Symmetric Functions and Infinite Dimensional Algebras

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Declaration

Except as stated herein this thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

Timothy H. Baker

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Abstract

Infinite-dimensional algebras and symmetric functions arise in many diverse areas of mathematics and physics. In this thesis, several problems in these two areas are studied.

We investigate the concept of replicated and q-replicated arguments in Schur and Hall-Littlewood symmetric functions. A description of "dual" compound symmetric functions is obtained with the help of functions of a replicated argument, while Schur and Hall-Littlewood functions of a q-replicated argument are both shown to be related to Macdonald's symmetric functions.

Various tensor product decompositions and winding subalgebra branching rules for the $N = 1$ and $N = 2$ superconformal algebras are examined by using the triple and quintuple product identities, and various generalizations thereof, concentrating on the particular cases when these decompositions are finite or multiplicity-free.

The boson-fermion correspondence is utilized to develop an algorithm for the calculation of outer products of Schur and Q-functions with power sum symmetric functions, and general (outer) multiplication of S-functions. A procedure is also developed for the evaluation of (outer) plethysms of Schur functions and power sums. A few examples are given which demonstrate the usefulness of this method for calculating plethysms between Schur functions. By examining the vertex operator realization of Hall-Littlewood functions we are also able to generate an algorithm for expressing Hall-Littlewood functions in terms of Schur functions. The operation of outer plethysm is defined for Hall-Littlewood functions and the algorithm developed for S-functions is extended to this case as well.

Kerov's generalized symmetric functions are used to provide a realization for level k Fock space representations of the quantum affine algebra $U_q(\widehat{sl(2)})$. Using these functions, we derive a generalized Macdonald identity which enables the regularized trace of a product of $U_q(\widehat{sl(2)})$ currents to be calculated.

Contents

"There is truth, my boy. But the doctrine you desire, absolute, perfect dogma that alone provides wisdom, does not exist. Nor should you long for a perfect doctrine, my friend. Rather, you should long for the perfection of yourself. The deity is within *you*, not in ideas and books. Truth is lived, not taught."

—*The Glass Bead Game*, Hermann Hesse

"The Truth is out there. I know this as that's where I was when I lost it."

— David Shanahan

"The boy reached through to the Soul of the World, and saw that it was a part of the Soul of God. And he saw that the Soul of God was his own soul."

—*The Alchemist*, Paulo Coelho

"Because little boys will always think they can fly."

— advertisement for *Dettol*

"Whither will my path yet lead me ? This path is stupid, it goes in spirals, perhaps in circles, but whichever way it goes, I will follow it."

—*Siddhartha*, Hermann Hesse

Chapter 1

Introduction

In this thesis, we study some particular problems in the theory of symmetric functions, and in the representation theory of certain infinite-dimensional Lie algebras, namely the $N=1$ and $N=2$ superconformal algebras in two dimensions, and the quantum affine algebra $U_q(\widehat{sl(2)})$. In this introductory chapter, we shall review the areas of the literature relevant to the contents of this thesis.

1.1 Symmetric functions

The symmetric functions which we call S-functions $s_\lambda(x)$, were first studied by Jacobi [1], who defined them in terms of ratios of antisymmetric determinants. S-functions are functions of a countable number of indeterminates $x = (x_1, x_2, \cdots)$, which are indexed by a partition λ of some integer n and which are homogeneous of degree n

$$s_\lambda(cx) = c^n s_\lambda(x).$$

The theory of these functions was developed by Schur [2] who used them as a means of studying the characters χ^λ_μ of irreducible representations of the symmetric group S_n. A decade later [3], he investigated projective representations of S_n and some new symmetric functions (Q-functions) which enabled him to calculate the spin-characters of these types of representation. Frobenius [4] also employed symmetric functions to investigate characters of the symmetric group, his efforts culminating in the celebrated Frobenius formula for the characters of S_n. Explicit formulae for these characters (in reduced notation [5,6]) in terms of partition labels, have been calculated by Specht [7]. The set of all S-functions forms a (graded) ring and the structure constants $c^\lambda_{\mu\nu}$ of this ring are called Littlewood-Richardson coefficients. A combinatorial recipe for their calculation was given in [8] although it was not until much later that a complete proof of the validity of this procedure was given [9]. S-functions also have a combinatorial description in terms of tableaux, however we shall concern ourselves just with their algebraic aspects.

S-functions are also important in the representation theory of finite-dimensional (classical) Lie algebras, in that the character of the irreducible representation of $gl(n)$ labelled by a standard partition λ, is just the S-function $s_\lambda(x)$. The integers which label irreducible representations of $gl(n)$ need not be positive however. For the general case where the parts of λ are not necessarily positive, the character of the corresponding representation can be succinctly described in terms of *composite* S-functions [10], which are functions of the variables $\{x_i, x_i^{-1}\}$. From this relationship, the expansion of the product of two S-functions s_μ and s_ν in terms of S-functions s_λ carries precisely the same information as the decomposition of the tensor product of the irreducible representations (μ), (ν) of $gl(n)$ in terms of the irreducible representations (λ). Another type of multiplication (called the inner product) of S-functions exists, in which the coefficients $\gamma^\lambda_{\mu\nu}$ which occur in the product are given by the coefficients which occur in the tensor product decomposition of irreducible representations of the symmetric group. There is a generic formula for these coefficients in terms of characters of S_n (see (2.25)), however Littlewood [11] discovered remarkable formulae which connect outer and inner products of S-functions. This enables $\gamma^\lambda_{\mu\nu}$ to be calculated in terms of $c^\lambda_{\mu\nu}$, and the procedure is much more efficient in certain cases. Littlewood's formulae can even be used to develop some explicit formulae in the case where the partitions μ and ν have a particular shape [12, 13]. Thus the decompositions of irreducible representations of the Lie algebra $gl(n)$ are connected to those of the symmetric group S_n.

Symmetric functions are directly connected with the Quantum Hall effect [14–17]. Given a system of massive fermions in a magnetic field in two spatial dimensions, it is well-known that the ratio of the wavefunction of an excited state, to the wavefunction of the ground state is a symmetric function. It has also been shown that the infinite-dimensional algebra $W_{1+\infty}$ is the underlying symmetry algebra of this system [16, 18–20]. Due to the existence of free fermionic realizations of this algebra [21], it is perhaps not too surprising that symmetric functions appear, due to the boson-fermion correspondence and its connection with S-functions (see Chapter 5).

There exist several different generalizations of S-functions. Firstly, as previously mentioned, there are Q-functions which are associated with projective representations of S_n [22, 23]. Hall-Littlewood functions $P_\lambda(x;t)$ were introduced implicitly by P. Hall [24] in connection with the enumeration of the subgroups of finite Abelian p-groups, and later in a more explicit form by Littlewood [25]. These functions are also homogeneous and are identical to S-functions when $t = 0$, while Q-functions correspond to the case when $t = -1$. The case when t is a root of unity was investigated by Morris [26] who obtained some analogous properties to the $t = -1$ case. There exist functions $X^\lambda_\mu(t)$ whose $t \to 0$ limit is the S_n character χ^λ_μ, but these are not the characters of some t-deformation of the symmetric group. They are, however, directly related to certain polynomials called Green's polynomials which are used in the evaluation of the irreducible characters of the group $GL(n, t)$ over a finite field of t elements [27]. In analogue to Specht's treatment of the characters of S_n, Morris [28] was able to use certain properties of Hall-Littlewood functions to provide explicit formulae for $X^\lambda_\mu(t)$ for certain partitions λ. The coefficients $f^\lambda_{\mu\nu}(t)$ which arise from the expansion of the product of two Hall-Littlewood functions have been examined [29–31], and although no combinatorial description for them exists, as in the $t = 0$ case, certain special cases are known explicitly, such as when the skew diagram $\lambda - \mu$ is a horizontal or vertical strip [32,33], or when ν is a rectangular block and λ is of a particular form [34].

There are numerous other generalizations of S-functions which have various applications in mathematics. Firstly, there are Jack symmetric functions [23,35,36], which depend on a parameter α such that when $\alpha = 1$, they become ordinary S-functions. These functions originally arose in statistics, but they also occur in statistical mechanics [37,38], where they are just the eigenfunctions for certain Coulomb systems. Jack symmetric functions are just a special limit of a generalized Hall-Littlewood function $P_\lambda(x; q, t)$ considered by Macdonald [39,40]. These Macdonald's functions can also be considered as a generalization of Hall-Littlewood functions in the sense that when $q = 0$, ordinary Hall-Littlewood functions $P_\lambda(x; t)$ are recovered. Macdonald's functions are defined in terms of certain properties which they are required to possess with repect to an inner product on the ring of symmetric functions over the field $F = \mathbb{Q}(q, t)$. In [40] the properties of these functions were extensively examined. There exists another formulation of Macdonald's polynomials as the trace of an interwiner (algebra homomorphism) of modules over the quantum group $U_q(sl(n))$ [41]. This was generalized to affine Macdonald's polynomials in [42], by considering homomorphisms of the affine quantum group $U_q(\widehat{sl(n)})$. Finally, a whole series of generalizations of S-functions was considered in [43] and their properties compared.

1.2 Finite-dimensional algebras

Although the algebras considered in this thesis are infinite-dimensional, it is useful to remind ourselves of their finite-dimensional counterparts, which are usually involved in their definition anyway. Moreover the methods used to study the finite-dimensional case often (although not always) can be carried over to the infinite-dimensional case.

Lie algebras have long surpassed their original role in classifying symmetries of ordinary differential equations, and have become vitally important in describing the symmetries of many physical systems. The finite-dimensional semi-simple Lie algebras and their representations have long since been classified. As was mentioned in the previous section, the characters of the irreducible representations of $gl(n)$ are just given by (composite) S-functions. More recently, Lie superalgebras ($\mathbb{Z}_2$ graded Lie algebras with a graded commutator) were introduced and classified [44]. The representation theory of these algebras proved to be similar for certain representations known as *typical* representations [45], but somewhat more difficult for *atypical* representations [46,47]. The characters of these Lie superalgebras can be expressed in terms of symmetric functions, which allow various branching, modification and Kronecker product rules for representations of these algebras to be derived (see [48,49] for comprehensive reviews). Lie algebras and superalgebras also have useful realizations in terms of bosonic and fermionic creation and annihilation operators [50,51] which enables one to ascertain the kinematical or dynamical symmetries of many physical systems (e.g. supersymmetric quantum mechanics [52,53]).

Another type of algebra, which has appeared in the last decade and has had a wide variety of applications in physics, is the so-called *quantum group* which may be regarded as a deformation, depending on a parameter q, of the universal enveloping algebra of a semi-simple Lie algebra. Thus they are not finite-dimensional algebras, but are finitely generated. They were first constructed as an algebra by Kulish and Reshetikhin [54] and as a Hopf algebra by Sklyanin [55], and subsequently generalized by Drinfeld [56] and Jimbo [57]. Their representation theory for q not a root of unity was found to be simliar to the corresponding semi-simple Lie algebra [58]. Complications arise when q is a root of unity due to the fact that the centre of the algebra becomes larger, as it is augmented by certain powers of all the generators. Nevertheless, it is often these representations that are useful in physics [59–69], and their representation theory (especially in the simplest case $U_q(sl(2))$) has been investigated by several authors [70–74].

Quantum groups are an example of quasi-triangular Hopf algebras and as such, for each quantum group there exists a *universal* R-matrix which intertwines with the action of the coproduct. For the quantum group $U_q(sl(2))$ this was first given by Drinfeld [75], for $U_q(sl(n))$ by Rosso [76], and for $U_q(g)$ with g any complex simple Lie algebra by Kirillov and Reshetikhin [77]. Khoroshkin and Tolstoy subsequently gave explicit formulae for the univeral R-matrices of all quantum (super)groups which possess a symmetrizable Cartan matrix [78] and then for untwisted quantum affine algebras [79]. This enables one to construct representations of the braid group, and hence to go on to generate knot invariants [80–82]. Once one has a braid group generator, one can try to construct solutions of the Yang-Baxter equation, either directly [83] or by using additional relations which it satisfies, for example the defining relations of the Temperley-Lieb algebra [84], the Birman-Wenzl-Murakami algebra [85], the BH algebras [86], or the braid monoid algebras [87]. The importance of discovering solutions of the Yang-Baxter equation lies in the fact that every solution gives rise to an integrable vertex model of a two-dimensional (classical) statistical mechanical system (or equivalently, a one-dimensional quantum spin chain). For a review see reference [88].

By considering q-deformations of simple Lie superalgebras, one comes across the concept of quantum *supergroups*. The definition of these graded Hopf algebras is slightly more involved than for their non-graded counterparts due to the need for extra Serre relations [78, 89, 90]. R-matrices have also been investigated for quantum supergroups [91–95]. There exist various realizations of quantum (super)groups in terms of q-deformed bosonic and fermion oscillators, as well as *anyonic* oscillators [96] and these realizations have found use in quantum-mechanical applications [97–99].

1.3 Infinite-dimensional algebras

Of central importance in many diverse areas of mathematics and physics are a particular class of infinite-dimensional algebras called (affine) Kac-Moody algebras, which were introduced in the late 1960's [100–102]. For a review on Kac-Moody algebras and their relevance to physics see [103, 104]. As in the case of for their finite-dimensional counterparts, the representation theory has been developed [105] and various realizations have been constructed. Unlike the finite-dimensional case, where the simple Lie algebras can be realized in terms of a finite number of fermionic/bosonic modes, simple Kac-Moody algebras have various *vertex operator* realizations: that is, in terms of a finite number of bosonic *free fields*, the modes of which, generate a Heisenberg algebra. There are numerous inequivalent vertex operator realizations, depending on how the Heisenberg subalgebra is embedded in the Kac-Moody algebra [106, 107], the two main ones being the principal realization [108, 109] and the homogeneous realization [110, 111] (see also [112]).

