Is there a nice theory of representations of inverse semigroups in special inverse algebras? A discussion paper

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1 Representations of inverse semigroups

Inverse semigroups generalise both groups and semilattices, and describe partial symmetries just as groups do for total symmetries; they also arise naturally in appropriately rich categories, essentially as monoids of partial automorphisms. Analogy with the group case suggests that the use of inverse semigroups will be assisted by the development of a workable representation theory. Moreover the representation theory of some operator algebras turns out to be linked with that of inverse semigroups [refs].

We do have one situation where, thanks to Boris Schein's work, the theory is well-developed: effective representations in the symmetric inverse monoid $\mathcal{I}_X$ decompose to a 'sum' of transitive ones, and every transitive one has an 'internal' description in terms of appropriately defined cosets of closed inverse subsemigroups [ref]. In [3] the authors introduced the categorical dual of $\mathcal{I}_X$, named as 'the' *dual symmetric inverse monoid*, $\mathcal{I}_X^*$, but also describable as the inverse semigroup of all *block multipermutations* of a set X, and as a semigroup of special binary relations with a variant multiplication. Maltcev [4] has independently discovered this semigroup as, apart from an extra zero, a maximal inverse subsemigroup of the composition (or partition) monoid, and so calls it the *inverse partition monoid*. Surely the protean character of this monoid argues for its importance. Maltcev also describes maximal subsemigroups of $\mathcal{I}_X^*$, its automorphisms and its transitive embedding in an $\mathcal{I}_Y$ [4].

In [3] the authors also found a dual of the Wagner-Preston embedding, and went on to distinguish some classes of inverse semigroups by their embedding properties in $\mathcal{I}_X^*$. There remains the problem of elaborating an analogue, for $\mathcal{I}_X^*$, of the 'classical' (Schein) theory. It is not at first clear what would be appropriate analogues of the concepts of 'effective' or 'transitive' for block multipermutations.

Consideration of this issue would also be a useful test case in developing theories for representation in partial automorphism monoids of entities such as (perhaps) graphs, modules, etc. The point is that many of these partial automorphism monoids (including prototypically $\mathcal{I}_X$ itself) are significantly richer in structure as a result of their categorical properties—they are actually *inverse algebras* (see later). So I contend that the representation question requires taking account of the properties of inverse algebras, and working with those with helpful properties. These remarks motivate the content of this discussion paper—to find appropriate generalizations of the features of the classical theory, and apply them where possible in other settings. This would be worthwhile if it were to prove more than a mere 'abstractification' of the classical theory. We begin by rehearsing some terminology and then move to considering the concept of an effective representation.

2 Atomistic inverse algebras

An *inverse algebra* $A = (A, \cdot, ^{-1}, \wedge)$ is an inverse monoid $(A, \cdot, ^{-1})$ in which the natural ordering is a semilattice order; equivalently, in which each element x possesses a maximum idempotent $f[x]$ beneath x in the natural order. Inverse algebras were introduced and elucidated in [1] and the reader is referred to that paper for further discussion and examples. Here it suffices to point out that $\mathcal{I}_X$ and $\mathcal{I}_X^*$ (and others) are examples which are important for representations and which have significant extra properties.

A is *complete* if each of its subsets has an infimum in the natural ordering; equivalently, if its idempotents $E = E(A)$ form a complete semilattice. A complete inverse algebra has a multiplicative zero element, denoted 0. In a complete inverse algebra A, *Ehresmann's lemma* ([1], section 1.28) holds: if $X \subseteq A$ and X is bounded above by $u \in A$, then X has a least upper bound, $\sup X$, given by $\sup X = (\sup_{x \in X} xx^{-1})\, u = u\, (\sup_{x \in X} x^{-1}x)$.

[Proof: $x \leq u$ for all $x \in X$, so $x = xx^{-1}u \leq (\sup_{x\in X} xx^{-1})\, u$. On the other hand, if v be another upper bound for X, then $\sup_{x\in X} xx^{-1} \leq uv^{-1}$ for all $x \in X$, and

$$\left(\sup_{x\in X} xx^{-1}\right) uv^{-1} = \left(\sup_{x\in X} xx^{-1}\right) = \left(\sup_{x\in X} xx^{-1}\right) u \left(\left(\sup_{x\in X} xx^{-1}\right) u\right)^{-1},$$

i.e., $(\sup_{x\in X} xx^{-1})\, u \leq v$.]

