve superalgebraic symmetric structure constant $d_{abc} = STr(\lambda_a \{\lambda_b, \lambda_c\})$

involving three Lie algebra even indices (abc). Our new interpretation of this well known result is that the chiral supertrace, in the sense of Adler-Bell-Jackiw: left minus right Fermions, is equivalent to the charge supertrace in the sense of Kac (1.2), so the supertrace terms in (5.1) matches the definition of the even part of the super-Casimir (A.7). In a similar way, the counterterm to the scalar and tensor interactions (4.6) involve algebraic antisymmetric structure constants $f_{aij} = Tr(\lambda_a [\lambda_i, \lambda_j])$ although (ij) are odd indices. Our result is that all these unwanted terms cancel out if all the even generators are traceless (6.3). This is true in particular in the standard model of the fundamental interactions when we apply the BIM [9] mechanism whereby each lepton family is balanced by its pair of quarks (section 7). The complete rank-3 super-Casimir tensor (A.7) vanishes and only the representation independent universal couplings $f_{abc} = Tr(\lambda_a [\lambda_b, \lambda_c])$ and $d_{aij} = STr(\lambda_a \{\lambda_i, \lambda_j\})$ survive. The resulting scalar-vector-tensor theory is therefore, at one-loop, superalgebraic and anomaly-free.

Another very interesting consequence of our superchiral structure is the induction by the Fermion loops of a new scalar-vector-tensor triple interaction (4.8) which reproduces, if and only if we apply the BIM mechanism, the structure of the square of the geometric supercurvature (2.11). Once again, Differential Geometry and Quantum Field Theory agree, conditional on the elimination of the Adler-Bell-Jackiw anomaly.

These results are tantalizing from a theoretical point of view, yet very surprising: the coupling (2.8)–(2.10) of the vectors $\chi\lambda_a$ and of the scalars $\lambda_i(1 \pm \chi)$ are outside the naive superalgebra generated by the (λ_a, λ_i) matrices (appendix A); the couplings to the Fermions are all even (they transform Fermions into Fermions, not Fermions into Bosons as in Wess-Zumino supersymmetry); yet the signs induced by the helicity of the Fermion propagators restore the superalgebraic structure (4.6), if and only if the model is anomaly-free. The expected minimal coupling of the vectors to the scalars and the tensors, via the covariant derivatives, is also necessarily completed by a new scalar-vector-tensor vertex (4.8) which modifies the asymptotic behavior of the coupling of the scalars and tensors to the Fermions. A deeper understanding of these equations must be possible.

These results are also curious from a phenomenological point of view, even if the superalgebraic structure is a direct consequence of the experimentally verified BIM mechanism whereby the chiral quantum anomalies are canceled by the balances of the leptons against the quarks. However, the model is highly constrained and offers no choice. The field content is defined by the differential geometry, the dynamics are induced by the Fermion loops, and there are no free parameters, except that the Cabbibo-Kobayashi-Maskawa angles can be understood as specifying the details of the 3-generations indecomposable representations of $SU(2/1)$ (see [18, 19] and [6], appendix H).

Many problems remain. Having established the self interactions of the Boson fields (4.9), one has to examine if the theory is renormalizable and in particular if the counterterms involving Boson loops have the correct Lorentz structure, which seems likely, and the correct algebraic structure, which is non-trivial as we only have Lie-algebra Ward identities. The scalar potential has to be evaluated. The symmetry breaking pattern of the model must be studied. Finally, the crucial open question is the eventual existence of a symmetry associated to the odd-generators of the superalgebra.

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A Definition of a chiral superalgebra

For completeness, this section is copied from appendix A of [6].

Let us define, using the notations of [6], a chiral-superalgebra as a finite dimensional basic classical Lie-Kac superalgebra [3], graded by chirality. For example, we could take a superalgebra of type $SU(m/n)$, or $OSp(m/2n)$ or a product of Lie algebras and superalgebras like the $SU(2/1)SU(3)$ superalgebra of the standard model.