All simple (twisted and untwisted) Kac-Moody algebras can be embedded in the infinite-dimensional algebra $gl(\infty)$ of infinite matrices with a finite number of non-zero entries, which has a simple realization in terms of generators of a Clifford algebra. There is fundamental link between representations of $gl(\infty)$ and the Kadomtsev-Petviashvili (KP) hierarchy of nonlinear partial differential equations (see [113]). Namely, the equations in the hierarchy are described by the group orbit of the highest weight vector of the vacuum representation of $gl(\infty)$. An alternative construction using Wronskians has been described by Nimmo [114], which relies on the properties of supersymmetric polynomials, and has the added advantage of generating only the non-trivial equations in the hierarchy. Symmetric functions are also involved in the description of the tau functions for the KP hierarchy (functions which are a solution to every equation in the hierarchy) because it has been shown [113] that every *Schur polynomial* (which are just S-functions in disguise) is a tau function for the KP hierarchy. Indeed, not only are Schur polynomials solutions of the KP hierarchy, but the Hirota polynomials, which describe the individual equations in the hierarchy, can also be succinctly expressed as various combinations of Schur polynomials [115, 116]. By considering the various embeddings of affine Lie algebras in $gl(\infty)$, one obtains the various reductions of the KP hierarchy. Thus for example, the subalgebra $A_1^{(1)}$ of $gl(\infty)$ gives rise to the KdV hierarchy (of which the KdV equation is the simplest element). Similarly one can consider the algebras B_∞ and C_∞ (which are subalgebras of $gl(\infty)$) and construct the corresponding hierarchies. In particular with the BKP hierarchy (associated with the algebra B_∞), the role played by Schur polynomials is taken over by Schur Q-polynomials (again, closely related to Q-functions) [117, 118].

Another type of infinite-dimensional algebra which arises in different areas of physics is the Virasoro algebra [119] which is the algebra of conformal transformations in two-dimensions. In the seminal work of Belavin, Polyakov and A. B. Zamolodchikov [120] it was shown how the operator algebra structure of two-dimensional conformally-invariant quantum field theories is determined by the representation theory of the Virasoro algebra. Thus the complete classification of the unitary, irreducible highest weight representations of this algebra became an important problem. The first piece of the puzzle was solved with the calculation of the determinant of the inner product on the Verma module [121–123]. This allowed Friedan, Qiu and Shenker [124] to determine which values of central charge c and highest weight h are necessary for the representation to be unitary. At that stage however, it was not a sufficiency condition, and it was not until the Goddard-Kent-Olive construction [125] was discovered, that it was proven [126] that all the representations listed in the discrete series [124] were, in fact, unitary. The knowledge of the Kac-determinant also enabled the location of all Verma module singular vectors (vectors in the Verma module which are annihilated by all the Virasoro raising generators and which hence generate submodules) to be determined [123] and the embedding diagram [127] to be written down. This allowed the irreducible characters of the discrete series of unitary representations to be computed [128] (although the characters for the irreducible $c = 1$ representations had been known previously [122]). Knowledge of the explicit structure of the singular vectors [129–132] in the Verma module enables one (in principle) to derive the partial differential equations satisfied by the correlation functions in any two-dimensional conformal field theory. There are also Fock space representations of the Virasoro algebra, although in this case, the embedding structure of the submodules generated by the singular vectors is more complicated [133]. Nevertheless, the explicit form of the Fock space singular vectors can be derived [134–136] using vertex operator techniques (and in some cases directly [137]), and it turns out that the singular vectors can be succinctly expressed in terms of Schur polynomials ! Moreover, there is an amazing connection between the Fock space representations of the Virasoro algebra at $c = 1$ and the modified KP hierarchies which we shall now describe. The Fock representation $V(h = n^2/4, c = 1)$ is completely reducible and decomposes into irreducible Verma modules as

$$V(n^2/4, 1) = \bigoplus_{r=0}^{\infty} M((n + 2r)^2/4, 1).$$

It was shown in references [111,138] that there is a one-to-one correspondence between the vectors in $M((n + 2r)^2/4, 1)$ for $r \geq 1$ (the highest weight vector being given by a Schur polynomial) and the Hirota polynomials occuring in the n-th modified KP hierarchy. Thus there is an intimate connection between S-functions, hierarchies of nonlinear partial differential equations and the Virasoro algebra !

Just after the Virasoro algebra was introduced, two supersymmetric extensions of it were proposed, the Ramond [139] and the Neveu-Schwarz superalgebras [140], both collectively known as either the super-Virasoro algebras or the $N = 1$ superconformal algebras. In analogy with the ordinary ($N = 0$) case, the determinant formula was computed [122, 141–143] and the necessary [124] and sufficient [126] conditions on the values of central charge and highest weight for the representations to be unitary were found. Using these results, the embedding diagram for the Verma submodules generated by the singular vectors was obtained [144,145] and hence also the irreducible characters. Explicit forms for some of the singular vectors in Verma modules of the Neveu-Schwarz superalgebra were given in [146,147] and for the Ramond superalgebra in [148], while for Fock representations, singular vectors were contructed for the $c = 1$ [149] (where they were described in terms of super-Schur polynomials in commuting and anti-commuting variables) and the discrete series of representations [142,150,151].

As was shown in references [120, 124, 152], the critical exponents for the bulk correlation functions in a two-dimensional conformally invariant statistical mechanical system are governed by the highest weights of irreducible representations of the Virasoro algebra. In the case of quantum spin chains at criticality, there is a systematic procedure for extracting the *operator content* of the chains. That is, the irreducible representations describing the spectrum of the Hamiltonian of a chain of infinite length can be deduced from the eigenvalues of the Hamiltonian of a finite-size chain [153]. However, the allowed values of the highest weight h which describe the spectrum are limited by unitarity constraints, so that this process becomes much easier once the central charge associated with the model is known. This can also be computed from finite size chains, by looking at the ground state energy per unit site of the chain [154, 155].

The Ramond and Neveu-Schwarz superalgebras are not the only supersymmetric extensions of the Virasoro algebra. In fact the possible extensions were discussed by Ademollo et al. [156, 157] who wrote down explicitly the (anti-)commutation relations satisfied by $N = 2$, $N = 3$ and $N = 4$ superconformal algebras. In a similar vein to the $N = 1$ case, for the $N = 2$ superconformal algebras the Kac determinant [143, 158–161], the embedding diagram and character formulae [162–164] and Fock space singular vectors [135, 150] have be written down. For the $N = 3$ case, the determinant formulae have been conjectured [165], although a proof of them still seems to be lacking. For the $N = 4$ superconformal algebra, the Kac determinant was conjectured by Kent and Riggs [166] and partially proved by Matsuda [167] and conditions for the existence of unitary representations [168, 169] and character formulae [170] have been examined. This programme has also been extended to the $\widehat{so(4)}$-extended $N = 4$ superconformal algebras [171–174]. Another way of extending the Virasoro algebra is to include primary fields of spin greater than 2, resulting in a class of algebras called $\mathcal{W}$-algebras. These are nonlinear algebras, in the sense that the commutation relations do not close on the algebra itself, but on its universal enveloping algebra [175, 176].

The final algebras we shall mention are the quantum affine algebras, which are q-deformations of Kac-Moody algebras $\widehat{g}$. These algebras can either be defined by Chevally generators associated to the simple roots of the Dynkin diagram of $\widehat{g}$ [56, 57], or by the deformation of the central extension of the loop algebra $g \otimes \mathbb{C}[t, t^{-1}]$ [177]. However, in this latter description, the expression for the coproduct of the raising and lowering generators is not quite complete. In analogy with the undeformed case, vertex operator realizations for level one representations of $U_q(\widehat{sl(2)})$ were constructed [178] using one set of deformed bosonic oscillators. This was extended to the case of arbitrary level k by using *three* sets of deformed oscillators [179–182] These various realizations were shown to be equivalent, and the relations between them were established by Bougourzi [183]. Finally, arbitrary level representations of $U_q(\widehat{sl(n)})$ were constructed [184] using $n^2 - 1$ commuting sets of deformed bosons.

The main application of quantum affine algebras has been to studying the degeneracies in the spectrum of the (anti-ferromagnetic) XXZ quantum spin chain Hamiltonian in the thermodynamic limit (i.e. as the length of the chain tends to infinity) [185–189]. In this scheme, the Hamiltonian of the spin 1/2 XXZ chain is related to the derivation operator of $U_q(\widehat{sl(2)})$ and the space of states (realized by the infinite tensor product of two-dimensional vector spaces) is isomorphic to the tensor product of certain irreducible $U_q(\widehat{sl(2)})$ modules. The isomorphism is accomplished by certain "vertex operators" which are defined as intertwiners (algebra homomorphisms) between irreducible representations. From the definition, it is possible to obtain explicit realizations of these intertwiners acting on Fock representations [190, 191]. Physically relevant quantities are then given by ratios of weighted traces of these intertwiners over the appropriate irreducible representations [188].

1.4 Structure of the thesis

We begin in Chapter 2 with a review of the symmetric functions we shall be dealing with in this thesis: S-functions, Q-functions, Hall-Littlewood functions, Jack functions, and Macdonald's functions. In addition to reviewing some of the standard results concerning these functions, we provide a new proof of one of Littlewood's results concerning the inner product of S-functions, along with some generalizations. The inner product of Hall-Littlewood functions is defined, and Littlewood's result is generalized to this case as well, providing an efficient procedure for calculating these inner product coefficients. In the last part of this chapter, we examine the relationship between Macdonald's functions associated with different parameters, obtaining some explicit results with the aid of some identities involving basic hypergeometric series.

In Chapter 3 we define the concept of S-functions of a replicated argument which turn out to be related to Jack symmetric functions. We utilize these functions in answering a question concerning functions dual to compound and supersymmetric S-functions under an induced inner product. All of these results are then extended to the Hall-Littlewood case. In the final section, we introduce q-replicated Schur and Hall-Littlewood functions and show how they are related to Macdonald's functions. All of the standard bases for the ring of symmetric functions spanned by Macdonald's functions are described, while the relations between them are listed in Appendix F. Throughout this chapter, various interesting combinatorial identities arise and are examined.

The calculation of branching rules for the $N = 1$ and $N = 2$ superconformal algebras is the objective of Chapter 4. Identities such as the Jacobi triple product identity and Watson's quintuple product identity are used to derive branching rules for certain winding subalgebras of the $N = 1$ superconformal algebra. This is extended to the $N = 2$ superconformal algebras, where new identities are derived to tackle this branching rule problem, as well as the determination of certain decompositions of the tensor product of two irreducible representations of these algebras.

Various applications of the boson-fermion correspondence to the theory of symmetric functions are examined in Chapter 5. After a brief review of the classical boson-fermion correspondence, we derive a result concerning the multiplication and skewing of S-functions by power sum symmetric functions. This is mildly generalized to an identity involving S-functions and elementary Hall-Littlewood functions. We look at the Q-function case and derive similar results with regards to the multiplication and skewing of Q-functions by power sums. The boson-fermion correspondence is then applied to developing an algorithm for S-function multiplication, where we derive some explicit results for decomposing the product of S-functions in terms of non-standard S-functions. We then turn our attention to the calculation of outer plethysms of the form $s_\lambda(x^r) = s_\lambda(x) \otimes p_r(x)$, which enable us to derive some explicit results concerning more general plethysms.

We then generalize this by finding applications of Jing's generalized boson-fermion correspondence to the theory of Hall-Littlewood functions. Firstly, we derive an algorithm for decomposing Hall-Littlewood functions in terms of S-functions and exhibit this procedure through some explicit results for one and two-part, and one-hook Hall-littlewood functions. Secondly, we introduce the concept of the (outer) plethysm of two Hall-Littlewood functions and, as in the S-function case, we use the aforementioned generalized boson-fermion correspondence in deriving some explicit formulae concerning the calculation of these plethysms.

In Chapter 6 we use symmetric function techniques to carry out some calculations involving the quantum affine algebra $U_q(\widehat{sl(2)})$. We define an inner product on the power sums which allows us to realize the level one representation of $U_q(\widehat{sl(2)})$ on the ring of symmetric functions Λ_F, where $F = \mathbb{Q}(q)$. The matrix elements of the currents of this algebra are then described in terms of a set of symmetric functions defined by Kerov [192]. Traces of products of these currents can then be calculated following a procedure introduced by King [193]. These results are then extended to level k representations of $U_q(\widehat{sl(2)})$ using the realization given by Matsuo [182].

Conclusions are presented in Chapter 7 where we summarize the new results obtained and outline possible directions for future research.

Several appendices are given at the end where detailed proofs of some of the results described in this thesis are given, as well as providing convenient summaries of other results.

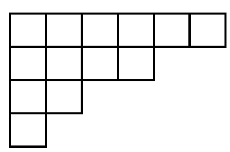
Chapter 2

Symmetric Functions

In this chapter we shall review the parts of the theory of symmetric functions which we shall use in this thesis. Most of the material covered here can be found in [33, 39, 194–197]. In addition, we derive some new results concerning inner products of Schur and Hall-Littlewood functions, and some identities between Macdonald's functions.