A subset X of A is *compatible* if $x, y \in X$ implies $xx^{-1}y = yy^{-1}x$ (when the common value is $x \wedge y$). A is *hyper-complete* if every compatible subset has a supremum (this implies completeness and is the strongest form of completeness, since any set bounded above is compatible). A subset X of A is [*completely*] *distributive* if $x(y \vee z) = xy \vee xz$ for all $x, y, z \in X$ with y, z bounded above in A $[x(\bigvee_{y\in Y} y) = \bigvee_{y\in Y} xy$ for all $x \in X$ and all $Y \subseteq X$ such that Y has an upper bound in $A]$. (Note, the calculations are in A, not necessarily in X. And *bounded above in* A may be replaced by *compatible* for the pair or subset.) As in any ordered set, $\downarrow X = \{y \in A : y \leq x \text{ for some } x \in X\}$ denotes the order ideal (or down-set) generated by X, and $\uparrow X = \{y \in A : y \leq x \text{ for some } x \in X\}$ the filter (or up-set) generated by X.

Many of the order properties of A are linked with those of $E(A)$. A itself is [completely] distributive if, and only if, $E(A)$ is a distributive lattice [meet distributes over arbitrary joins in $E(A)$]. A is *Boolean* if $E(A)$ is a Boolean lattice. A is *atomistic* if each element is the join of the atoms below it; for a Boolean A, this is equivalent to A being atomic, that is, each element is above an atom.

The symmetric inverse monoid $\mathcal{I}_X$ is hyper-complete, Boolean and atomistic. The *dual symmetric inverse monoid*, $\mathcal{I}_X^*$ is also hyper-complete and atomistic, and $E(\mathcal{I}_X^*)$ is the dually-ordered lattice of equivalence relations on X, so that the semilattice meet operation in $E(\mathcal{I}_X^*)$ is the join of equivalences. The zero and identity elements of $\mathcal{I}_X^*$ are the universal and identity equivalence relations on X, denoted ∇ and Δ respectively.

As I am viewing the atoms of A as important for the purpose, consider the following technical observations as a preliminary step.

Proposition 1 *a is an atom of A if, and only if, aa^{-1} is an atom of $E(A)$.*

Proof. Clearly $a \neq 0$ if, and only if, $aa^{-1} \neq 0$. Now if aa^{-1} is an atom of $E(A)$ and $x \leq a$, then $x = xa^{-1}x$ and $xa^{-1} \leq aa^{-1}$. So either $xa^{-1} = 0$, when $x = 0$, or $xa^{-1} = aa^{-1}$, when $x \geq xa^{-1}a = a$ and $x = a$. Conversely, if a is an atom and $y \leq aa^{-1}$, then $ya \leq a$ and $y = yaa^{-1}$. So either $ya = 0$, when $y = 0$, or else $ya = a$, when $y = aa^{-1}$. $\square$

Thus the atoms of A are precisely the elements in the $\mathcal{D}$-classes containing the primitive idempotents. Let the set of primitive idempotents of A (atoms of $E(A)$) be denoted by P.

Proposition 2 *A is atomistic if, and only if, $E(A)$ is an atomistic lattice.*

Proof. One direction is immediate: the atoms below $e \in E(A)$ are *ipso facto* idempotent, so if A is atomistic, $E(A)$ is too. Conversely if $E(A)$ is atomistic and $x \in A$, let X be the set of atoms of A beneath x. Then $a \in X$ if, and only if, $a = px$ for some $p \in P$ such that $p \leq xx^{-1}$. Now X has an upper bound x, and by Ehresmann's lemma

$$\begin{aligned} \bigvee X &= \bigvee \left\{pxx^{-1}p : p \leq x^{-1}x\right\} \cdot x \\ &= \bigvee \left\{p : p \leq x^{-1}x\right\} \cdot x = (xx^{-1})x = x. \ \square \end{aligned}$$

Proposition 3 *If a is an atom of A, and $a \leq xy$ for $x, y \in A$, then there exists an atom b of A such that $b \leq x$ and $a = by$; and an atom $c \leq y$ such that $a = bc$.*

Proof. Since $a = aa^{-1}xy$, $xx^{-1}a = xx^{-1}aa^{-1}xy = aa^{-1}xy = a$. Take $b = aa^{-1}x$; then $b \leq x$ and $a = by$. Moreover, $bb^{-1} = aa^{-1}(xx^{-1}a)a^{-1} = aa^{-1}$ is an atom, so by Proposition 1, b is indeed an atom. Applying the dual we obtain the second statement. $\square$

Proposition 4 (Ehresmann for atoms) . *For a set P' of primitive idempotents, and any $x \in A$,*

$$\bigvee \{px : p \in P'\} = \left(\bigvee \{p : p \in P', px \neq 0\}\right) x.$$