The superalgebra acts on a finite dimensional space of massless Fermion states graded by their helicity. The chirality matrix χ is diagonal, with eigenvalue 1 on the left Fermions and -1 on the right Fermions. It defines the supertrace

$$STr(\dots) = Tr(\chi \dots). \quad (\text{A.1})$$

Each generator is represented by a finite dimensional matrix of complex numbers (we do not need anticommuting Grassmann numbers). The even generators are denoted λ_a and the odd generators λ_i . χ commutes with the λ_a and anticommutes with the λ_i

$$[\chi, \lambda_a]_- = \{\chi, \lambda_i\}_+ = 0. \quad (\text{A.2})$$

The λ matrices close under (anti)-commutation

$$[\lambda_a, \lambda_b]_- = f_{ab}^c \lambda_c, \quad [\lambda_a, \lambda_i]_- = f_{ai}^j \lambda_j, \quad \{\lambda_i, \lambda_j\}_+ = d_{ij}^a \lambda_a, \quad (\text{A.3})$$

and satisfy the super-Jacobi relation with 3 cyclic permuted terms:

$$(-1)^{AC} \{\lambda_A, \{\lambda_B, \lambda_C\}\} + (-1)^{BA} \{\lambda_B, \{\lambda_C, \lambda_A\}\} + (-1)^{CB} \{\lambda_C, \{\lambda_A, \lambda_B\}\} = 0. \quad (\text{A.4})$$

The quadratic Casimir tensor (g_{ab}, g_{ij}) , also called the super-Killing metric, is defined as

$$\begin{aligned} g_{ab} &= \frac{1}{2} STr(\lambda_a \lambda_b), \\ g_{ij} &= \frac{1}{2} STr(\lambda_i \lambda_j). \end{aligned} \quad (\text{A.5})$$

The even part g_{ab} of the metric is as usual symmetric, but because the odd generators anticommute (A.2) with the chirality hidden in the supertrace (A.1), its odd part g_{ij} is antisymmetric. The structure constants can be recovered from the supertrace of a product of 3 matrices

$$\begin{aligned} f_{abc} &= g_{ae} f_{bc}^e = \frac{1}{2} STr(\lambda_a [\lambda_b, \lambda_c]_-), \\ d_{aij} &= g_{ae} d_{ij}^e = \frac{1}{2} STr(\lambda_a \{\lambda_i, \lambda_j\}_+). \end{aligned} \quad (\text{A.6})$$

The cubic Casimir tensor is defined as

$$\begin{aligned} C_{abc} &= \frac{1}{2} STr(\lambda_a \{\lambda_b, \lambda_c\}_+), \\ C_{aij} &= \frac{1}{2} STr(\lambda_a [\lambda_i, \lambda_j]_-). \end{aligned} \tag{A.7}$$

The Casimirs use the ‘wrong’ type of commutator, otherwise, using equation (A.3), they could be simplified. We have $g_{ai} = C_{abi} = C_{ijk} = 0$ since the diagonal elements of the product of an odd number of odd matrices necessarily vanish. Using these tensors, we can construct the super-Casimir operators

$$K_2 = g^{AB} \lambda_A \lambda_B, \quad K_3 = C^{ABC} \lambda_A \lambda_B \lambda_C, \tag{A.8}$$

where the upper index metric g^{AB} is the inverse of the lower metric g_{AB} , summation over the repeated indices is implied and ranges over even and odd values $A, B = a, b, \dots, i, j, \dots$, and the indices of C^{ABC} are raised using g^{AB} . The Casimir operators K_2 and K_3 commute with all the generators of the superalgebra. In an irreducible representation, they are represented by a multiple of the identity matrix. In $SU(2/1)$, which has rank 2, they form a basis of its enveloping superalgebra.