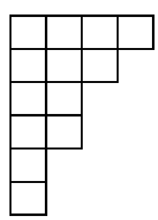
2.1 S-functions

Let us review some terminology. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a partition of weight n, (we shall alternatively write $|\lambda| = n$ or $\lambda \vdash n$) and length p ($\ell(\lambda) = p$). That is $\sum_{i=1}^{p} \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$. To a given partition λ, one can ascribe a Young diagram with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. Thus the partition $\lambda = (6, 4, 2, 1)$ has the Young diagram



(2.1)

Let λ' denote the conjugate partition of λ. That is, the partition obtained by reflecting the Young diagram about the main diagonal. Thus, in the above example, we have $\lambda' = (4, 3, 2^2, 1^2)$



An alternative notation for the partition λ is the Frobenius notation. Suppose the Young diagram of λ has r boxes along its main diagonal. Then we say λ has Frobenius *rank* r. Let a_i be the number of boxes to the right of the i'th diagonal box, and b_i the number of boxes directly below it. This specifies uniquely the partition λ, and so we can write

$$\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}, \tag{2.2}$$

where $a_1 > a_2 > \cdots > a_r \geq 0$, $b_1 > b_2 > \cdots > b_r \geq 0$. Sometimes we shall also write $\lambda = (a_1, \ldots, a_r | b_1, \ldots, b_r)$. Thus, the Young diagram (2.1) represents the partition $(5, 2|3, 1)$ in Frobenius notation. The advantage of this notation is that the conjugate of the partition (2.2), is

$$\lambda' = \begin{pmatrix} b_1 & b_2 & \cdots & b_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}.$$

A partition with Frobenius rank one is called a (one-)hook, while a rank 2 partition is called a two-hook etc.

Let the ring of symmetric polynomials in the indeterminates $x_1, x_2, \ldots, x_n$ with integer coefficients be denoted by Λ_n. That is

$$\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}.$$

Given a partition λ, define the monomial functions m_λ by

$$m_\lambda(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(n)}^{\lambda_n} \ ,$$

where if $\ell(\lambda) = p < n$, we set $\lambda_{p+1} = \cdots = \lambda_n = 0$. Then these functions, where λ runs over all partitions of length less than or equal to n, form a $\mathbb{Z}$-basis for Λ_n. That is, every symmetric function in Λ_n can be written as a linear combination of the m_λ with integer coefficients. It is often more useful to work with a (countably) infinite number of variables and then specialize by setting all but a finite number of them to zero. This is achieved by working with the ring Λ which is defined to be the inverse limit of the (graded)-rings Λ_n (see [33] for details).

Given $p \geq 0$, let $e_n(x)$ denote the n'th *elementary* symmetric function which has the generating function

$$\sum_{n=0}^{\infty} e_n(x)\, z^n = \prod_{i=1}^{\infty} (1 + x_i z) \ , \tag{2.3}$$

so that one can write

$$e_n(x) = m_{(1^n)}(x) = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}.$$

Similarly let $h_n(x)$ denote the n'th *complete* symmetric function with generating function

$$\sum_{n=0}^{\infty} h_n(x)\, z^n = \prod_{i=1}^{\infty} (1 - x_i z)^{-1}. \tag{2.4}$$

From (2.3) and (2.4) we have the important relation

$$\sum_{j=0}^{n} (-1)^j e_j(x)\, h_{n-j}(x) = 0 \ . \tag{2.5}$$

Finally we need the notion of the n'th power sum $p_n(x)$ defined by

$$p_n(x) = m_{(n)}(x) = \sum_{i=1}^{\infty} x_i^n. \tag{2.6}$$

A very important relation we shall find use for time and again, is

$$\prod_i (1 - x_i z)^{-\alpha} = \exp\left(\alpha \sum_{n=1}^{\infty} p_n(x) z^n\right), \qquad \alpha \in \mathbb{R}. \tag{2.7}$$

Given a partition λ of length $\ell(\lambda) \leq p$, let Δ_λ be the following determinant,

$$\Delta_\lambda = \begin{vmatrix} x_1^{\lambda_1+p-1} & x_1^{\lambda_2+p-2} & \cdots & x_1^{\lambda_p} \\ x_2^{\lambda_1+p-1} & x_2^{\lambda_2+p-2} & \cdots & x_2^{\lambda_p} \\ \vdots & \vdots & \ddots & \vdots \\ x_p^{\lambda_1+p-1} & x_p^{\lambda_2+p-2} & \cdots & x_p^{\lambda_p} \end{vmatrix},$$

with $\Delta_0 = \prod_{1 \leq i < j \leq p}(x_i - x_j)$ being the Vandermonde determinant. For a finite number of indeterminates $x_1, \ldots, x_p$ one defines the S-function $s_\lambda(x_1, \ldots, x_p)$ to be

$$s_\lambda(x_1, \ldots, x_p) = \frac{\Delta_\lambda}{\Delta_0}. \tag{2.8}$$

This has the useful property that [48]

$$s_\lambda(x_1, \ldots, x_p, 0, \ldots, 0) = \begin{cases} s_\lambda(x_1, \ldots, x_p), & \text{if } \ell(\lambda) \leq p, \\ 0, & \text{if } \ell(\lambda) > p, \end{cases}$$

which allows one to talk about S-functions in a countable number of indeterminates x. The S-functions s_λ form a $\mathbb{Z}$-basis for the ring Λ. There are actually several standard bases for Λ. For $\lambda = (\lambda_1, \ldots, \lambda_p)$ let $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_p}$, and similarly for e_λ and p_λ. Then it is known that h_λ and e_λ also form $\mathbb{Z}$-bases for Λ, while the p_λ form a $\mathbb{Q}$-basis.

There is an involutive endomorphism ω on Λ defined by its action on the power sums $\omega(p_n) = (-1)^{n-1} p_n$ which has the remarkable property that

$$\omega(s_\lambda(x)) = s_{\lambda'}(x). \tag{2.9}$$

One important identity involving S-functions and the complete symmetric functions h_λ is the Jacobi-Trudi identity

$$s_\lambda(x) = \det\left(h_{\lambda_i - i + j}(x)\right)_{1 \leq i,j \leq n}, \tag{2.10}$$

where $n \geq \ell(\lambda)$ is arbitrary. In the above determinant, it is understood that $h_n = 0$ if $n < 0$. In particular, we have $h_n = s_{(n)}$, $e_n = s_{(1^n)}$ and the useful formula for one-hook S-functions

$$s_{(n-r,1^r)}(x) = \sum_{j=0}^{r} (-1)^{j+r} h_{n-j}(x) e_j(x)\ . \tag{2.11}$$

The equation (2.10) can be used to derive *modification rules* which enable non-standard S-functions (functions s_λ where the elements of λ are not in non-increasing order) to be expressed in terms of standard S-functions.

- $\{\lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_p\} = -\{\lambda_1, \ldots, \lambda_{i+1}-1, \lambda_i+1, \ldots, \lambda_p\}$ (interchanging two consecutive rows in the determinant),
- if $\lambda_{i+1} = \lambda_i$ then $s_\lambda = 0$ (two consecutive rows are equal),
- if $\lambda_p < 0$ then $s_\lambda = 0$ (the last row is a row of zeros).

Another determinantal formula for S-functions exists for λ written in Frobenius notation, the Giambelli formula: if $\lambda = (a_1, \ldots, a_r | b_1, \ldots, b_r)$, then

$$s_\lambda(x) = \det(s_{(a_i|b_j)}(x))_{1 \le i,j \le r} \ , \tag{2.12}$$

where $s_{(a|b)} = s_{(a+1,1^b)}$ is a one-hook S-function. This provides us with the alternative modification rules

- $\{a_1, \ldots, a_i, a_{i+1}, \ldots, a_r | b_1, \ldots, b_r\} = -\{a_1, \ldots, a_{i+1}, a_i, \ldots, a_r | b_1, \ldots, b_r\}$,
- $\{a_1, \ldots, a_i, a_i, \ldots, a_r | b_1, \ldots, b_r\} = 0$,

with similar rules for the labels b_i.

As previously mentioned, one expression for the one-hook S-function $s_{(a|b)}(x)$ is just

$$s_{(a|b)} = h_{a+1}e_b - h_{a+2}e_{b-1} + \cdots + (-1)^b h_{a+b+1}. \tag{2.13}$$

From this formula, Macdonald [33, p 30, Ex. 9] defines $s_{(a|b)}$ for *arbitrary* integers a and b and concludes that $s_{(a|b)} = 0$ except when $a + b = -1$, in which case

$$s_{(a|b)} = (-1)^b. \tag{2.14}$$

We would like to point out that (2.14) is valid *only* in the case $a < 0$, $b \ge 0$, because there are $b + 1$ terms in (2.13). To get an expression in the case $a \ge 0$, $b < 0$, we use (2.5) to rewrite (2.13) in the form

$$s_{(a|b)} = h_a e_{b+1} - h_{a-1}e_{b+2} + \cdots + (-1)^a e_{a+b+1},$$

which is a sum involving $a + 1$ terms. Thus when $a \ge 0$, $b < 0$ with $a + b = -1$, we should define $s_{(a|b)} = (-1)^a = (-1)^{b+1}$. Hence the correct definition of $s_{(a|b)}$ in the case where one of a or b is negative is

$$s_{(a|b)} = \begin{cases} (-1)^b, & b \ge 0,\ a < 0,\ a + b = -1, \\ (-1)^{b+1}, & b < 0,\ a \ge 0,\ a + b = -1, \\ 0, & \text{else.} \end{cases}$$

This allows us to give a third modification rule, namely that if $a_r + b_r = -1$, then

$$\{a_1, \ldots, a_{r-1}, a_r | b_1, \ldots, b_{r-1}, b_r\} = \begin{cases} (-1)^{b_r}\{a_1, \ldots, a_{r-1} | b_1, \ldots, b_{r-1}\}, & b_r \ge 0, \\ (-1)^{b_r+1}\{a_1, \ldots, a_{r-1} | b_1, \ldots, b_{r-1}\}, & b_r < 0, \\ 0 & \text{else.} \end{cases}$$

Multiplication of S-functions (also called the *outer* product of S-functions) is expressed by means of coefficients $c_{\mu\nu}^{\lambda}$ such that

$$s_\mu \, s_\nu = \sum_\lambda c_{\mu\nu}^{\lambda} \, s_\lambda. \tag{2.15}$$

The coefficients $c_{\mu\nu}^{\lambda}$ are called Littlewood-Richardson coefficients, and there is a combinatorial rule for their determination, the details of which, we refer the reader to [33].

S-functions and power sums are related by the characters of the symmetric group S_n by the Frobenius formulae

$$p_\lambda = \sum_\rho \chi_\lambda^\rho \, s_\rho, \qquad s_\lambda = \sum_\rho z_\rho^{-1} \chi_\rho^\lambda \, p_\rho, \tag{2.16}$$

where χ_ρ^λ is the character of the representation (λ) of S_n evaluated on elements of the class ρ, and $z_\lambda = \prod_i i^{m_i} m_i!$ for a partition of the form $\lambda = (n^{m_n}, \ldots, 2^{m_2}, 1^{m_1})$. These S_n characters obey the orthogonality relations

$$\sum_{\rho \vdash n} z_\rho^{-1} \chi_\rho^\mu \chi_\rho^\nu = \delta_{\mu\nu}, \qquad \sum_{\lambda \vdash n} \chi_\rho^\lambda \chi_\sigma^\lambda = z_\rho \delta_{\rho\sigma}. \tag{2.17}$$

Moreover, there are the useful relations

$$\chi_\rho^{\lambda'} = \varepsilon_\rho \, \chi_\rho^\lambda, \qquad \varepsilon_\rho = (-1)^{|\rho| - \ell(\rho)}. \tag{2.18}$$

An important ingredient for discussing S-functions is the fact that one can define an inner product on Λ by requiring that for the functions $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$ we have

$$\langle \, p_\lambda, p_\mu \, \rangle = z_\lambda \, \delta_{\lambda\mu}. \tag{2.19}$$

From the explicit form of the power sums (2.6), it follows that [33]

$$\sum_\lambda \frac{1}{z_\lambda} \, p_\lambda(x) \, p_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}.$$

By using the Frobenius formulae (2.16) and then the orthogonality relations (2.17) in the above equation, we end up with the Cauchy identity

$$\sum_\lambda s_\lambda(x) s_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}. \tag{2.20}$$

Macdonald has shown [33] that if (u_λ) and (v_λ) are $\mathbb{Z}$-bases of Λ, then the following are equivalent

$$\langle \, u_\lambda, v_\mu \, \rangle = \delta_{\lambda\mu}, \quad \forall \, \lambda, \mu \qquad \Longleftrightarrow \qquad \sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \tag{2.21}$$

It then follows from (2.20) that the S-functions s_λ are self-dual under the inner product (2.19):

$$\langle \, s_\lambda, s_\mu \, \rangle = \delta_{\lambda\mu}. \tag{2.22}$$

Using the Littlewood-Richardson coefficients one can define skew S-functions

$$s_{\lambda/\mu}(x) = \sum_\nu c^\lambda_{\mu\nu} s_\nu(x),$$

having the property that under the inner product (2.19),

$$\langle\, s_{\lambda/\mu}, s_\nu \,\rangle = \langle\, s_\lambda, s_\mu s_\nu \,\rangle.$$

Equation (2.20) allows one to show [33] that S-functions in the indeterminates $(x_1, x_2, \ldots, y_1, y_2, \ldots)$ (which we shall call *compound* S-functions) can be expressed in the following form

$$s_\lambda(x,y) = \sum_\sigma s_{\lambda/\sigma}(x) s_\sigma(y) = \sum_{\rho,\sigma} c^\lambda_{\rho\sigma} s_\rho(x) s_\sigma(y). \tag{2.23}$$

Finally, there is the notion of a supersymmetric S-function

$$s_\lambda(x/y) = \sum_\rho (-1)^{|\rho|} s_{\lambda/\rho}(x) s_{\rho'}(y),$$

where ρ' is the conjugate partition of ρ. We shall discuss more about compound and supersymmetric S-functions in Chapter 3.