Proof. For $\{px : p \in P'\}$ is bounded above by x, and so by Ehresmann's lemma has a supremum $(\bigvee \{pxx^{-1} : p \in P'\})\, x$. But since $px \neq 0$ if and only if $pxx^{-1} = p$,

$$\left(\bigvee \left\{pxx^{-1} : p \in P'\right\}\right) x = \left(\bigvee \{p : p \in P', \quad px \neq 0\}\right) x. \quad \square$$

3 Effectiveness and transitivity considered

Let us turn to consider how the concepts *effective* and *transitive* might be rendered in $\mathcal{I}_X^*$ or indeed in other inverse algebras. First effectiveness: the idea is that no "smaller" $\mathcal{I}_X$ can be used, i.e. that there are no redundant elements of X. Recall that for each $e \in E(A)$, eAe is itself an inverse algebra, called a *local subalgebra* of A. The local subalgebra eAe is proper if and only if $e \neq 1$. Local subalgebras eAe are always order-ideals, and are atomistic if A is.

In $\mathcal{I}_X$, note that for $Y \subseteq X$, $\iota_Y\left(\mathcal{I}_X\right)\iota_Y \cong \mathcal{I}_Y$; similarly for an equivalence θ on X, the local subalgebra $\theta\mathcal{I}_X^*\theta \cong \mathcal{I}_{X/\theta}^*$ is another algebra "of the same kind".

Let A be a complete atomistic inverse algebra, with its set of primitive idempotents (atoms of $E(A)$) denoted by $P = P\,(A)$. Write $P^0 = P \cup \{0\}$. Let S be an inverse subsemigroup of A. Then S acts on P^0 by conjugation: $\gamma_s : p \mapsto s^{-1}ps$, so $\gamma_s \in \Phi(P^0)$, in the notation of [5], section IV.2.1. When A is $\mathcal{I}_X$, P consists of the singletons of the diagonal, $\{(x,x)\}$. When A is $\mathcal{I}_X^*$, P consists of the rank-2 partitions of X, each of which has the form $Y \times Y \cup \overline{Y} \times \overline{Y}$, where Y is a proper nonempty subset of X.

It is easier for the moment to describe ineffective subsemigroups. In $\mathcal{I}_X$, a subsemigroup S is ineffective if (a) there exists a proper local subalgebra $\mathcal{I}_Y$ containing S; equivalently if (b) there is at least one primitive idempotent $\{(x,x)\}$ such that x is in the domain of no member of S, that is, $\{(x,x)\}\,s = \varnothing$, which is the zero of $\mathcal{I}_X$.

Now in the general case, if A is atomistic, (a) implies (b): if (a) holds, there there exist $e \neq 1$ with $S \subseteq eAe$, and $p \in P$ with $p \nleq e$ (otherwise, $e = \vee P = 1$). Thus $pe = 0$, but then $ps = pes = 0$ for all $s \in S$.

Strategically, it is perhaps better to work at first with the stronger sense of effectiveness, which is the complement of property (b). Thus we make the definition that the subsemigroup S of A is *(strongly) effective* if there is no $p \in P$ such that $ps = 0$ for all $s \in S$. (That way we should find it easier to prove things; what is lost is that it is no longer the case that any representation can be "pared back" to an effective one by taking the smallest local subalgebra.) The other definition is that the subsemigroup S of A is *weakly effective* if the only local subalgebra containing S is A itself: $S \leq eAe$ implies $e = 1$.

Now some technical background bits about atoms and primitive idempotents. S is an inverse semigroup throughout.

Lemma 5 *For $p, q \in P$ and $s \in S$, the following are equivalent:*

(1) $q = s^{-1}ps$;

(2) $ps = sq \neq 0$;

(3) $psq = ps = sq \neq 0$;

(4) $psq \neq 0$.

[Note $p \leq ss^{-1}$ iff $ps \neq 0$.]

Proof. (1) $\Rightarrow sq = ss^{-1}ps = ps$ and $ps \neq 0$ (else $q = 0$) $\Rightarrow$ (2) $\Rightarrow psq = ps \Rightarrow$ (3) $\Rightarrow$ (4)$\Rightarrow s^{-1}psq \neq 0 \Rightarrow s^{-1}psq = q \Rightarrow$(1). □

We attempt to develop and illuminate the concepts with a series of simple related examples.