B Pauli matrices

Because $SO(4)$ is isomorphic to $SU(2) + SU(2)$, the $SO(1, 3)$ spinors split into left and right doublets. The projectors are traditionally represented as

$$P_L = \frac{1}{2}(1 - \gamma_5), \quad P_R = \frac{1}{2}(1 + \gamma_5) \tag{B.1}$$

and we have

$$P_L + P_R = 1, \quad P_L P_R = P_R P_L = 0, \quad P_L P_L = P_L, \quad P_R P_R = P_R. \tag{B.2}$$

The spin-one Pauli matrices σ map the right spinors on the left spinors and the $\bar{\sigma}$ matrices map the left spinors on the right spinors.

$$\sigma = P_L \sigma P_R, \quad \bar{\sigma} = P_R \bar{\sigma} P_L. \tag{B.3}$$

In Minkowski space they can be represented as:

$$\begin{aligned} \sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \bar{\sigma}_0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & \bar{\sigma}_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \bar{\sigma}_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \bar{\sigma}_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{B.4}$$

They are Hermitian and satisfy the chiral Clifford-Weyl relations

$$\begin{aligned} \sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu &= 2g_{\mu\nu} P_L, \\ \bar{\sigma}_\mu \sigma_\nu + \bar{\sigma}_\nu \sigma_\mu &= 2g_{\mu\nu} P_R, \end{aligned} \tag{B.5}$$

where $g_{\mu\nu} = g^{\mu\nu}$ denotes the diagonal Minkowski metric $(-1, 1, 1, 1)$. Importantly, if we compute the trace of the product of four σ matrices we find a tensor with mixed symmetry

$$\begin{aligned} Tr(\sigma_\mu \bar{\sigma}_\nu \sigma_\rho \bar{\sigma}_\sigma) &= 2(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho} + i \epsilon_{\mu\nu\rho\sigma}), \\ Tr(\bar{\sigma}_\mu \sigma_\nu \bar{\sigma}_\rho \sigma_\sigma) &= 2(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho} - i \epsilon_{\mu\nu\rho\sigma}), \end{aligned} \tag{B.6}$$

where the g terms are symmetric, and ϵ is fully antisymmetric in $\mu\nu\rho\sigma$ with $\epsilon_{0123} = 1$, hence $\epsilon^{0123} = -1$ and $\epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = -24$. If we now consider the antisymmetric products of several Pauli matrices, we can derive by inspection several interesting identities:

$$\begin{aligned} i \epsilon_{\mu\nu\rho\sigma} \bar{\sigma}^\rho \sigma^\sigma &= \bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu, & i \epsilon_{\mu\nu\rho\sigma} \sigma^\rho \bar{\sigma}^\sigma &= -\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu, \\ i \epsilon_{\mu\nu\rho\sigma} \sigma^\nu \bar{\sigma}^\rho \sigma^\sigma &= -6 \sigma_\mu, & i \epsilon_{\mu\nu\rho\sigma} \bar{\sigma}^\nu \sigma^\rho \bar{\sigma}^\sigma &= 6 \bar{\sigma}_\mu, \\ i \epsilon_{\mu\nu\rho\sigma} \sigma^\mu \bar{\sigma}^\nu \sigma^\rho \bar{\sigma}^\sigma &= 24 P_L, & i \epsilon_{\mu\nu\rho\sigma} \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho \sigma^\sigma &= -24 P_R. \end{aligned} \tag{B.7}$$

Two other useful identities are the contractions

$$\begin{aligned} g_{\mu\nu} \sigma^\mu \bar{\sigma}^\alpha \sigma^\nu &= -2\sigma^\alpha, & g_{\mu\nu} \sigma^\mu (\bar{\sigma}^\alpha \sigma^\beta - \bar{\sigma}^\beta \sigma^\alpha) \bar{\sigma}^\nu &= 0, \\ g_{\mu\nu} \bar{\sigma}^\mu \sigma^\alpha \bar{\sigma}^\nu &= -2\bar{\sigma}^\alpha, & g_{\mu\nu} \bar{\sigma}^\mu (\sigma^\alpha \bar{\sigma}^\beta - \sigma^\beta \bar{\sigma}^\alpha) \sigma^\nu &= 0. \end{aligned} \tag{B.8}$$