Given partitions λ, μ and ν of weight n, define coefficients $\gamma^\lambda_{\mu\nu}$ by the decomposition of characters

$$\chi^\mu \chi^\nu = \sum_\lambda \gamma^\lambda_{\mu\nu}\, \chi^\lambda. \tag{2.24}$$

That is, $\gamma^\lambda_{\mu\nu}$ is the multiplicity of the irreducible S_n representation (λ) in the Kronecker product decomposition of $(\mu) \otimes (\nu)$. Using the orthogonality relations (2.17), we see that

$$\gamma^\lambda_{\mu\nu} = \sum_\rho \frac{1}{z_\rho} \chi^\lambda_\rho \chi^\mu_\rho \chi^\nu_\rho\ . \tag{2.25}$$

Note that they are symmetric under the interchange of any of the three indices λ, μ and ν, and have the useful property that $\gamma^\lambda_{\mu\nu} = \gamma^\lambda_{\mu'\nu'}$, which follows from application of (2.18). These coefficients are used to define the *inner* product $s_\mu \circ s_\nu$ of two S-functions,

$$s_\mu \circ s_\nu = \sum_\lambda \gamma^\lambda_{\mu\nu}\, s_\lambda\ . \tag{2.26}$$

The label "inner" is just there to distinguish it from the "outer" product of S-functions given by (2.15) and should not be confused with the inner product (bilinear form) (2.19). Although (2.25) gives an explicit formula for calculating the inner product coefficients, it is not particularly efficient for that purpose. Littlewood [11] has given an alternative procedure for their evaluation, using the theory of induced representations. It is based on the formula

$$(s_\lambda s_\mu) \circ s_\nu = \sum_\sigma (s_\lambda \circ s_{\nu/\sigma})(s_\mu \circ s_\sigma)\ . \tag{2.27}$$

which Remmel [12] has generalized to the case where the S-functions appearing here are replaced by skew S-functions. We would like to point out another proof of (2.27) based on generating function techniques. First note that this is equivalent to proving that

$$\sum_{\rho} c^{\rho}_{\lambda\mu}\gamma^{\sigma}_{\rho\nu} = \sum_{\alpha\beta\eta\rho} c^{\nu}_{\rho\eta}c^{\sigma}_{\alpha\beta}\gamma^{\alpha}_{\lambda\rho}\gamma^{\beta}_{\mu\eta}, \tag{2.28}$$

holds for all partitions σ, λ, μ and ν. Call the expression on the left of (2.28) $I_{\sigma\lambda\mu\nu}$ and that on the right $J_{\sigma\lambda\mu\nu}$. Upon using the fact that

$$s_\lambda(xy) = \sum_{\mu,\nu} \gamma^{\lambda}_{\mu\nu}\, s_\mu(x)s_\nu(y), \tag{2.29}$$

for the set of indeterminates $xy = (x_1y_1, x_1y_2, \ldots, x_2y_1, x_2y_2, \ldots)$ we can form the generating function

$$\sum_{\sigma\lambda\mu\nu} I_{\sigma\lambda\mu\nu}s_\sigma(u)s_\lambda(v)s_\mu(x)s_\nu(y) = \prod_{i,j,k}(1-u_iv_jy_k)^{-1}\prod_{l,m,n}(1-u_lx_my_n)^{-1}.$$

However, utilizing (2.29) once again we see that the product on the right is also the generating function for $\sum_{\sigma\lambda\mu\nu} J_{\sigma\lambda\mu\nu}s_\sigma(u)s_\lambda(v)s_\mu(x)s_\nu(y)$. Thus $I_{\sigma\lambda\mu\nu} = J_{\sigma\lambda\mu\nu}$. Hence (2.28) is proved.

The above method can be used for other identities of the same ilk, so that we can prove that, for example

$$\sum_{\rho\sigma} c^{\rho}_{\lambda\mu}c^{\sigma}_{\rho\nu}\gamma^{\alpha}_{\sigma\tau} = \sum_{\beta\eta\rho\sigma} \gamma^{\lambda}_{\beta_1\eta_1}\gamma^{\mu}_{\beta_2\eta_2}\gamma^{\nu}_{\beta_3\eta_3}c^{\sigma}_{\beta_1\beta_2}c^{\tau}_{\beta_3\sigma}c^{\rho}_{\eta_1\eta_2}c^{\alpha}_{\eta_3\rho},$$

is true for all partitions λ, μ, ν, τ and α. Hence it follows that

$$(s_\lambda s_\mu s_\nu)\circ s_\tau = \sum_{\alpha\beta}(s_\lambda\circ s_\alpha)(s_\mu\circ s_{\beta/\alpha})(s_\nu\circ s_{\tau/\beta})\ . \tag{2.30}$$

Similarly one can prove

$$\sum_{\alpha} \gamma^{\lambda_1}_{\alpha_1\alpha_2}c^{\alpha_1}_{\lambda_2\lambda_3}c^{\alpha_2}_{\lambda_4\lambda_5} = \sum_{\alpha\beta\eta\sigma} \gamma^{\alpha_1}_{\beta_1\eta_1}\gamma^{\alpha_2}_{\beta_2\eta_2}\gamma^{\alpha_3}_{\beta_3\eta_3}\gamma^{\alpha_4}_{\beta_4\eta_4}c^{\sigma_1}_{\alpha_1\alpha_2}c^{\sigma_2}_{\alpha_3\alpha_4}c^{\lambda_1}_{\sigma_1\sigma_2}c^{\lambda_2}_{\beta_1\beta_2}c^{\lambda_3}_{\beta_3\beta_4}c^{\lambda_4}_{\eta_1\eta_3}c^{\lambda_5}_{\eta_2\eta_4}\ ,$$

which corresponds to the identity

$$(s_{\lambda_2}s_{\lambda_3})\circ(s_{\lambda_4}s_{\lambda_5}) = \sum_{\beta_1\beta_2\eta_1\eta_2}(s_{\beta_1}\circ s_{\lambda_4/\eta_1})(s_{\lambda_2/\beta_1}\circ s_{\lambda_5/\eta_2})(s_{\beta_2}\circ s_{\eta_1})(s_{\lambda_3/\beta_2}\circ s_{\eta_2})\ . \tag{2.31}$$

The identities (2.30), (2.31) appear to be new.

2.2 Q-functions

The study of projective representations of S_n led Schur to introduce some more symmetric functions, which are called Q-functions. Elementary Q-functions $q_n(x)$ are defined by the generating function

$$\sum_{n=0}^{\infty} q_n(x)z^n = \prod_i \left(\frac{1+x_iz}{1-x_iz}\right) = \exp\left(2\sum_{n\text{ odd}}\frac{1}{n}p_n(x)z^n\right)\ , \tag{2.32}$$

so that $q_n(x) \in \mathbb{Q}[p_1, p_3, p_5, \ldots]$. One consequence of (2.32) is that it leads to a recursive definition of $q_n(x)$ [197]

$$k\, q_k(x) = 2 \sum_{\substack{i+j=k \\ j \text{ odd}}} q_i(x) p_j(x) \ ,$$

which, by Cramer's rule, provides us with determinant formulae which express the q_n in terms of p_{2k+1}: $k \geq 0$

$$q_{2n} = \frac{2^{2n-1}}{(2n)!} \begin{vmatrix} 2p_1^2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ p_3 & p_1 & -\frac{3}{2} & 0 & \cdots & 0 & 0 \\ 2p_3p_1 & 0 & p_1 & -2 & \cdots & 0 & 0 \\ p_5 & p_3 & 0 & p_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \\ p_{2n-1} & p_{2n-3} & 0 & \cdots & 0 & p_1 & -k+\frac{1}{2} \\ 2p_{2n-1}p_1 & 0 & p_{2n-3} & \cdots & p_3 & 0 & p_1 \end{vmatrix}_{2n-1} ,$$

$$q_{2n+1} = \frac{4^n}{(2n+1)!} \begin{vmatrix} 2p_1^2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ p_3 & p_1 & -\frac{3}{2} & 0 & \cdots & 0 & 0 \\ 2p_3p_1 & 0 & p_1 & -2 & \cdots & 0 & 0 \\ p_5 & p_3 & 0 & p_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \\ 2p_{2n-1}p_1 & 0 & p_{2n-3} & \cdots & 0 & p_1 & -k \\ p_{2n+1} & p_{2n-1} & 0 & \cdots & p_3 & 0 & p_1 \end{vmatrix}_{2n} ,$$

(the subscripts on the determinants in the above equations represent the size of the determinant) which are similar to those which express the complete and elementary symmetric functions h_n and e_n in terms of power sums p_n [33, p. 20].

The original definition for $Q_{(\lambda_1,\ldots,\lambda_p)}(x_1,\ldots,x_n)$ for a finite number of arguments takes the form [3]

$$Q_{(\lambda_1,\ldots,\lambda_p)}(x_1,\ldots,x_n) = 2^p \sum_{i_1,\ldots,i_p=1}^{n} \frac{x_{i_1}^{\lambda_1} \cdots x_{i_p}^{\lambda_p}}{u_{i_1} \cdots u_{i_p}} \mathcal{A}(x_{i_p},\ldots,x_{i_2},x_{i_1}) \ , \tag{2.33}$$

where

$$\mathcal{A}(w_1,\ldots,w_p) = \prod_{1 \leq i < j \leq p} \frac{w_i - w_j}{w_i + w_j} \ ,$$

and

$$u_i = \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{x_i - x_j}{x_i + x_j} \ .$$

These functions obey a Cauchy identity

$$\sum_{\lambda \in DP} 2^{-\ell(\lambda)} Q_\lambda(x) Q_\lambda(y) = \prod_{i,j=1}^{n} \left(\frac{1 + x_i y_j}{1 - x_i y_j} \right) \ ,$$

where DP is the set of all partitions with distinct parts.

Another (equivalent) way of defining Q-functions is through Pfaffians. Before we give it, note that (2.32) entails the constraint

$$\sum_{j=0}^{n}(-1)^j q_j(x)\, q_{n-j}(x) = 0\ . \tag{2.34}$$

For a two part partition (m,n) (strict or non-strict), it follows from (2.33) that

$$Q_{(m,n)}(x) = q_m(x)q_n(x) + 2\sum_{j=1}^{n}(-1)^j q_{m+j}(x)q_{n-j}(x).$$

Observe that (2.34) implies that $Q_{(m,n)}(x) = -Q_{(n,m)}(x)$. If $\mathrm{Pf}(A)$ denotes the Pfaffian of the anti-symmetric matrix A, then for any partition λ with distinct parts $\lambda_1 > \lambda_2 > \cdots > \lambda_p$, $Q_\lambda(x)$ can equivalently be defined by

$$Q_\lambda(x) = \begin{cases} \mathrm{Pf}(Q_{\lambda_i\lambda_j}(x)), & p \text{ even},\\ \mathrm{Pf}(Q_{\tilde{\lambda}_i\tilde{\lambda}_j}(x)), & p \text{ odd},\end{cases}$$

where $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_p, 0)$. This definition has the advantage that it can be used when the number of indeterminates is not necessarily finite.

More recently, an analogue of (2.8) has been proven [118] with the result that for $n > p$, p and n even,

$$Q_{(\lambda_1,\ldots,\lambda_p)}(x_1,\ldots,x_n) = 2^p\,\frac{\mathrm{Pf}(A')}{\mathrm{Pf}(A)}\ ,$$

where

$$A_{ij} = \left(\frac{x_i - x_j}{x_i + x_j}\right), \qquad A' = \begin{pmatrix} A & C \\ -C^T & 0 \end{pmatrix}, \qquad C_{ij} = x_i^{\lambda_j}\ .$$

If p is odd one must replace λ by $\tilde{\lambda}$, and if n is odd, $(x_1,\ldots,x_n)$ by $(x_1,\ldots,x_n,0)$.

2.3 Hall-Littlewood functions

One particular generalization of the idea of S-functions is that of the Hall-Littlewood function [24,25] in the variables $x_1, x_2, \ldots, x_n$ defined for a partition of length $\ell(\lambda) \le n$ by

$$Q_\lambda(x_1,\ldots,x_n;t) = (1-t)^{\ell(\lambda)} \sum_{\sigma\in S_n} \sigma\left(x_1^{\lambda_1}\cdots x_n^{\lambda_n} \prod_{1\le i<j\le n} \frac{x_i - t x_j}{x_i - x_j}\right), \tag{2.35}$$

where σ acts as $\sigma(x_1^{\lambda_1}\cdots x_n^{\lambda_n}) = x_{\sigma(1)}^{\lambda_1}\cdots x_{\sigma(n)}^{\lambda_n}$, and t is some parameter. When $t=0$, Q_λ reduces to the S-function s_λ, while the Q-functions introduced in the previous section are just $Q_\lambda(x;-1)$.

There exists a modification rule for Hall-Littlewood functions $Q_\lambda(x;t)$, where λ is a non-standard partition as follows. Let $s<r$ and $m=[\frac{1}{2}(r-s)]$. Then[1]

$$Q_{(s,r)} = \begin{cases} tQ_{(r,s)} + \displaystyle\sum_{i=1}^{m}(t^{i+1}-t^{i-1})Q_{(r-i,s+i)}, & r-s=2m+1, \\ tQ_{(r,s)} + \displaystyle\sum_{i=1}^{m-1}(t^{i+1}-t^{i-1})Q_{(r-i,s+i)} + (t^m-t^{m-1})Q_{(r-m,s+m)}, & r-s=2m. \end{cases} \tag{2.36}$$

Given a field F, let $\Lambda_F = \Lambda \otimes_{\mathbb{Z}} F$ be the ring of symmetric functions over F. In the case of the Hall-Littlewood functions (2.35) in an infinite number of indeterminates, it is known [33] that they form a basis for Λ_F where $F=\mathbb{Q}(t)$, the field of rational functions in t.