Examples. Consider a semigroup S which is a 0-direct sum of a 5-element aperiodic Brandt semigroup with a 2-element semilattice. It may be embedded in $\mathcal{I}_4^*$ as the subsemigroup

$$S_1 = \left\{\nabla, \delta, \alpha, \alpha^{-1}, \alpha\alpha^{-1}, \alpha^{-1}\alpha\right\},$$

where

$$\alpha = \left(\begin{array}{c|c} 12 & 34 \\ 13 & 24 \end{array}\right) \text{ and } \delta = \left(\begin{array}{c|c} 1 & 234 \\ 1 & 234 \end{array}\right).$$

The idempotents of S_1 are $\alpha\alpha^{-1} = (12|34)$, $\alpha^{-1}\alpha = (13|24)$, and $\delta = (1|234)$, which are all members of P. Checking condition (b) applied to the other members of P, note

$$(2|134)\,\nabla = (2|134)\,\delta = (2|134)\,\alpha = (2|134)\,\alpha^{-1} = \nabla,$$

so that $(2|134)\,S_1 = \{\nabla\}$ and (b) is satisfied, so S_1 is ineffective in that sense. But condition (a) above is not satisfied: the only local subalgebra containing S_1 is $\mathcal{I}_4^*$ itself, because the l.u.b of $\alpha\alpha^{-1}$ and $\alpha^{-1}\alpha$ is Δ. So S_1 is weakly effective, but not effective in the strong sense.

We can also embed S in $\mathcal{I}_4^*$ as the subsemigroup

$$S_2 = \left\{\nabla, \delta, \beta, \beta^{-1}, \beta\beta^{-1}, \beta^{-1}\beta\right\},$$

where

$$\beta = \left(\begin{array}{r|r} 12 & 34 \\ 2 & 134 \end{array} \right)$$

and δ is as before. This time, (a) is satisfied since S_2 is contained in the local subalgebra whose identity is $(1|2|34)$ and so (b) is satisfied too; S_2 is ineffective on either criterion. In fact $p = (4|123)$ has $pS_2 = \{\nabla\}$. But we can modify this example to embed S in $\mathcal{I}_3^*$ as

$$S_3 = \left\{\nabla, \varepsilon, \gamma, \gamma^{-1}, \gamma\gamma^{-1}, \gamma^{-1}\gamma\right\}$$

by lumping vertices $3, 4$ together and treating the 3-element set as the quotient of $\{1, 2, 3, 4\}$ by the equivalence generated by $(3, 4)$. Thus:

$$\gamma = \left(\begin{array}{r|r} 12 & 3 \\ 2 & 13 \end{array} \right), \varepsilon = \left(\begin{array}{r|r} 1 & 23 \\ 1 & 23 \end{array} \right)$$

and now all the primitive idempotents in $\mathcal{I}_3^*$ occur already in S_3, whence $p \in pS_3$ for all $p \in P$ and so this S_3 is effective. Similar examples could be given for subsemigroups in (say) the inverse semigroup of partial automorphisms of a vector space. □

Define a relation $\mathcal{T} = \mathcal{T}_S$ on the set P as follows: for $p, q \in P$, $p\mathcal{T}_S q$ if there exists $s \in S$ such that $p = s^{-1}qs$ (or, any of the equivalents 1)—4) of Lemma 5). This relation $\mathcal{T}_S$ is symmetric ($psq \neq 0$ implies $qs^{-1}p \neq 0$) and transitive ($p = s^{-1}qs$ and $q = t^{-1}rt$ imply $p = (ts)^{-1}rts$). In general, $\mathcal{T}_S$ is only a partial equivalence, that is, an equivalence on its domain $\mathbf{dom}\mathcal{T}_S = \{p \in P : ps \neq 0 \text{ for some } s \in S\}$. (However if S is (strongly) effective, $\mathcal{T}_S$ is also reflexive —if $ps \neq 0$ for some $s \in S$, then $p = s(s^{-1}ps)s^{-1}$ with $s^{-1}ps \in P$ —and so an equivalence relation on P, partitioning P into orbits under the action by S.) So far, this is similar to the classical theory.

Let the $\mathcal{T}_S$-classes into which $\mathbf{dom}\mathcal{T}_S$ is thus partitioned by $\mathcal{T}_S$ be indexed by I, and denoted by P_i $(i \in I)$.

Examples continued. In the first example above (S_1), there are two $\mathcal{T}$-classes, $P_1 = \{\alpha\alpha^{-1}, \alpha^{-1}\alpha\} = \{(12|34), (13|24)\}$ and $P_2 = \{\delta\} = \{(1|234)\}$.

For the third example (S_3) the two $\mathcal{T}$-classes are $P_1 = \{(12|3), (2|13)\}$, $P_2 = \{(1|23)\}$.