As a final example, the trace of 6 Pauli matrices contains 15 triple g contractions, plus 15 terms in $i g \epsilon$, alternating the signs and conjugating depending on the nature of the epsilon contracted Pauli matrices ($\epsilon \sigma \bar{\sigma} \sigma \bar{\sigma}$ or $\epsilon \bar{\sigma} \sigma \bar{\sigma} \sigma$). The identity looks like

$$Tr(\sigma_\lambda \bar{\sigma}_\mu \sigma_\nu \bar{\sigma}_\rho \sigma_\sigma \bar{\sigma}_\tau) = 2(g_{\lambda\mu} g_{\nu\rho} g_{\sigma\tau} - g_{\lambda\nu} g_{\mu\rho} g_{\sigma\tau} + i g_{\lambda\mu} \epsilon_{\nu\rho\sigma\tau} - i g_{\lambda\nu} \epsilon_{\mu\rho\sigma\tau} + \dots) \tag{B.9}$$

The trace of 8 Pauli matrices has 105 term in g^4 , 210 terms in $g g \epsilon$, and 70 terms in $\epsilon \epsilon$. We recommend using a computer!

C Hodge duality

In Minkowski space, with signature $(-+++)$, we can define the Hodge dual of a p -form ($p = 0, 1, 2, 3, 4$) as

$$\begin{aligned} *(1) &= +\frac{i}{24} \epsilon_{\mu\nu\rho\sigma} dx^\mu dx^\nu dx^\rho dx^\sigma, \\ *(dx_\mu) &= +\frac{i}{6} \epsilon_{\mu\nu\rho\sigma} dx^\nu dx^\rho dx^\sigma, \\ *(dx_\mu dx_\nu) &= -\frac{i}{2} \epsilon_{\mu\nu\rho\sigma} dx^\rho dx^\sigma, \\ *(dx_\mu dx_\nu dx_\rho) &= -i \epsilon_{\mu\nu\rho\sigma} dx^\sigma, \\ *(dx_\mu dx_\nu dx_\rho dx_\sigma) &= +i \epsilon_{\mu\nu\rho\sigma}, \end{aligned} \tag{C.1}$$

where ϵ is the totally antisymmetric symbol with $\epsilon_{0123} = 1$ and the (exterior) differentials dx^α anticommute with each other. Our sign choices, the $\pm i$ factors, are explained below. The definition extends by linearity to a general polyform of mixed degree. By inspection,

we can verify that $** = 1$. Since the dual of a 2-form is a 2-form, we can define the self-dual, and the anti-self-dual, projectors

$$(P^\pm)_{\mu\nu\rho\sigma} = \frac{1}{4}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} \mp i \epsilon_{\mu\nu\rho\sigma}), \quad (\text{C.2})$$

and verify by inspection that they split the space of 2-forms

$$(P^+)^2 = P^+, \quad (P^-)^2 = P^-, \quad P^+P^- = P^-P^+ = 0, \quad P^+ + P^- = 1. \quad (\text{C.3})$$

Let us now consider a superconnection polyform \tilde{A} mixing exterior-forms of all degrees:

$$\tilde{A} = \phi + a_\mu dx^\mu + \frac{1}{2}b_{\mu\nu}dx^\mu dx^\nu + \frac{1}{6}c_{\mu\nu\rho}dx^\mu dx^\nu dx^\rho + \frac{1}{24}e \epsilon_{\mu\nu\rho\sigma}dx^\mu dx^\nu dx^\rho dx^\sigma, \quad (\text{C.4})$$

where we wrote the 4-form as $e_{\mu\nu\rho\sigma} = e \epsilon_{\mu\nu\rho\sigma}$. Using (C.1), its dual reads

$$\begin{aligned} * \tilde{A} = & -ie - \frac{i}{6}c^{\mu\nu\rho}\epsilon_{\mu\nu\rho\sigma} dx^\sigma - \frac{i}{4}b^{\mu\nu}\epsilon_{\mu\nu\rho\sigma} dx^\rho dx^\sigma \\ & + \frac{i}{6}a^\mu\epsilon_{\mu\nu\rho\sigma} dx^\nu dx^\rho dx^\sigma + \frac{i}{24}\phi\epsilon_{\mu\nu\rho\sigma} dx^\mu dx^\nu dx^\rho dx^\sigma. \end{aligned} \quad (\text{C.5})$$