An inner product $\langle\,\cdot,\cdot\,\rangle_t$ can be defined on the space Λ_F so that the power sum symmetric functions p_λ obey the orthogonality conditions

$$\langle\, p_\lambda(x), p_\mu(x)\,\rangle_t = \delta_{\lambda\mu} z_\lambda(t), \qquad \text{where} \quad z_\lambda(t) = z_\lambda \prod_i (1-t^{\lambda_i})^{-1}, \tag{2.37}$$

with z_λ being defined as on page 14. Given a partition $\lambda=(n^{m_n},\ldots,1^{m_1})$ let $b_\lambda(t) = \prod_i \gamma_{m_i}(t)$ where

$$\gamma_j(t) = (1-t)(1-t^2)\cdots(1-t^j).$$

Let $P_\lambda(x;t) = b_\lambda^{-1}(t)\,Q_\lambda(x;t)$. Then these functions obey the generalized Cauchy identity

$$\sum_\lambda P_\lambda(x;t)Q_\lambda(x;t) = \prod_{i,j}\frac{1-tx_iy_j}{1-x_iy_j}. \tag{2.38}$$

By an argument similar to that occuring for the S-function case (*c.f.* (2.21)), it follows that under the inner product (2.37) we have the important orthogonality condition

$$\langle\, P_\lambda(x;t), Q_\mu(x;t)\,\rangle_t = \delta_{\lambda\mu}\,.$$

A function of particular interest is the elementary Hall-Littlewood function $q_n(x;t) \equiv Q_{(n)}(x;t)$ which has the generating function

$$\sum_{n=0}^{\infty} q_n(x;t)z^n = \prod_i \frac{1-tx_iz}{1-x_iz}, \tag{2.39}$$

so that, in particular $q_n(x;0)=h_n(x)$. From the determinantal expansion of $h_n(x)$ in terms of powers sums [33], we have

$$q_n(x;t) = \frac{1}{n!}\begin{vmatrix} (1-t)p_1 & -1 & 0 & \cdots & 0 & 0 \\ (1-t^2)p_2 & (1-t)p_1 & -2 & 0 & \cdots & 0 \\ (1-t^3)p_3 & (1-t^2)p_2 & (1-t)p_1 & -3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ (1-t^{n-1})p_{n-1} & (1-t^{n-2})p_{n-2} & \cdots & \cdots & (1-t)p_1 & -n+1 \\ (1-t^n)p_n & (1-t^{n-1})p_n & \cdots & \cdots & \cdots & (1-t)p_1 \end{vmatrix}.$$

[1]This corrects a misprint in Macdonald [33, p. 109].

The function $q_n(x;t)$ also has a simple expansion in terms of $h_n(x)$ and $e_n(x)$,

$$q_n(x;t) = \sum_{j=0}^{n} (-t)^j h_{n-j}(x) e_j(x), \tag{2.40}$$

which follows directly from the generating functions (2.39), (2.3) and (2.4). This relation provides us with an elementary way of proving the result [22] [33, p. 110]

$$P_{(n)}(x;t) = \sum_{j=0}^{n-1} (-t)^j \, s_{(n-j,1^j)}(x) \ . \tag{2.41}$$

To prove (2.41), we can use (2.11) to write the right-hand side of the above equation as

$$\begin{aligned}
\text{r.h.s.} &= \sum_{j=0}^{n-1} \sum_{k=0}^{n-1-j} (-1)^j t^k h_{n-j} e_j \\
&= (1-t)^{-1} \left(\sum_{j=0}^{n-1} (-t)^j h_{n-j} e_j - t^n \sum_{j=0}^{n-1} (-1)^j h_{n-j} e_j \right) \\
&= (1-t)^{-1} \sum_{j=0}^{n} (-t)^j h_{n-j} e_j = (1-t)^{-1} q_n(x;t) = \quad \text{l.h.s.} \ ,
\end{aligned}$$

where in the last line we have used (2.5).

Multiplication of Hall-Littlewood functions $P_\lambda(x;t)$ is defined by means of coefficients $f^\lambda_{\mu\nu}(t)$ such that

$$P_\mu(x;t) P_\nu(x;t) = \sum_\lambda f^\lambda_{\mu\nu}(t) P_\lambda(x;t),$$

where $f^\lambda_{\mu\nu}(t) \in \mathbb{Z}[t]$ and are non-zero if and only if $|\lambda| = |\mu| + \nu|$. There are several explicit formulae known for these coefficients, especially when $\lambda - \mu$ is a horizontal or vertical strip [32, 33]. Using the identity (2.38), Hall-Littlewood functions with compound arguments $(x_1, x_2, \ldots, y_1, y_2, \ldots)$ can be expressed as

$$\begin{aligned}
Q_\lambda(x,y;t) &= \sum_{\rho\sigma} f^\lambda_{\rho\sigma}(t) Q_\rho(x;t) \, Q_\sigma(y;t), \\
P_\lambda(x,y;t) &= \sum_{\rho\sigma} \bar{f}^\lambda_{\rho\sigma}(t) P_\rho(x;t) \, P_\sigma(y;t),
\end{aligned}$$

where $\bar{f}^\lambda_{\rho\sigma}(t) = \frac{b_\rho(t) b_\sigma(t)}{b_\lambda(t)} f^\lambda_{\rho\sigma}(t)$. Similarly skew Hall-Littlewood functions can be defined

$$\begin{aligned}
P_{\mu/\nu}(x;t) &= \sum_\rho \bar{f}^\mu_{\nu\rho}(t) \, P_\rho(x;t), \\
Q_{\mu/\nu}(x;t) &= \sum_\rho f^\mu_{\nu\rho}(t) \, Q_\rho(x;t).
\end{aligned} \tag{2.42}$$

Following [33] we can define an operator D on Λ_F, such that $D(f)$ is the adjoint of multiplication by f

$$\langle \, D(f) \, m, g \, \rangle_t = \langle \, m, f g \, \rangle_t \ .$$

Then it is clear, using the inner product (2.37), that the skew functions (2.42) can equivalently be defined in terms of this adjoint operator,

$$\begin{aligned} D(Q_\nu)\,P_\mu &= P_{\mu/\nu}, & \qquad D(Q_\nu)\,Q_\mu = b_\nu Q_{\mu/\nu}, \\ D(P_\nu)\,Q_\mu &= Q_{\mu/\nu}, & \qquad D(P_\nu)\,P_\mu = b_\nu^{-1} P_{\mu/\nu}. \end{aligned}$$

The transformation between the power sums and the Hall-Littlewood functions $P_\lambda(x;t)$ is accomplished by polynomials $X^\lambda_\mu(t)$, such that

$$p_\mu(x) = \sum_\lambda X^\lambda_\mu(t)\, P_\lambda(x;t), \tag{2.43}$$

which clearly have the property that $X^\lambda_\mu(0) = \chi^\lambda_\mu$. Like the characters of S_n, these polynomials obey certain orthogonality relations,

$$\sum_{\rho\vdash n} z_\rho^{-1}(t) X^\mu_\rho(t) X^\nu_\rho(t) = \delta_{\mu\nu}\, b_\nu(t), \qquad \sum_{\lambda\vdash n} b_\lambda^{-1}(t) X^\lambda_\rho(t) X^\lambda_\sigma(t) = z_\rho(t)\delta_{\rho\sigma}, \tag{2.44}$$

which can be derived from the relation (2.38). The polynomials $X^\lambda_\mu(t)$ are related to Green's polynomials $Q^\lambda_\mu(q) = q^{n(\lambda)} X^\lambda_\mu(q^{-1})$, $n(\lambda) = \sum_i (i-1)\lambda_i$ which have been tabulated by Morris [28].

Given partitions μ, ν, λ and ρ of n, define coefficients $\Gamma^\lambda_{\mu\nu}(t)$ by

$$X^\mu_\rho(t)\; X^\nu_\rho(t) = \sum_\lambda \Gamma^\lambda_{\mu\nu}(t)\, X^\lambda_\rho(t), \qquad \forall \rho \vdash n. \tag{2.45}$$

Applying the orthogonality relations (2.44) to (2.45) furnishes the formula

$$\Gamma^\lambda_{\mu\nu}(t) = \frac{1}{b_\lambda(t)} \sum_\rho z_\rho^{-1}(t) X^\lambda_\rho(t) X^\mu_\rho(t) X^\nu_\rho(t)\ . \tag{2.46}$$

Notice the symmetry

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} = \frac{b_\mu(t)}{b_\lambda(t)}\ \Gamma^\mu_{\lambda\nu}\ .$$

Tables of the polynomials $\Gamma^\lambda_{\mu\nu}(t)$ are given in Appendix E for partitions of weight ≤ 4. From these tables, it appears that $\Gamma^\lambda_{\mu\nu}(t) \in \mathbb{Z}[t]$ although it is not clear whether this is valid in general. If it were true, it would be natural to ask whether or not there was a combinatorial description of these polynomials as there is for the Kostka-Foulkes polynomials $K_{\lambda\mu}(t)$, which are the elements of the transition matrix between Hall-Littlewood and S-functions.

Using these coefficients, one can define the *inner* product of Hall-Littlewood functions

$$P_\mu(x;t) \circ P_\nu(x;t) = \sum_\lambda \Gamma^\lambda_{\mu\nu}(t) P_\lambda(x;t)\ . \tag{2.47}$$

Note that from (2.43) and (2.45) it follows that for the argument $xy = (x_1y_1, x_1y_2, \ldots, x_2y_1, x_2y_2, \ldots)$ we have

$$P_\lambda(xy;t) = \sum_{\mu\nu} \Gamma^\lambda_{\mu\nu}(t)\, P_\mu(x;t) P_\nu(y;t). \tag{2.48}$$

Using (2.48) and generating function techniques, one can derive an analogue of equation (2.27) in the form

$$\sum_\rho f^\rho_{\lambda\mu}(t)\Gamma^\rho_{\sigma\nu}(t) = \sum_{\alpha\beta\eta\rho} f^\nu_{\beta\rho}(t) f^\sigma_{\alpha\eta}(t)\Gamma^\lambda_{\alpha\beta}(t)\Gamma^\mu_{\eta\rho}(t). \tag{2.49}$$

Unfortunately, it is *not* possible to express either side of the the above equation in terms of inner or outer products of Hall-Littlewood function, as in (2.27). Nevertheless, we can still use (2.49) to calculate $\Gamma^\lambda_{\mu\nu}(t)$ in some simple cases, in a similar manner to (2.27).

Example
From the definition (2.46) and the orthogonality relations (2.44), we know that $\Gamma^{(n)}_{\rho\sigma}(t) = \delta_{\rho\sigma} b_\rho(t)/(1-t)$. Suppose we wish to calculate $\Gamma^{(n-1,1)}_{\rho\sigma}(t)$. Since

$$P_{(n-1)}\, P_{(1)} = \begin{cases} P_{(n-1,1)} + P_{(n)}, & n > 2, \\ (1+t)P_{(1^2)} + P_{(2)}, & n = 2, \end{cases}$$

we can substitute $\lambda = (n-1)$, $\mu = (1)$ in (2.49) when $n \neq 2$ to obtain

$$\Gamma^{(n-1,1)}_{\sigma\nu} = \sum_{\alpha\vdash n-1} f^\nu_{\alpha(1)}(t) f^\sigma_{\alpha(1)}(t) \frac{b_\alpha(t)}{(1-t)} \;-\; \frac{b_\nu(t)}{(1-t)}\,\delta_{\sigma\nu}.$$

The advantage of this expression over the direct formula (2.46), is that it involves a sum involving $p(n-1)+1$ terms (where $p(n)$ is the number of partitions of n), instead of $p(n)$ terms. Thus, for example, using $f^{(1^3)}_{(1^2)(1)}(t) = 1+t+t^2$, $f^{(1^3)}_{(2)(1)}(t) = 0$, we have

$$\Gamma^{(21)}_{(1^3)(1^3)}(t) = (1+t+t^2)^2(1-t^2) - (1-t^2)(1-t^3) = t(t+1)(t+2)(1-t^3),$$

which agrees with the entry appearing in the tables of Appendix E, which were generated by direct calculations using (2.46).

2.4 Macdonald functions

Finally, we discuss another set of symmetric functions over the field $F = \mathbb{Q}(q,t)$, which are generalizations of the Hall-Littlewood functions. We first define an inner product on the power sum symmetric functions by

$$\langle\, p_\lambda(x), p_\mu(x)\,\rangle_{(q,t)} = z_\lambda(q,t)\,\delta_{\lambda\mu}, \tag{2.50}$$

where

$$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \tag{2.51}$$

Then it was shown by Macdonald [39], that there exist unique symmetric functions $P_\lambda(x;q,t)$ satisfying the following conditions:

$$\begin{aligned} P_\lambda(x;q,t) = m_\lambda(x) &\quad+\quad \sum_{\mu<\lambda} u_{\lambda\mu}(q,t)\, m_\mu(x), \\ \langle\, P_\lambda(x;q,t), P_\mu(x;q,t)\,\rangle_{(q,t)} &\quad=\quad 0, \qquad \text{for } \lambda \neq \mu, \end{aligned} \tag{2.52}$$

where the order $<$ is any total order *compatible* with the dominance partial order $\preceq$, defined by

$$\lambda \preceq \mu \Longleftrightarrow \lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k \qquad \forall k \geq 1. \tag{2.53}$$

By compatible, we mean that if λ precedes μ under the dominance order $\preceq$ then it precedes it under the total order $<$.