In each case, the local subalgebra generated by P_1 contains the local subalgebra generated by P_2, whereas in the classical theory, the local subalgebras generated by distinct orbits would intersect in $\{\varnothing\}$, the trivial subalgebra. $\square$

Define (for $i \in I$)

$$e_i = \bigvee \{p : p \in P_i\} = \bigvee P_i$$

and the local subalgebra $A_i = e_i A e_i$. Also define the mapping $\phi_i : S \to A$ by

$$s\phi_i = \bigvee \{ps : p \in P_i\}.$$

By Prop. 4 and Lemma 5,

$$\begin{aligned} s\phi_i &= \left(\bigvee \left\{pss^{-1} : p \in P_i\right\}\right) s = \left(\bigvee \{p \in P_i : ps \neq 0\}\right) s \\ &= s\left(\bigvee \left\{s^{-1}ps : p \in P_i\right\}\right) = s\left(\bigvee \{q \in P_i : sq \neq 0\}\right), \end{aligned}$$

for $s \in S$. Clearly $s\phi_i \leq s$, also

$$e_i (s\phi_i) = e_i \left(\bigvee \left\{pss^{-1} : p \in P_i\right\}\right) s = \bigvee \left\{pss^{-1} : p \in P_i\right\} s = s\phi_i,$$

and sim. $(s\phi_i) e_i = s\phi_i$, so the image $S\phi_i$ is a subset of A_i.

Moreover $\bigvee \{s\phi_i : i \in I\} \leq s$. On the other hand, let $a \in A$ be an atom such that $a \leq s$. Then $aa^{-1} \leq ss^{-1}$ and $aa^{-1} \in \mathbf{dom}\mathcal{T}_S$, so there is $i \in I$ such that $q := aa^{-1} \in P_i$ and $a = qs \neq 0$, so that $a \leq (\bigvee \{p \in P_i : ps \neq 0\}) s = s\phi_i \leq \bigvee \{s\phi_i : i \in I\}$. We conclude that $s = \bigvee \{s\phi_i : i \in I\}$.

So far so good. Next, take $s, t \in S$ and suppose $pstq \neq 0$ for some $p, q \in P_i$. Then $ps \neq 0$ and $tq \neq 0$ so that $pstq \leq (s\phi_i)(t\phi_i)$ and hence $(st)\phi_i \leq (s\phi_i)(t\phi_i)$. This makes ϕ_i a *prehomomorphism*. To prove ϕ_i a homomorphism, it would be enough to prove that $(e\phi_i)(f\phi_i) \leq (ef)\phi_i$ by Lawson, p. 80 (in Theorem 3.1.5). This happens if $\downarrow S$ is distributive, e.g. in $\mathcal{I}_X$. Is there some lattice condition, weaker than distributivity, e.g. modularity, which ensures the ϕ_i are homomorphisms?

Example, part three. Continuing the notation of the examples, in the first example (S_1),

$$e_1 = \vee P_1 = \vee \left\{\alpha\alpha^{-1}, \alpha^{-1}\alpha\right\} = (12|34) \cap (13|24) = \triangle$$

and $e_2 = \vee P_2 = \delta = \{(1|234)\}$. The maps ϕ are $\phi_1 = \phi$ and

$$\phi_2 = \begin{pmatrix} \nabla & \delta & \alpha & \alpha^{-1} & \alpha\alpha^{-1} & \alpha^{-1}\alpha \\ \nabla & \delta & \nabla & \nabla & \nabla & \nabla \end{pmatrix},$$

The third example (S_3) is similar, $e_1 = (12|3) \cap (2|13) = \triangle, e_2 = \{(1|23)\}$, $\phi_1 = \phi$ and again $\delta\phi_2 = \delta$ and $t\phi_2 = \nabla$ if $t \neq \delta$.
These ϕ_i are actually homomorphisms. □

Let us now turn to considering the meaning of *transitivity* of subsemigroups in $\mathcal{I}_X^*$. Schein, in the context of the semigroup $\mathcal{B}_X$ of binary relations, says that a subsemigroup S is transitive if, given any $x, y \in X$, there is $s \in S$ with $(x, y) \in s$. But an abstract version for inverse algebras of the classical definition would state that S is *(strongly) transitive* in A if there is only one orbit of $\mathcal{T}_S$; i.e., each atom of A is underneath some element of S. This has implications for the structure of A:

Lemma 6 *S is strongly transitive if, and only if, for each pair $p, q \in P$ there exists $a \in A$ such that $p = a^{-1}a$, $q = aa^{-1}$, and $a \leq s$ for some $s \in S$; that is, the $\mathcal{H}$-class $R_p \cap L_q$ contains an element beneath some element of S. In particular, all atoms of A form one $\mathcal{D}$-class.*

Proof. Ad $\Rightarrow$: let $p, q \in P$, so by transitivity, there exists $s \in S$ such that $p = s^{-1}qs$. Then take $a = sp = qs$ so $a^{-1}a = s^{-1}qs = p$, $0 \neq aa^{-1} = qss^{-1}q$ and $a \leq s$ as required. Ad $\Leftarrow$: if $a \leq s$ then $p = a^{-1}a \leq s^{-1}s \in S$ and $s^{-1}qs = s^{-1}(aa^{-1})^2 s = a^{-1}a = p$. □

Applied to $\mathcal{I}_X^*$, this definition of transitivity is quite a strong requirement: it means that, given any proper subsets Y, Z of X, there is $s \in S$ contained in $(Y \times Z) \cup (\bar{Y} \times \bar{Z})$. That is, it applies the usual sort of idea not just to individual points of X, but to all proper subsets.