In components, the self-duality condition $\tilde{A} = * \tilde{A}$ expands as

$$\begin{aligned} \phi = -ie, \quad a_\mu = (*c)_\mu &= -\frac{i}{6}\epsilon_{\mu\nu\rho\sigma} c^{\nu\rho\sigma}, \\ b_{\mu\nu} = (*b)_{\mu\nu} &= -\frac{i}{2}\epsilon_{\mu\nu\rho\sigma} b^{\rho\sigma}, \\ e = i\phi, \quad c_{\mu\nu\rho} = (*a)_{\mu\nu\rho} &= -i\epsilon_{\mu\nu\rho\sigma} a^\sigma. \end{aligned} \quad (\text{C.6})$$

By inspection, these equations are consistent and allow to eliminate c and e . Consider now the part of the Dirac operator associated to the superconnection,

$$\begin{aligned} \tilde{\mathcal{A}} = & \phi + a_\mu(\sigma^\mu + \bar{\sigma}^\mu) + \frac{1}{2}b_{\mu\nu}(\bar{\sigma}^\mu\sigma^\nu + \sigma^\mu\bar{\sigma}^\nu) + \frac{1}{6}c_{\mu\nu\rho}(\sigma^\mu\bar{\sigma}^\nu\sigma^\rho + \bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho) \\ & + \frac{1}{24}e \epsilon_{\mu\nu\rho\sigma}(\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho\sigma^\sigma + \sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\sigma). \end{aligned} \quad (\text{C.7})$$

Using the Pauli matrices identities (A.7), this equation can be rewritten as

$$\begin{aligned} \tilde{\mathcal{A}} = & (\phi + ie)P_R + (\phi - ie)P_L \\ & + (a_\mu - \frac{i}{6}c^{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma\mu})\sigma^\mu + (a_\mu + \frac{i}{6}c^{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma\mu})\bar{\sigma}^\mu \\ & + \frac{1}{4}(b_{\mu\nu} + \frac{i}{2}b^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu})\bar{\sigma}^\mu\sigma^\nu + \frac{1}{4}(b_{\mu\nu} - \frac{i}{2}b^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu})\sigma^\mu\bar{\sigma}^\nu. \end{aligned} \quad (\text{C.8})$$

Remarkably, we can recognize in the terms acting on the right Fermions the self-duality conditions (C.6). Therefore, if \tilde{A} is self-dual, then $\tilde{\mathcal{A}}$ annihilates the right Fermions:

$$\begin{aligned} \tilde{\mathcal{A}} = * \tilde{\mathcal{A}} \Rightarrow \tilde{\mathcal{A}} \psi_R &= 0, \\ \tilde{\mathcal{A}} \psi_L &= (2\phi + 2a_\mu\bar{\sigma}^\mu + \frac{1}{4}(b_{\mu\nu} + *b_{\mu\nu})\sigma^\mu\bar{\sigma}^\nu) \psi_L. \end{aligned} \quad (\text{C.9})$$

Mutatis mutandis, if \tilde{A} is anti-self-dual, \tilde{A} annihilates the left Fermions.

$$\begin{aligned} \tilde{A} &= -^* \tilde{A} \Rightarrow \tilde{A} \psi_L = 0, \\ \tilde{A} \psi_R &= (2\phi + 2a_\mu \sigma^\mu + \frac{1}{4}(b_{\mu\nu} - ^* b_{\mu\nu}) \bar{\sigma}^\mu \sigma^\nu) \psi_R. \end{aligned} \tag{C.10}$$

Let us now consider the freedom of the sign choices in the definition of the Hodge dual (C.1). There are 5 equations. Asserting that the Hodge duality is an involution (** = 1), the last 2 signs, for the 3 and 4-forms, are related to the first 2 choices for the 0 and 1-forms and the i factors are needed in Minkowski space-time (they would disappear in Euclidean space). We remain with 3 optional choices. The choice for the 2-form selects which of $b \pm^* b$ acts on the left spinors, i.e. is proportional to $\sigma \bar{\sigma}$. In quantum field theory this translate into the orientation of the $\bar{B}B$ propagator: i.e. choosing whether B is emitted by a left or by a right spinor (section 3). *In fine*, it is a choice of names. Similarly, the choice of the signs for the 0 and 4-forms relates to the orientation of the $\bar{\Phi}\Phi$ propagator (section 3 and 4). Finally, the choice of sign for the 1 and 3-forms, joined to the superchirality constraint $\tilde{A} = ^* \tilde{A} \chi$, correlates the superalgebra to the spinor helicities. It is a phenomenological choice. It tells us, in the $SU(m/n)$ case, whether the left or the right spinors interact with the $SU(m)$ vectors, or reciprocally, whether the left spinors interact with the $SU(m)$ or the $SU(n)$ vectors. We chose our signs so that in the lepton/quark model [6], the left electrons and quarks are $SU(2)$ doublets and emit Φ and B fields.