Letting $b_\lambda^{-1}(q,t) \equiv \langle\, P_\lambda(q,t), P_\lambda(q,t)\,\rangle_{(q,t)}$, define $Q_\lambda(q,t) = b_\lambda(q,t)P_\lambda(q,t)$, so that from (2.52) we have

$$\langle\, P_\lambda(q,t), Q_\mu(q,t)\,\rangle_{(q,t)} = \delta_{\lambda\mu}. \tag{2.54}$$

Define

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad\qquad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i),$$

where we understand $(a;q)_0 = 1$. Then from the explicit form of the power sums, it follows that they obey the identity [39]

$$\sum_\lambda \frac{1}{z_\lambda(q,t)}\, p_\lambda(x)\, p_\lambda(y) = \prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}.$$

From (2.54), the functions P_λ, Q_λ are dual bases for Λ_F. It follows then that the Macdonald functions $P_\lambda(x;q,t)$ obey the identity

$$\sum_\lambda P_\lambda(x;q,t)\, Q_\lambda(y;q,t) = \prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}. \tag{2.55}$$

There is a beautiful formula for the specialization $x = (1,t,t^2,\ldots,t^{n-1})$ of the function $P_\lambda(x;q,t)$. Define $\varepsilon_{u,t} \in \mathrm{End}(\Lambda_F)$ by its action on the power sums

$$\varepsilon_{u,t}(p_r(x)) = \left(\frac{1-u^r}{1-t^r}\right)\, p_r(x), \tag{2.56}$$

so that $\varepsilon_{t^{n-1},t}(p_r(x)) = p_r(1,t,\ldots,t^{n-1})$. Then Macdonald has shown that

$$\varepsilon_{u,t}(P_\lambda(x;q,t)) = \prod_{x\in\lambda} \frac{q^{a'(x)}u - t^{l'(x)}}{q^{a(x)}t^{l(x)+1} - 1}, \tag{2.57}$$

where the above product is over all nodes x in the diagram of λ, and $a'(x)$, $a(x)$, $l'(x)$ and $l(x)$ are respectively the number of squares to the left, to the right, above and below the node x. In particular it follows that for one variable z

$$P_{(m)}(z;q,t) = z^m. \tag{2.58}$$

Macdonald has defined "integral forms" $J_\lambda(x;q,t)$ of the functions $P_\lambda(x;q,t)$ by

$$J_\lambda(x;q,t) = c_\lambda(q,t) P_\lambda(x;q,t), \tag{2.59}$$

where

$$c_\lambda(q,t) = \prod_{x\in\lambda} (1 - q^{a(x)} t^{l(x)+1}),$$

with $a(x)$, $l(x)$ defined above. They are "integral" in the sense that if they are expanded in terms of the functions $S_\lambda(x;t) = \det(q_{\lambda_i - i+j}(x;t))$

$$J_\lambda(x;q,t) = \sum_\mu K_{\mu\lambda}(q,t)\, S_\mu(x;t), \tag{2.60}$$

then it is conjectured that the functions $K_{\mu\lambda}(q,t)$ are polynomials in q and t with *integral* coefficients. In the case where $\ell(\lambda) \le 2$ or $\ell(\lambda') \le 2$, then Stembridge [198] has proven that $K_{\mu\lambda}(q,t)$ is a polynomial, while when λ is a hook-partition, then this is a polynomial with non-negative coefficients. If $X^\lambda_\sigma(q,t)$ are defined by the expansion

$$J_\lambda(x;q,t) = \sum_\sigma \frac{1}{z_\sigma(t)} X^\lambda_\sigma(q,t)\, p_\sigma(x), \tag{2.61}$$

then Macdonald has shown that

$$X^\lambda_\rho(q,t) = \varepsilon_\rho\, X^{\lambda'}_\rho(t,q), \qquad \varepsilon_\rho = (-1)^{|\rho|-\ell(\rho)}. \tag{2.62}$$

Another set of symmetric functions which we need to introduce are the Jack symmetric functions $P^{(\alpha)}_\lambda(x)$, which are defined as a particular limit of a Macdonald function

$$P^{(\alpha)}_\lambda(x) = \lim_{t\to 1} P_\lambda(x; t^\alpha, t).$$

For these functions, there is an inner product $\langle \cdot, \cdot \rangle_\alpha$ on the ring Λ_G of symmetric functions with coefficients in $G = \mathbb{Q}(\alpha)$ which is defined by

$$\langle\, p_\lambda(x), p_\mu(x)\, \rangle_\alpha = \delta_{\lambda\mu}\, z_\lambda\, \alpha^{\ell(\lambda)}, \tag{2.63}$$

under which, the Jack symmetric functions obey the orthogonality relation

$$\langle\, P^{(\alpha)}_\lambda, P^{(\alpha)}_\mu\, \rangle_\alpha = \delta_{\lambda\mu}\, j_\lambda,$$

where the numerical factor j_λ is calculated in [199, Theorem 5.8]. Let $g^{(\alpha)}_n(x) = P^{(\alpha)}_{(n)}(x)/j_{(n)}$ denote the elementary Jack function, which has the generating function

$$\sum_{n=0}^\infty g^{(\alpha)}_n(x)\, z^n = \prod_i (1 - x_i z)^{1/\alpha}. \tag{2.64}$$

It was shown by Macdonald [40] that the functions

$$T^{(\alpha)}_\lambda = \det(g^{(\alpha)}_{\lambda_i - i+j}) \tag{2.65}$$

form a basis for Λ_G, dual to the S-functions under the inner product (2.63). We shall see in Chapter 3 that the functions $T_\lambda^{(\alpha)}$ are intimately related to S-functions with a replicated argument.

We shall now describe how it is possible to derive some relations between one-row and two-row Macdonald functions associated with different parameters. It follows from (2.55) and (2.58) that the generating function for one-row Macdonald functions is

$$\sum_{n=0}^{\infty} Q_{(n)}(x;q,t)z^n = \prod_i \frac{(tx_iz;q)_\infty}{(x_iz;q)_\infty} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\frac{1-t^n}{1-q^n}\, p_n(x)\, z^n\right). \qquad (2.66)$$

From this generating function, we see that (dropping the argument x for convenience)

$$\begin{aligned} \sum_{p=0}^{\infty} Q_{(p)}(q,t^2)z^n &= \prod_i \frac{(t^2x_iz;q)_\infty}{(tx_iz;q)_\infty} \prod_i \frac{(tx_iz;q)_\infty}{(x_iz;q)_\infty} \\ &= \left(\sum_{m=0}^{\infty} Q_{(m)}(q,t)\,(tz)^m\right)\left(\sum_{n=0}^{\infty} Q_{(n)}(q,t)\, z^n\right), \end{aligned}$$

so that

$$Q_{(p)}(q,t^2) = \sum_{m=0}^{p} Q_{(m)}(q,t)Q_{(p-m)}(q,t)\, t^m. \qquad (2.67)$$

In a similar manner it can be seen that for a positive integer k, we have

$$Q_{(p)}(q,t^k) = \sum_{n_2,\dots,n_k} Q_{(p-n_2-\dots-n_k)}(q,t)Q_{(n_2)}(q,t)\cdots Q_{(n_k)}(q,t)\, t^{n_2+2n_3+\cdots(k-1)n_k}. \qquad (2.68)$$

Now, the Pieri formulae for one-row Macdonald functions, which expressses the product of one-row functions in terms of two-row functions, has the form [200]

$$Q_{(n)}Q_{(m)} = \sum_{j=0}^{m} d_j^p\, Q_{(n+j,m-j)}, \qquad (2.69)$$

where $p = n - m \geq 0$, $d_0^p = 1$ and for $j > 0$

$$d_j^p = \frac{(t;q)_j}{(q;q)_j}\,\frac{(q^{p+j+1};q)_j}{(q^{p+j}t;q)_j}. \qquad (2.70)$$

It turns out that by applying the formula (2.69) to the functions appearing in (2.67), we can reexpress them in terms of the functions $Q_{(n,m)}(q,t)$ in the form

$$Q_{(2p)}(q,t^2) = \sum_{j=0}^{p} t^{p-j}\frac{(t^2;q)_{2j}}{(t;q)_{2j}} Q_{(p+j,p-j)}(q,t), \qquad (2.71)$$

$$Q_{(2p-1)}(q,t^2) = \sum_{j=1}^{p} t^{p-j}\frac{(t^2;q)_{2j-1}}{(t;q)_{2j-1}} Q_{(p+j-1,p-j)}(q,t). \qquad (2.72)$$

Let us prove this for (2.72). From (2.67) we write

$$\begin{aligned}
Q_{(2p-1)}(q,t^2) &= \sum_{k=1}^{p}(t^{p-k}+t^{p-1+k})\; Q_{(p+k-1)}(q,t)Q_{(p-k)}(q,t) \\
&= \sum_{k=1}^{p}(t^{p-k}+t^{p-1+k})\left(\sum_{l=0}^{p-k} d_l^{2k-1}\right) Q_{(p+k-1+l,p-k-l)}(q,t) \\
&= \sum_{j=1}^{p} t^{p-j} f_j Q_{(p+j-1,p-j)}(q,t),
\end{aligned}$$

where

$$f_j = \sum_{n=0}^{j-1}(t^n + t^{2j-1-n})\; d_n^{2j-1-2n}.$$

Although it is not apparent, we can rewrite this as a terminating basic hypergeometric series of the type ${}_2\phi_1$, which is defined by

$${}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}\; z^n.$$

Indeed,

$$f_j = {}_2\phi_1(t,q^{1-2j};q^{2-2j}t^{-1};q,q) = t^{2j-1}\frac{(q^{2-2j}t^{-2};q)_{2j-1}}{(q^{2-2j}t^{-1};q)_{2j-1}}, \tag{2.73}$$

where we have used the q-Vandermonde summation formula

$${}_2\phi_1(a,q^{-n};c;q,q) = a^n\; \frac{(c/a;q)_n}{(c;q)_n}. \tag{2.74}$$

From (2.73) the result (2.72) follows, and the proof of (2.71) is similar.

The question then arises as to whether the function $Q_\lambda(q,t^k)$ can be expanded in the form

$$Q_\lambda(q,t^k) = \sum_{\mu} a_{\lambda\mu}(q,t,k)\; Q_\mu(q,t) \tag{2.75}$$

where the functions $a_{\lambda\mu}(q,t,k) \in \mathbb{Z}(q,t)$, the field of quotients of integer-valued polynomials in q and t. For the case of $|\lambda| = 3$ we have explicitly

$$\begin{aligned}
Q_{(3)}(q,t^k) = {} & \frac{(1-t^k)(1-qt^k)(1-q^2t^k)}{(1-t)(1-qt)(1-q^2t)}\; Q_{(3)}(q,t) \\
& + \frac{(1-t^k)(t-t^k)(1-qt^k)}{(1-t)^2(1-qt^2)}\; Q_{(21)}(q,t) + \frac{(1-t^k)(t-t^k)(t^2-t^k)}{(1-t)(1-t^2)(1-t^3)}\; Q_{(1^3)}(q,t),
\end{aligned}$$

$$\begin{aligned}
Q_{(21)}(q,t^k) &= \frac{(1-t^k)^2(1-q^2t)(1-qt^{2k})}{(1-t)^2(1-qt^2)(1-q^2t^k)}\; Q_{(21)}(q,t) \\
&\quad + \frac{(1-t^k)^2(t-t^k)(1-qt)(1+t+qt+t^k+qt^k+qt^{k+1})}{(1-t)(1-t^2)(1-t^3)(1-q^2t^k)}\; Q_{(1^3)}(q,t), \\
Q_{(1^3)}(q,t^k) &= \frac{(1-t^k)(1-t^{2k})(1-t^{3k})(1-qt)(1-qt^2)}{(1-t)(1-t^2)(1-t^3)(1-qt^k)(1-qt^{2k})}\; Q_{(1^3)}(q,t).
\end{aligned}$$

There are two observations we can make from the above example. Firstly, the coefficient $a_{(1^3)(1^3)}(q,t,k)$ appears to contain some general structure. This is not surprising however, as the function $Q_{(1^n)}(x;q,t)$ is proportional to the elementary symmetric function $e_n(x)$. In fact,

$$Q_{(1^n)}(q,t) = b_{(1^n)}(q,t)\ P_{(1^n)}(q,t) = b_{(1^n)}(q,t)\ e_n, \tag{2.76}$$

where $b_{(1^n)}(q,t) = (t;t)_n/(q;t)_n$. Thus

$$Q_{(1^n)}(q,t^k) = \frac{(t^k;t^k)_n\ (q;t)_n}{(q;t^k)_n\ (t;t)_n} Q_{(1^n)}(q,t). \tag{2.77}$$

Secondly, for $k \geq 3$ the coefficients do not factor into products of the form $(1-q^a t^b)$, where $a,\ b \in \mathbb{Z}_+$. In the case $k=2$, we have checked for partitions of weight ≤ 4 and all coefficients $a_{\lambda\mu}(q,t)$ are of this form. It remains to be seen what is so special about this case.