Awkward, perhaps, but I don't see any other good way of doing it, assuming that the orbit equivalence $\mathcal{T}$ is as important as it seems. Perhaps it would be useful to say that S is *weakly transitive* if $\mathcal{T}_S$ has just one class. That is, for each pair $p, q \in P$ such that $pS \neq \{0\}$ and $q \neq \{0\}$, $p = s^{-1}qs$ for some $s \in S$.

The relationship between transitivity and effectiveness. The sum of subsemigroups $S_\alpha \leq A_\alpha$ (for α in an index set I and $S_\alpha \cap S_\beta \subseteq \{\nabla\}$ when

$\alpha \neq \beta$) is $\oplus S_\alpha \leq \oplus A_\alpha$. Note that $\oplus \mathcal{I}_{X_\alpha} \cong \mathcal{I}_{\oplus X_\alpha}$ where $\oplus X_\alpha$ is the sum in **Set** ($\sqcup X_\alpha$). Now if each S_α is transitive, then $\oplus S_\alpha$ should be effective (in A_α and $\oplus A_\alpha$ respectively).

Special things continue to happen if $\downarrow S$ is distributive: then, $s\phi_i = e_i s = se_i$, each $S\phi_i$ is transitive in A_i, and $A_i A_j = \{0\}$ if $i \neq j$ holds.

4 Transitive representations in $\mathcal{I}_X^*$ — a speculation

In view of Theorem 4.1 of [3], I reckon it is a fair bet that there will be an $\mathcal{I}_X^*$ version of the Schein transitive representation in some $\mathcal{I}_X$ by closed subsemigroups (see e.g. [5] IV.4.7 for an account). This may actually explain to us what transitivity ought to mean.

It would go something like this.

Let Σ be an inverse subsemigroup of an inverse algebra A. For a non-empty subset K of Σ, the following are equivalent:

1. $KK^{-1}K = K = \uparrow K$;
2. $K = \uparrow Ha$ for some inverse subsemigroup H of Σ and some $a \in \Sigma$ with $aa^{-1} \in H$.

Such a K is called an $\uparrow$-*coset* or *strong coset* in Σ. Continuing this notation, we have

Proposition Let $p, q \in P(A)$ and set

$$K = K_{p,q} = \{s \in \Sigma : s^{-1}ps = q\}.$$

Then K is, if non-empty, an $\uparrow$-coset in Σ.

Proof. Clearly $K \subseteq KK^{-1}K$. Suppose that $s, t, u \in K$ so that

$$s^{-1}ps = q = t^{-1}pt = u^{-1}pu,$$

and then $u^{-1}ts^{-1}pst^{-1}u = u^{-1}tqt^{-1}uq = u^{-1}pu = q$, which gives $st^{-1}u \in K$ and hence $KK^{-1}K \subseteq K$. If $t \geq s$ for some $t \in \Sigma$, then $pt \geq ps$ but pt is an atom; so $pt = ps$ and $t \in K$, i.e. $K = \uparrow K$. $\square$

Corollaries If $p = q$ in the proposition, $H = K_{p,p}$ is a closed inverse subsemigroup of Σ corresponding to K. And if $\phi : S \to A$ is a homomorphism, and $p, q \in P(A)$, then $K = K_{p,q} = \{s \in S : (s^{-1}\phi)\, p\,(s\phi) = q\}$ is an $\uparrow$-coset of S and $H = K_{p,p}$ its corresponding closed inverse subsemigroup. □

The obvious way to proceed to this author seems to be to consider the Schein representation as a cut-back of the representation of $s \in S$ by the translation $\rho_s :\uparrow Ha \mapsto \uparrow Has$ in the transformation semigroup $\mathcal{T}_{\mathcal{X}}$ to a partial bijection (by restricting ρ_s to the range of the inverse $\rho_{s^{-1}}$). Note that ρ_s needs to use *weak cosets* since $\uparrow Has$ need not satisfy the equivalent conditions for strong cosets even if $\uparrow Ha$ does, since $as\,(as)^{-1}$ need not be in H. However, the cut-back of ρ_s to a partial bijection α_s, done as follows:

$$(\uparrow Ha, \uparrow Hb) \in \alpha_s \text{ iff } \uparrow Has = \uparrow Hb \text{ and } \uparrow Ha = \uparrow Hbs^{-1},$$

guarantees that $\uparrow Ha$ and $\uparrow Hb$ are strong cosets. However when we need to generate a biequivalence from ρ_s, we will continue to need all the weak cosets.