D The ‘t Hooft integrals

To compute the pole part of the 1-loop divergent Feynman integrals, we need a regularization scheme. Following, but not exactly, ‘t Hooft and Veltman diagrammar [21], we want to define dimensional regularization in an axiomatic way just from dimensional analysis and the linearity of the integrand, and of the integration variable. If we denote $I(f(k))$ a loop integral of a function f of the momentum k , we postulate that

1. UV convergent integrals vanish.
2. The integral I is a linear operator.

$$I(af + bg) = aI(f) + bI(g), \quad a, b \text{ in } \mathbb{C}. \tag{D.1}$$

3. We can freely perform a linear change of integration variable

$$\int d^d k f(k) = \int d^d (ak + b) f(ak + b). \tag{D.2}$$

4. The integral is d dimensional

$$\int d^d (ak) f(k) = a^d \int d^d k f(k). \tag{D.3}$$

It follows that the integral of a pure power rescales as

$$I(k^p) = \int d^d k k^p = \int d^d(ak) (ak)^p = a^{d+p} \int d^d k k^p = a^{d+p} I(k^p). \quad (\text{D.4})$$

Hence, if $d + p \neq 0$, the integral of a power of k vanishes. In negative dimension $d = -1$, we recover the Berezin integral

$$\int d^{-1} \theta \theta = I_1. \quad (\text{D.5})$$

In 4 dimensions we obtain the ‘t Hooft integral

$$\int d^4 k \frac{1}{(k^2)^2} = I_1. \quad (\text{D.6})$$

In the renormalization procedure, we treat I_1 as a non standard number. In any equation, all terms linear in I_1 must be isolated and treated separately from the other terms just like we treat the imaginary part of a complex equation, or the term proportional to $\sqrt{5}$ in $\sqrt{5}\mathbb{Q}$ arithmetic. If in the Lagrangian, the global sum of all the counterterms proportional to I_1 cancels out, the theory is called renormalizable.

Let us now consider the wave function renormalization of a Fermion. Factorizing out the Dirac matrix and the sign conventions which play no role in the present discussion we get an integral of the form

$$Z(p_\mu) = \int d^4 k \frac{k_\mu}{k^2 (k+p)^2}. \quad (\text{D.7})$$

To evaluate this integral, we complete the square in the numerator

$$\begin{aligned} 0 &= \int d^4 k \frac{1}{k^2} = \int d^4 k \frac{(k+p)^2}{k^2 (k+p)^2} = \\ &= \int d^4 k \frac{k^2}{k^2 (k+p)^2} + 2p_\mu \int d^4 k \frac{k_\mu}{k^2 (k+p)^2} + p^2 \int d^4 k \frac{1}{k^2 (k+p)^2} = \\ &= 0 + 2 p_\mu Z(p_\mu) + p^2 I_1 \end{aligned} \quad (\text{D.8})$$

Hence, we conclude that

$$Z(p_\mu) = \int d^4 k \frac{k_\mu}{k^2 (k+p)^2} = -\frac{1}{2} p_\mu I_1. \quad (\text{D.9})$$

More generally, by dimensional analysis, we have 3 kinds of divergent diagrams, a) the Boson Fermion vertices and the 4 Bosons interactions which are independent of the external momenta; b) the triple Boson vertices and the Fermion propagators which are linear in the incoming momenta (p, q) ; c) the Boson propagators which are quadratic in p .