Another result stemming from the Pieri formula (2.69) concerns the expansion of $Q_{(p)}(q,t)$ in terms of functions $Q_{(n,m)}(q^2,t)$. By considering the generating function (2.66) and splitting the infinite product into factors involving even and odd powers of q, we obtain the relation

$$Q_{(p)}(q,t) = \sum_{k=0}^{p} q^k Q_{(p-k)}(q^2,t) Q_{(k)}(q^2,t). \tag{2.78}$$

Similarly we can derive the result

$$Q_{(p)}(q,t) = \sum_{n_2,\ldots,n_k} Q_{(p-n_2-\cdots-n_k)}(q^k,t) Q_{(n_2)}(q^k,t) \cdots Q_{(n_k)}(q^k,t)\ q^{n_2+2n_3+\cdots(k-1)n_k}. \tag{2.79}$$

Again, by applying the Pieri formula (2.69) to (2.78), resumming the resulting expression, and then using the identity

$$_2\phi_1(t,q^{-2n};q^{2-2n}/t;q^2,q^3/t) = (-q;q)_n\ \frac{(t;q)_n}{(t;q^2)_n}, \tag{2.80}$$

one can show that

$$\begin{aligned} Q_{(2p-1)}(q,t) &= \sum_{j=1}^{p} q^{p-j} \frac{(-q;q)_{2j-1}(t;q)_{2j-1}}{(t;q^2)_{2j-1}}\ Q_{(p+j-1,p-j)}(q^2,t), &(2.81)\\ Q_{(2p)}(q,t) &= \sum_{j=0}^{p} q^{p-j} \frac{(-q;q)_{2j}(t;q)_{2j}}{(t;q^2)_{2j}}\ Q_{(p+j,p-j)}(q^2,t). &(2.82) \end{aligned}$$

To prove the summation formula (2.80)[2], one first transforms this $_2\phi_1$ with base q^2 to a $_8\phi_7$ with base q, using [201, Eq. (3.5.5)],

$$_2\phi_1(a^2,b^2;a^2q^2b^{-2};q^2;x^2q^2b^{-4}) = G\ .\ _8\phi_7 \left[\begin{matrix} c_1,\ldots,c_8 \\ d_1,\ldots,d_7 \end{matrix} ;q;-xq^{\frac{1}{2}}b^{-1} \right], \tag{2.83}$$

[2] I take great pleasure in thanking Prof. M. Rahmen for pointing out a proof of this identity

where

$$G=\frac{(qa^2x^2b^{-2};q^2)_\infty(q^2a^2x^2b^{-4};q^2)_\infty(-q^{\frac{1}{2}}xb^{-1};q)_\infty(-q^{\frac{3}{2}}axb^{-3};q)_\infty}{(qx^2b^{-2};q^2)_\infty(q^2x^2b^{-4};q^2)_\infty(-q^{\frac{1}{2}}axb^{-1};q)_\infty(-q^{\frac{3}{2}}a^2xb^{-3};q)_\infty},$$

and

$$\begin{aligned}
&c_1=-q^{\frac{1}{2}}a^2xb^{-3},\quad c_2=q\sqrt{c_1},\quad c_3=-q\sqrt{c_1},\quad c_4=a,\quad c_5=q^{\frac{1}{2}}ab^{-1},\\
&c_6=-q^{\frac{1}{2}}ab^{-1},\quad c_7=-qab^{-2},\quad c_8=-q^{\frac{1}{2}}xb^{-1},\\
&d_1=\sqrt{c_1},\quad d_2=-\sqrt{c_1},\quad d_3=-q^{\frac{3}{2}}axb^{-3},\quad d_4=-qaxb^{-2},\\
&d_5=qaxb^{-2},\quad d_6=q^{\frac{1}{2}}axb^{-1},\quad d_7=qa^2b^{-2},
\end{aligned}$$

and then summing the resulting series using Jackson's sum of a terminating, very-well-poised, balanced ${}_8\phi_7$ series [201, Eq. (2.6.2)]

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}a,qa^{1/2},-qa^{\frac{1}{2}},b,c,d,e,q^{-n}\\ a^{\frac{1}{2}},-a^{\frac{1}{2}},aqb^{-1},aqc^{-1},aqd^{-1}aqe^{-1},aq^{n+1}\end{matrix};q;q\right]\\
&\qquad=\frac{(aq;q)_n(aqb^{-1}c^{-1})_n(aqb^{-1}d^{-1})_n(aqc^{-1}d^{-1})_n}{(aqb^{-1})_n(aqc^{-1})_n(aqd^{-1})_n(aqb^{-1}c^{-1}d^{-1})_n},
\end{aligned}$$

provided $a^2q^{n+1}=bcde$.

Again, we may ask the question as to whether the functions $Q_\lambda(q,t)$ can be expanded in terms of the functions $Q_\mu(q^k,t)$ where the coefficients in the expansion have a nice factorized form. Taking the $|\lambda|=3$ case we have

$$\begin{aligned}
Q_{(3)}(q,t)&=\frac{(1-q^k)(1-q^{2k})(1-q^{3k})(1-qt)(1-q^2t)}{(1-q)(1-q^2)(1-q^3)(1-q^kt)(1-q^{2k}t)}Q_{(3)}(q^k,t)\\
&\quad+\frac{(1-q^k)^2(q-q^k)(1-qt)(1+q+qt+q^k+q^kt+q^{k+1}t)}{(1-q)(1-q^2)(1-q^3)(1-q^kt^2)}Q_{(21)}(q^k,t)\\
&\quad+\frac{(1-q)(q-q^k)(q^2-q^k)}{(1-q)(1-q^2)(1-q^3)}Q_{(1^3)}(q^k,t),\\
Q_{(21)}(q,t)&=\frac{(1-q^k)^2(1-q^{2k}t)(1-q^2t)}{(1-q)^2(1-q^2t)(1-q^kt^2)}Q_{(21)}(q^k,t)\\
&\quad+\frac{(1-q^k)(q-q^k)(1-q^kt)}{(1-q)^2(1-q^2t)}Q_{(1^3)}(q^k,t),\\
Q_{(1^3)}(q,t)&=\frac{(1-q^k)(1-q^kt)(1-q^kt^2)}{(1-q)(1-qt)(1-qt^2)}Q_{(1^3)}(q^k,t).
\end{aligned}$$

This last equation follows from (2.76), in that, for all $n>0$,

$$Q_{(1^n)}(q,t)=\frac{(q^k;t)_n}{(q;t)_n}Q_{(1^n)}(q^k,t).$$

We note once again that for $k>2$, the above coefficients do not necessarily factorize, but when $k=2$ they do. We have also checked this for partitions of weight 4, and the same conclusion applies, and we conjecture that this behaviour is true for partitions of any weight.

Chapter 3

Symmetric Functions of a Replicated Argument

In this chapter, we discuss symmetric functions of a replicated argument, and their applicability in the problem of determining dual functions of compound and super-symmetric symmetric functions under an induced inner product. After first providing some motivation for the problem, we define replicant S-functions and investigate their dual basis under the normal Schur function inner product. The dual compound problem is then solved with the aid of these replicant functions. These results are then extended to the Hall-Littlewood case and some interesting combinatorial identities are derived in the process.

We then introduce q-replicant symmetric functions and generalize the identities of the previous section. The relationship of these q-replicant symmetric functions to Macdonald's symmetric functions is then examined, and some standard bases for the symmetric functions over the field $F = \mathbb{Q}(q,t)$ are listed, with the transition matrices between them being calculated in Appendix F.

3.1 Motivation

As mentioned in Chapter 1, infinite dimensional Heisenberg algebras play a central role in applying symmetric function techniques to various problems in mathematical physics. This algebra is generated by operators $\{\alpha_i | \; i \in \mathbb{Z}\}$ obeying the commutation relations

$$[\alpha_n, \alpha_m] = n\, \delta_{n+m,0}. \tag{3.1}$$

These algebras can be realized on the space of symmetric functions by the association

$$\alpha_{-n} = p_n(x), \qquad \alpha_n = n\frac{\partial}{\partial p_n(x)}, \qquad n > 0, \tag{3.2}$$

with the central element α_0 acting as a constant. An alternative basis to that consisting of monomials in the creation operators α_{-n}, which corresponds to the power sum basis $p_\lambda(x)$, is the basis consisting of all Schur functions $s_\lambda(x)$. The S-function basis has proven convenient for carrying out calculations in bosonic Fock spaces, using the realization (3.2).

We now pose the following question: suppose we have 2 commuting copies $\{\alpha_n^{(i)}\}$ of the Heisenberg algebra (3.1) (or k copies in general) realized on the space $\Lambda(x)\times\Lambda(y)$. Then for a state

$$|v\rangle = \alpha_{-n_1}^{(1)} \cdots \alpha_{-n_p}^{(1)} \alpha_{-m_1}^{(2)} \cdots \alpha_{-m_q}^{(2)} |0\rangle \,,$$

we know that

$$||\,|v\rangle\,||^2 = ||\, \alpha_{-n_1}^{(1)} \cdots \alpha_{-n_p}^{(1)} |0\rangle\,||^2 \quad ||\, \alpha_{-m_1}^{(2)} \cdots \alpha_{-m_q}^{(2)} |0\rangle\,||^2.$$

In the language of symmetric functions this corresponds to using the inner product $\langle\,\cdot,\cdot\,\rangle_{\Lambda(x)\times\Lambda(y)}$ on the space $\Lambda(x)\times\Lambda(y)$. Compound S-functions form a basis for the subspace generated by monomials $(\alpha_n^{(1)}+\alpha_n^{(2)})$, which are only orthonormal under the inner product $\langle\,\cdot,\cdot\,\rangle_{\Lambda(x,y)}$. A natural question to ask then is, what is the nature of the functions dual to these compound S-functions, using the inner product $\langle\,\cdot,\cdot\,\rangle_{\Lambda(x)\times\Lambda(y)}$. It turns out that the key to this question is the notion of S-functions of a replicated argument. These functions formally extend the idea of compound S-functions $s_\lambda(x,x,\ldots,x)$ to the case where the argument x is repeated α times, where α is not necessarily an integer.

3.2 Replicant S-functions

Our starting point for the description of replicant S-functions is the equation [33]

$$s_\lambda(xy) = \sum_{\mu,\nu} \gamma^\lambda_{\mu\nu}\, s_\mu(x)s_\nu(y), \tag{3.3}$$

where the argument xy represents the set of indeterminates $xy=(x_1y_1,x_1y_2,\ldots,x_2y_1,x_2y_2,\ldots)$. Here the coefficients $\gamma^\lambda_{\mu\nu}$ denote the coefficients arising in the inner product of S-functions, defined by (2.26). Equation (3.3) can be derived from the observation that $p_\lambda(xy)=p_\lambda(x)p_\lambda(y)$, along with the Frobenius formula (2.16). First note that for an S-function with its arguments set equal to 1, we have [33]

$$s_\lambda(\underbrace{1,1,\ldots,1}_{n}) = \left(\begin{array}{c} n \\ \lambda' \end{array}\right), \tag{3.4}$$

where the generalized binomial coefficient associated to a partition λ is defined by

$$\left(\begin{array}{c} X \\ \lambda \end{array}\right) = \prod_{x\in\lambda} \frac{X-c(x)}{h(x)}. \tag{3.5}$$

The above product is over all nodes $x=(i,j)$ in the partition λ, and $c(x)=j-i$, $h(x)=\lambda_i+\lambda'_j-i-j+1$ are the content and hook-length respectively, of the node $x\in\lambda$. This definition reduces to the usual definition of a binomial coefficient when

$\lambda = (n)$, a one-part partition. With the particular choice $y = (\underbrace{1, 1, \ldots, 1}_{n})$ in (3.3), it follows that, for an S-function whose arguments are repeated n times,

$$s_\lambda(\underbrace{x_1, \ldots, x_1}_{n}, \underbrace{x_2, \ldots, x_2}_{n}, \ldots) = \sum_{\mu,\nu} \left(\begin{array}{c} n \\ \nu' \end{array} \right) \gamma^\lambda_{\mu\nu} s_\mu(x).$$

The generalized binomial coefficient (3.5) is just a polynomial in the variable X and can thus be evaluated for any $X \in \mathbb{R}$. Hence following [202], given $\alpha \in \mathbb{R}$ we define the replicated S-function $s_\lambda(x^{(\alpha)})$ to be

$$s_\lambda(x^{(\alpha)}) = \sum_\mu b_{\lambda\mu}(\alpha) s_\mu(x) \qquad \text{where} \quad b_{\lambda\mu}(\alpha) \equiv \sum_\rho \left(\begin{array}{c} \alpha \\ \rho' \end{array} \right) \gamma^\lambda_{\rho\mu}. \tag{3.6}$$

Certainly we have $s_\lambda(x^{(1)}) \equiv s_\lambda(x)$. In analogy to the Frobenius formulae (2.16), we can also define power sums with a replicated argument by

$$p_\lambda(x^{(\alpha)}) = \sum_\rho \chi^\rho_\lambda \, s_\rho(x^{(\alpha)}). \tag{3.7}$$

Let us now prove that

$$p_\lambda(x^{(\alpha)}) = \alpha^{\ell(\lambda)} p_\lambda(x), \qquad \forall \alpha \in \mathbb{R}. \tag{3.8}$$

Certainly this is true when $\alpha = n$, an integer. To prove it for arbitrary $\alpha \in \mathbb{R}$ is equivalent, using (3.6) and (2.16), to proving

$$\alpha^{\ell(\lambda)} = \sum_\rho \left(\begin{array}{c} \alpha \\ \rho' \end{array} \right) \chi^\rho_\lambda. \tag{3.9}$$

But this is just an identity of polynomials in the variable α, which we know holds for the "variable" n, hence (3.8) is true for all $\alpha \in \mathbb{R}$. If we now multiply both sides of (3.9) by $z_\lambda^{-1} \chi^\sigma_\lambda$ and use the orthogonality relations (2.17) we obtain the inverse relation

$$\left(\begin{array}{c} \alpha \\ \sigma' \end{array} \right) = \sum_\lambda z_\lambda^{-1} \chi^\sigma_\lambda \alpha^{\ell(\lambda)}. \tag{3.10}$$

Note that we can utilize (3.10) and the definition of $\gamma^\lambda_{\mu\nu}$ (see (2.24)) to rewrite the transition coefficients $b_{\lambda\mu}(\alpha)$ in the form

$$b_{\lambda\mu}(\alpha) = \sum_\sigma z_\sigma^{-1} \, \chi^\lambda_\sigma \chi^\mu_\sigma \, \alpha^{\ell(\sigma)}. \tag{3.11}$$

This gives us $b_{\lambda\mu}(1) = \delta_{\lambda\mu}$ as one expects.