So: for a closed inverse subsemigroup H of S, put $\mathcal{X} = \mathcal{X}_H = \{\uparrow Ha : a \in S\}$, removing the usual requirement that $aa^{-1} \in H$. Let

$$\beta^s = \left\{(\uparrow Ha, \uparrow Hb) \in \mathcal{X} \times \mathcal{X} :\uparrow Has = \uparrow Hbs^{-1}s\right\} = \left(\beta^{s^{-1}}\right)^{-1}.$$

Proposition The map $\beta = \beta_H : s \mapsto \beta^s$ is a representation of S in $\mathcal{I}^*_{\mathcal{X}}$.

Proof. The following is really just a translation of the proof in Theorem 4.1 of [3]. (1) Suppose $(\uparrow Ha, \uparrow Hb), (\uparrow Hc, \uparrow Hb), (\uparrow Hc, \uparrow Hd)$ all in $\beta = \beta^s$. Then

$$\uparrow Has = \uparrow Hbs^{-1}s = \uparrow Hcs = \uparrow Hds^{-1}s,$$

whence $(\uparrow Ha, \uparrow Hd) \in \beta$. That shows that $\beta \circ \beta^{-1} \circ \beta \subseteq \beta$, so $\beta \in \mathcal{I}^*_{\mathcal{X}}$.

(2) Let $(\uparrow Ha, \uparrow Hb) \in \beta^s$ and $(\uparrow Hb, \uparrow Hc) \in \beta^t$. Then $\uparrow Has = \uparrow Hbs^{-1}s$ and $\uparrow Hbt = \uparrow Hct^{-1}t$. In turn we have

$$\begin{aligned} \uparrow\ & Hast = \uparrow Hbs^{-1}st = \uparrow Hbtt^{-1}s^{-1}st \\ = \ & \uparrow Hct^{-1}tt^{-1}s^{-1}st = \uparrow Hc\,(st)^{-1}\,st, \end{aligned}$$

and so $(\uparrow Ha, \uparrow Hc) \in \beta^{st}$. Hence $\beta^s \circ \beta^t \subseteq \beta^{st}$ and $\beta^s\beta^t \subseteq \beta^{st}$ follows (the product in $\mathcal{I}^*_{\mathcal{X}}$). For the reverse inclusion, suppose $(\uparrow Ha, \uparrow Hc) \in \beta^{st}$, that is, $\uparrow Hast = \uparrow Hc\,(st)^{-1}\,stt^{-1}t = \uparrow Hct^{-1}s^{-1}st$. This means $(\uparrow Has, \uparrow Hct^{-1}s^{-1}st) \in \beta^t$, while $\uparrow Hastt^{-1} = \uparrow Hct^{-1}s^{-1}stt^{-1} = \uparrow Hct^{-1}s^{-1}s$. Now by definition we see

$$\begin{array}{cccc}
(\uparrow Ha, & \uparrow Has) & \in & \beta^s \\
 & \| & & \\
(\uparrow Hast, & \uparrow Has) & \in & \left(\beta^t\right)^{-1} \\
\| & & & \\
(\uparrow Hast, & \uparrow Hastt^{-1}) & \in & \beta^{t^{-1}} \\
 & \| & & \\
(\uparrow Hct^{-1}s^{-1}, & \uparrow Hct^{-1}s^{-1}s) & \in & \beta^s \\
\| & & & \\
(\uparrow Hct^{-1}s^{-1}, & \uparrow Hct^{-1}) & \in & \left(\beta^{s^{-1}}\right)^{-1} = \beta^s \\
 & \| & & \\
(\uparrow Hc, & \uparrow Hct^{-1}) & \in & \beta^{t^{-1}}
\end{array}$$

and so $(\uparrow Ha, \uparrow Hc) \in \left(\beta^s \circ \beta^t\right) \circ \left(\beta^s \circ \beta^t\right)^{-1} \circ \left(\beta^s \circ \beta^t\right) \subseteq \beta^s\beta^t$. Altogether we have that $\beta : s \mapsto \beta^s$ is a representation of S in $\mathcal{I}^*_{\mathcal{X}}$. □

Is $S\beta$ weakly transitive?