Let us first consider the log divergent integral, case a). By dimensional analysis, if the overall power of k under the integral is zero, they cannot depend on p or q and must be of the form

$$\int d^4 k \frac{k_a k_b k_c \dots k_n}{k^{n+4}} = \alpha (g_{ab} g_{c\dots} + g_{ac} g_{b\dots}) \quad (\text{D.10})$$

where the numerator is symmetric in $(abc\dots n)$, hence contains all possible pair contractions. The overall coefficient α is found by tracing all the free indices. One gets

$$\begin{aligned}
 \int d^4k \frac{k_a k_b}{k^6} &= \frac{1}{4} g_{ab}, \\
 \int d^4k \frac{k_a k_b k_c k_d}{k^8} &= \frac{1}{24} (g_{ab} g_{cd} + g_{ac} g_{bd} + g_{ad} g_{bc}), \\
 \int d^4k \frac{k_a k_b k_c k_d k_e k_f}{k^{10}} &= \frac{1}{192} (g_{ab} g_{cd} g_{ef} + g_{ac} g_{bd} g_{ef} + \dots \text{ (15 terms)}) \\
 \int d^4k \frac{k_a k_b k_c k_d k_e k_f k_g k_h}{k^{12}} &= \frac{1}{1920} (g_{ab} g_{cd} g_{ef} g_{gh} + \dots \text{ (105 terms)})
 \end{aligned}
 \tag{D.11}$$

Let us now consider the linear divergent integrals, case b). By dimensional analysis, the terms of the form $\int d^4k k^{2n-3}/k^{2a}(k+p)^{2b}(k+p+q)^{2c}$, $a+b+c=n$, must be of the form $(\alpha p + \beta q) I_1$. To find the coefficient (α, β) , we differentiate the integrand with respect to p and q and get

$$\begin{aligned}
 \int d^4k \frac{k_a}{k^2(k+p)^2} &= -\frac{1}{2} p_a, \\
 \int d^4k \frac{k_a k_b k_c}{k^2(k+p)^2(k+p+q)^2} &= -\frac{1}{12} ((2p+q)_a g_{bc} + (2p+q)_b g_{ca} + (2p+q)_c g_{ab}), \\
 \int d^4k \frac{k_a k_b k_c k_d k_e}{k^2(k+p)^2(k+p+q)^2(k+p+q+r)^2} &= -\frac{1}{96} ((3p+2q+r)_a g_{bc} g_{de} + \dots) \text{ (15 terms)},
 \end{aligned}
 \tag{D.12}$$

and so on. To compute the quadratic Boson propagators, we compute as in a Taylor series half the double derivative of the integrand and obtain

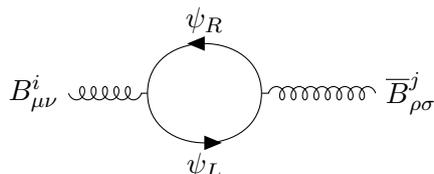
$$\int d^4k \frac{k_a k_b}{k^2(k+p)^2} = \frac{1}{3} \left(p_a p_b - \frac{1}{4} g_{ab} p^2 \right),
 \tag{D.13}$$

and so on. We verified all the coefficients with a dedicated C-program available on request.

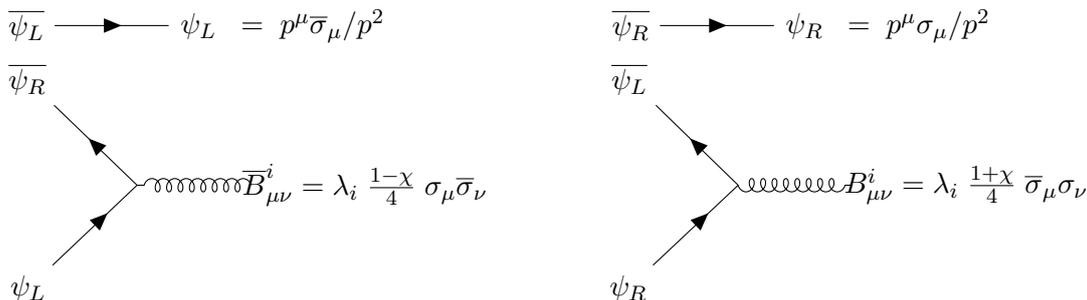
E Detailed calculation of the tensor propagator

In this appendix, we give in detail the quantum field theory evaluation of the pole-part of the B tensor 2-point function upon inclusion of a Fermion loop. The hope is to help the interested reader wishing to understand our notations and reproduce the results. These calculations are very delicate and feedback would be much appreciated.