Example, part the fourth. The closed subgroups of $S = S_1 = \{\nabla, \delta\} \cup \langle\alpha\rangle$ and their (strong and weak) cosets are:

- $H_0 = S = \uparrow \{\nabla\} = \uparrow \langle\alpha\rangle = \uparrow \{\nabla, \delta\} = \uparrow \{\nabla, \alpha\alpha^{-1}\} = \uparrow \{\nabla, \alpha^{-1}\alpha\}$, with trivial coset S;
- $H_1 = \uparrow \{\delta\} = \{\delta\}$, with strong coset $\{\delta\}$ and all weak cosets $= \uparrow \nabla = S$;
- $H_2 = \uparrow \{\alpha\alpha^{-1}\} = \{\alpha\alpha^{-1}\}$, with cosets $\{\alpha\alpha^{-1}\}$ and $\{\alpha\}$ (both strong) and others all $= S$;
- $H_3 = \uparrow \{\alpha^{-1}\alpha\} = \{\alpha^{-1}\alpha\}$,with cosets $\{\alpha^{-1}\alpha\}$ and $\{\alpha^{-1}\}$ (both strong) and others all $= S$.

The corresponding representations ρ_{H_i} and β_{H_i} are given next.

- H_0. For all $s \in S$, $\rho_s = \begin{pmatrix} S \\ S \end{pmatrix}$, the sole member of $\mathcal{T}_{\{S\}}$; and $\beta^s = \begin{pmatrix} S \\ S \end{pmatrix}$ also, the sole member of $\mathcal{I}^*_{\{S\}}$.

- H_1. $\rho_\delta = \begin{pmatrix} \{\delta\} & S \\ \{\delta\} & S \end{pmatrix} = \triangle_{\mathcal{X}}$, and for all $s \neq \delta$, $\rho_s = \begin{pmatrix} \{\delta\} & S \\ S & S \end{pmatrix}$. So $\beta^\delta = \left(\begin{array}{c|c} \{\delta\} & S \\ \{\delta\} & S \end{array}\right) = \triangle_{\mathcal{X}}$, and for all $s \neq \delta$, $\beta^s = \begin{pmatrix} \{\delta\} & S \\ \{\delta\} & S \end{pmatrix} = \nabla_{\mathcal{X}}$.

- H_2. $\rho_\alpha = \begin{pmatrix} \{\alpha\} & \{\alpha\alpha^{-1}\} & S \\ S & \{\alpha\} & S \end{pmatrix}$, $\rho_{\alpha^{-1}} = \begin{pmatrix} \{\alpha\} & \{\alpha\alpha^{-1}\} & S \\ \{\alpha\alpha^{-1}\} & S & S \end{pmatrix}$, $\rho_{\alpha\alpha^{-1}} = \begin{pmatrix} \{\alpha\} & \{\alpha\alpha^{-1}\} & S \\ S & \{\alpha\alpha^{-1}\} & S \end{pmatrix}$, $\rho_{\alpha^{-1}\alpha} = \begin{pmatrix} \{\alpha\} & \{\alpha\alpha^{-1}\} & S \\ \{\alpha\} & S & S \end{pmatrix}$, and $\rho_\nabla = \rho_\delta = \begin{pmatrix} \{\alpha\} & \{\alpha\alpha^{-1}\} & S \\ S & S & S \end{pmatrix}$. Then
$\beta^\alpha = \left(\begin{array}{c|c} S, \{\alpha\} & \{\alpha\alpha^{-1}\} \\ S, \{\alpha\alpha^{-1}\} & \{\alpha\} \end{array}\right)$, $\beta^{\alpha^{-1}} = \left(\begin{array}{c|c} S, \{\alpha\alpha^{-1}\} & \{\alpha\} \\ S, \{\alpha\} & \{\alpha\alpha^{-1}\} \end{array}\right)$,
$\beta^{\alpha\alpha^{-1}} = \left(\begin{array}{c|c} S, \{\alpha\} & \{\alpha\alpha^{-1}\} \\ S, \{\alpha\} & \{\alpha\alpha^{-1}\} \end{array}\right)$, $\beta^{\alpha^{-1}\alpha} = \left(\begin{array}{c|c} S, \{\alpha\alpha^{-1}\} & \{\alpha\} \\ S, \{\alpha\alpha^{-1}\} & \{\alpha\} \end{array}\right)$,
$\beta^\nabla = \beta^\delta = \begin{pmatrix} \{\alpha\}, & \{\alpha\alpha^{-1}\}, & S \\ \{\alpha\}, & \{\alpha\alpha^{-1}\}, & S \end{pmatrix} = \nabla$.

- For H_3 the results are similar to H_2, with just α and α^{-1} exchanged.$\square$

This reinforces the point that we will have to come up with some modification of the transitive property.

Concluding thought: One may have to surrender essential uniqueness with a relaxation of the effectiveness condition (which may be a kind of red herring anyway); in other words, use the relaxed definition of effectiveness and tolerate that $\mathcal{T}$ may be only a partial equivalence.

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