We want to compute the pole-part of the diagram



assuming the Feynman rules



Notice that we do not need to know in advance the Feynman rule for the propagator of the B field to compute its counterterm. We find

$$Z_{\mu\nu\rho\sigma}^{ij} = \frac{1}{4} \int d^4k \frac{k^\alpha (k+p)^\beta}{k^2 (k+p)^2} \text{Tr}(\sigma_\alpha \overline{\sigma}_\mu \sigma_\nu \overline{\sigma}_\beta \sigma_\rho \overline{\sigma}_\sigma) \text{Tr}\left(\lambda^i \frac{1-\chi}{2} \lambda^j \frac{1+\chi}{2}\right). \quad (\text{E.1})$$

Since χ satisfies $\chi^2 = 1$ and anticommutes with the odd matrices λ_i and λ_j , the charge trace can be rewritten as (3.7):

$$\kappa_{ij} = \text{Tr}\left(\lambda_i \frac{1-\chi}{2} \lambda_j \frac{1+\chi}{2}\right) = \frac{1}{2} \text{Tr}(\lambda_i \lambda_j) + \frac{1}{2} \text{STr}(\lambda_i \lambda_j). \quad (\text{E.2})$$

As discussed in section 3, the κ_{ij} propagator hesitates between a Lie algebra *Trace* and a Kac superalgebra *Supertrace*. The ambiguity is only lifted in section 6 when the precise Fermion content of the model is taken into account.

Using (D.11) and (D.13), the pole-part of the integral over the momenta gives:

$$\int d^4k \frac{k_\alpha (k+p)_\beta}{k^2 (k+p)^2} = \frac{1}{3} \left(p_\alpha p_\beta - \frac{1}{4} g_{\alpha\beta} p^2 \right) - \frac{1}{2} p_\alpha p_\beta = -\frac{1}{6} p_\alpha p_\beta - \frac{1}{12} g_{\alpha\beta} p^2. \quad (\text{E.3})$$

Finally the trace over the six Pauli matrices is the most complicated term (B.9), but it can be simplified. Observe first that the term in $g_{\alpha\beta}$ in (E.3) can be dropped thanks to the last identity (B.8) and the fact that $B_{\mu\nu}$ is an antisymmetric tensor. In the same way the terms proportional to $g_{\mu\nu}$ and to $g_{\rho\sigma}$ in the trace of the six Pauli matrices, a generalization of (B.6), can be dropped and the terms proportional to the epsilon symbol can be dropped if they contain $\alpha\beta$ or can be absorbed by the $\overline{B}B$ fields if they contain $\mu\nu$ or $\rho\sigma$. We are left with 6 equivalent contractions, 4 coming from the ggg trace, and 2 coming from $g\epsilon$, compensating the factor $1/6$ coming from the loop integral (E.3). Finally

$$Z_{\mu\nu\rho\sigma}^{ij} = -g^{ij} P_{\mu\nu\alpha\beta}^- p^\alpha p^\gamma g^{\beta\delta} P_{\gamma\delta\rho\sigma}^+, \quad (\text{E.4})$$

our equation (4.7), which is equivalent to (3.6), as the P^\pm projectors (C.2) can be absorbed by the (\overline{B}, B) fields.

The calculation of the Feynman diagrams leading to (4.6) and (4.8) are analogous, with traces involving up to 8 Pauli matrices and a flurry of ϵ symbols which can be all reabsorbed in the self-duality of the $\overline{B}B$ fields. The loop integral generates several terms like in (E.3), summing up to the only Lorentz invariant contraction $F^{\mu\nu} B_{\mu\nu}$ which is equivalent to $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} B_{\rho\sigma}$ since B is self-dual.

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