As alluded to in Chapter 2, there is a connection between S-functions of a replicated argument and Jack symmetric functions which we now point out. Using the fact that $\gamma^{(n)}_{\rho\mu} = \delta_{\rho\mu}$, which follows from (2.25) and the orthogonality relations (2.17) for S_n characters, we have

$$s_{(n)}(x^{(\alpha)}) = \sum_\mu \left(\begin{array}{c} \alpha \\ \mu' \end{array} \right) s_\mu(x).$$

However, the right hand side of the above equation is just the expansion, in terms of S-functions, of the elementary Jack symmetric function $g_n^{(\alpha^{-1})}(x)$ (see Stanley [199]). Thus $s_{(n)}(x^{(\alpha)}) = g_n^{(\alpha^{-1})}(x)$ and (2.63) provides us with the generating function

$$\sum_{n=0}^{\infty} s_{(n)}(x^{(\alpha)}) z^n = \prod_i (1 - x_i z)^{-\alpha}. \tag{3.12}$$

If we now take the Jack limit of the transition matrix between the S-functions s_λ and the functions $g_\lambda^{(1/\alpha)}(x) = g_{\lambda_1}^{(1/\alpha)}(x) g_{\lambda_2}^{(1/\alpha)}(x) \cdots$ appearing in the tables in Appendix F, we have

$$s_\lambda = \sum_{\rho\sigma} \frac{\alpha^{\ell(\rho)}}{z_\rho} \chi_\rho^\lambda \chi_\rho^\sigma \, K_{\mu\sigma}^{-1} \, g_\mu^{(1/\alpha)}, \tag{3.13}$$

where K^{-1} is the inverse Kostka matrix, which is the transition matrix between the monomial symmetric functions and the S-functions. By inserting (3.13) into (3.6) (using the definition of $b_{\lambda\mu}$ given by (3.11)) and using both of the orthogonality relations (2.17) we obtain the result

$$s_\lambda(x^{(\alpha)}) = \sum_\mu K_{\mu\lambda}^{-1} \, g_\mu^{(1/\alpha)}(x).$$

From this it follows that $s_\lambda(x^{(\alpha)}) = T_\lambda^{(1/\alpha)}(x)$, where the functions $T_\lambda^{(\alpha)}$ are defined by (2.65).

One question we may ask is, what are the functions dual to the functions $s_\lambda(x^{(\alpha)})$ under the ordinary S-function inner product $\langle \cdot, \cdot \rangle$? To answer this question, let us first note that, given a set of indeterminates $x_i y_j$, we see from (3.8) that

$$p_\lambda((xy)^{(\alpha\beta)}) = p_\lambda(x^{(\alpha)}) \, p_\lambda(y^{(\beta)}). \tag{3.14}$$

From Macdonald [40] we know that $g_n^{(\alpha)} = \sum_{\lambda \vdash n} z_\lambda^{-1} \alpha^{-\ell(\lambda)} p_\lambda$, from which it follows that

$$s_{(n)}(x^{(\alpha)}) = \sum_\mu z_\mu^{-1} \, p_\mu(x^{(\alpha)}). \tag{3.15}$$

Thus using the generating function (3.12), along with (3.14) and (3.15) we have

$$\prod_{i,j} (1 - x_i y_j)^{-\alpha\beta} = \sum_\lambda z_\lambda^{-1} \, p_\lambda(x^{(\alpha)}) \, p_\lambda(y^{(\beta)}).$$

However using (3.7) and the orthogonality relations (2.17), we have

$$\begin{aligned} \sum_\lambda z_\lambda^{-1} \, p_\lambda(x^{(\alpha)}) \, p_\lambda(y^{(\beta)}) &= \sum_{\lambda\rho\sigma} z_\lambda^{-1} \, \chi_\lambda^\rho \, \chi_\lambda^\sigma \, s_\rho(x^{(\alpha)}) s_\sigma(y^{(\beta)}) \\ &= \sum_\rho s_\rho(x^{(\alpha)}) s_\rho(y^{(\beta)}), \end{aligned}$$

which implies that

$$\prod_{i,j} (1 - x_i y_j)^{-\alpha\beta} = \sum_\rho s_\rho(x^{(\alpha)}) s_\rho(y^{(\beta)}). \tag{3.16}$$

Hence, from (2.21) we have

$$\langle\, s_\lambda(x^{(\alpha^{-1})}), s_\mu(x^{(\alpha)})\,\rangle = \delta_{\lambda\mu}. \tag{3.17}$$

This result can be proved another way, by observing that

$$p_\lambda(x^{(\alpha)}) = \varepsilon_\lambda\, \omega_\alpha(p_\lambda(x)),$$

where $\varepsilon_\lambda = (-1)^{|\lambda|-\ell(\lambda)}$ and the endomorphism ω_α is defined [40] on the power sums by $\omega_\alpha(p_n(x)) = (-1)^{n-1}\alpha\, p_n(x)$. Thus using the linear properties of ω_α we have

$$s_\lambda(x^{(\alpha)}) = \omega_\alpha(s_{\lambda'}(x)). \tag{3.18}$$

In analogy to (2.9), it follows from the definition (3.6) and the fact that $\gamma^\lambda_{\mu\nu} = \gamma^\lambda_{\mu'\nu'}$, that $s_{\lambda'}(x^{(\alpha)}) = \omega(s_\lambda(x^{(\alpha)}))$. So if we now use (3.18) along with the relation [40]

$$\langle\, \omega_{\alpha^{-1}} f, g\,\rangle_\alpha = \langle\, \omega f, g\,\rangle \qquad \text{for any } f, g \in \Lambda,$$

then

$$\begin{aligned}
\langle\, s_\lambda(x^{(\alpha)}), s_\mu(x^{(\alpha^{-1})})\,\rangle &= \langle\, \omega(s_{\lambda'}(x^{(\alpha)})), s_\mu(x^{(\alpha^{-1})})\,\rangle = \langle\, \omega_{\alpha^{-1}}(s_{\lambda'}(x^{(\alpha)})), s_\mu(x^{(\alpha^{-1})})\,\rangle_\alpha \\
&= \langle\, s_\lambda(x), \omega_{\alpha^{-1}}(s_\mu(x))\,\rangle_\alpha = \langle\, s_\lambda(x), s_\mu(x)\,\rangle = \delta_{\lambda\mu},
\end{aligned}$$

thus yielding (3.17).

Example
Consider $|\lambda| = 3$ and the following replicated S-functions

$$\begin{aligned}
s_{(3)}(x^{(\alpha)}) &= \frac{\alpha(\alpha+1)(\alpha+2)}{6} s_{(3)}(x) + \frac{\alpha(\alpha^2-1)}{3} s_{(21)}(x) + \frac{\alpha(\alpha-1)(\alpha-2)}{6} s_{(1^3)}(x), \\
s_{(21)}(x^{(\alpha)}) &= \frac{\alpha(\alpha^2-1)}{3} s_{(3)}(x) + \frac{\alpha(2\alpha^2+1)}{3} s_{(21)}(x) + \frac{\alpha(\alpha^2-1)}{3} s_{(1^3)}(x), \\
s_{(1^3)}(x^{(\alpha)}) &= \frac{\alpha(\alpha-1)(\alpha-2)}{6} s_{(3)}(x) + \frac{\alpha(\alpha^2-1)}{3} s_{(21)}(x) + \frac{\alpha(\alpha+1)(\alpha+2)}{6} s_{(1^3)}(x).
\end{aligned}$$

Using the self-dual nature of S-functions (see (2.22)) one can show explicitly that (3.17) is true for the above functions

3.2.1 Compound S-functions and duality

Next, we return to the question of compound S-functions and duality. Let $\Lambda(x, y)$ denote the ring of symmetric functions in the variables $(x_1, x_2, \ldots, y_1, y_2, \ldots)$. Then certainly

$$\langle\, s_\lambda(x, y), s_\mu(x, y)\,\rangle_{\Lambda(x,y)} = \delta_{\lambda\mu}.$$

Let us instead, consider the inner product on $\Lambda(x) \times \Lambda(y)$ induced by those of $\Lambda(x)$ and $\Lambda(y)$,

$$\langle\, f_1(x)g_1(y), f_2(x)g_2(y)\,\rangle_{\Lambda(x)\times\Lambda(y)} = \langle\, f_1(x), f_2(x)\,\rangle_{\Lambda(x)}\, \langle\, g_1(y), g_2(y)\,\rangle_{\Lambda(y)}. \tag{3.19}$$

We can ask ourselves the following question: what are the functions dual to the compound S-functions under the inner product (3.19)? In other words, we want to find functions $\tilde{s}(x,y) \in \Lambda(x,y)$ such that

$$(\tilde{s}_\lambda(x,y), s_\mu(x,y)) \equiv \langle\, \tilde{s}_\lambda(x,y), s_\mu(x,y)\,\rangle_{\Lambda(x)\times\Lambda(y)} = \delta_{\lambda\mu}. \tag{3.20}$$

Example
Consider the following compound S-functions:

$$\begin{aligned}
s_{(3)}(x,y) &= s_{(3)}(x) + s_{(2)}(x)s_{(1)}(y) + s_{(1)}(x)s_{(2)}(y) + s_{(3)}(y),\\
s_{(21)}(x,y) &= s_{(21)}(x) + s_{(2)}(x)s_{(1)}(y) + s_{(1)}(x)s_{(11)}(y)\\
&\quad + s_{(11)}(x)s_{(1)}(y) + s_{(1)}(x)s_{(2)}(y) + s_{(21)}(y),\\
s_{(1^3)}(x,y) &= s_{(1^3)}(x) + s_{(11)}(x)s_{(1)}(y) + s_{(1)}(x)s_{(11)}(y) + s_{(1^3)}(y).
\end{aligned}$$

Then it can be checked that the functions $\tilde{s}_\lambda(x,y)$ given by

$$\begin{pmatrix} \tilde{s}_{(3)}(x,y) \\ \tilde{s}_{(21)}(x,y) \\ \tilde{s}_{(1^3)}(x,y) \end{pmatrix} = \begin{pmatrix} \frac{5}{16} & -\frac{1}{8} & \frac{1}{16} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} \\ \frac{1}{16} & -\frac{1}{8} & \frac{5}{16} \end{pmatrix} \begin{pmatrix} s_{(3)}(x,y) \\ s_{(21)}(x,y) \\ s_{(1^3)}(x,y) \end{pmatrix},$$

satisfy (3.20). Let us explain where the entries of the above matrix come from.

Suppose we write

$$\tilde{s}_\lambda(x,y) = \sum_\mu a_{\lambda\mu} s_\mu(x,y) = \sum_{\rho\sigma} d^\lambda_{\rho\sigma} s_\rho(x) s_\sigma(y),$$

where $d^\lambda_{\rho\sigma} = \sum_\mu a_{\lambda\mu} c^\mu_{\rho\sigma}$. Then for the orthogonality condition (3.20) to hold, we require

$$\sum_{\rho\sigma} d^\lambda_{\rho\sigma} c^\mu_{\rho\sigma} = \delta_{\lambda\mu}. \tag{3.21}$$

Now, from (3.18) and the fact that $c^{\lambda'}_{\mu'\nu'} = c^\lambda_{\mu\nu}$, we know that S-functions with a replicated argument obey the Littlewood-Richardson rule. Thus, using (3.16), we have

$$\begin{aligned}
\sum_{\tau\nu} \delta_{\tau\nu}\, s_\tau(r) s_\nu(t) = \prod_{i,j}(1 - r_i t_j)^{-1} &= \sum_{\mu\nu\rho\sigma} c^\mu_{\rho\sigma} c^\nu_{\rho\sigma} s_\mu(r^{(1/2)}) s_\nu(t)\\
&= \sum_{\mu\nu\rho\sigma\eta\tau} c^\mu_{\rho\sigma} c^\nu_{\rho\sigma} \begin{pmatrix} 1/2 \\ \eta' \end{pmatrix} \gamma^\mu_{\eta\tau} s_\tau(r) s_\nu(t).
\end{aligned}$$

Hence, by comparison with (3.21), we can conclude that

$$d^\lambda_{\rho\sigma} = \sum_{\mu\eta} \begin{pmatrix} 1/2 \\ \eta' \end{pmatrix} c^\mu_{\rho\sigma} \gamma^\mu_{\eta\lambda},$$

which gives

$$a_{\lambda\mu} = \sum_\eta \begin{pmatrix} 1/2 \\ \eta' \end{pmatrix} \gamma^\mu_{\eta\lambda} = b_{\lambda\mu}(\tfrac{1}{2}) \quad . \tag{3.22}$$

Thus, returning to the above example, using $\gamma^{\rho}_{(3)\sigma} = \delta_{\rho\sigma}$, $\gamma^{(3)}_{(21)(21)} = \gamma^{(21)}_{(21)(21)} = \gamma^{(1^3)}_{(21)(21)} = 1$ and its symmetries, we have for example

$$\begin{aligned} a_{(3)(3)} &= \begin{pmatrix} 1/2 \\ (3)' \end{pmatrix} = \frac{5}{16}, \\ a_{(21)(21)} &= \begin{pmatrix} 1/2 \\ (3)' \end{pmatrix} + \begin{pmatrix} 1/2 \\ (21)' \end{pmatrix} + \begin{pmatrix} 1/2 \\ (1^3)' \end{pmatrix} = \frac{5}{16} + \frac{-1}{8} + \frac{1}{16} = \frac{1}{4}, \end{aligned}$$

which agrees with the above example.

We can extend the above result by considering n sets of variable x_i, $i = 1, \ldots n$, where $x_i = (x_{i,1}, x_{i,2}, \ldots)$. Then in the ring $\Lambda(x_1, x_2, \ldots, x_n)$, the functions dual to the compound S-functions $s_\lambda(x_1, \ldots, x_n)$ under the inner product on the space $\Lambda(x_1) \times \Lambda(x_2) \times \cdots \times \Lambda(x_n)$, are of the form

$$\tilde{s}_\lambda(x_1, \ldots, x_n) = \sum_\mu a_{\lambda\mu} s_\mu(x_1, \ldots, x_n),$$

where $a_{\lambda\mu} = b_{\lambda\mu}(1/n)$.

3.2.2 Supersymmetric S-functions

We can repeat the analysis of the previous section in the case of supersymmetric S-functions $s_\lambda(x/y)$. That is, under the inner product of $\Lambda(x) \times \Lambda(y)$ we want to find the functions $\tilde{s}_\lambda(x/y)$ which satisfy

$$(\tilde{s}_\lambda(x/y), s_\mu(x/y)) = \delta_{\lambda\mu}.$$

If we write $\tilde{s}_\lambda(x/y) = \sum_\sigma a_{\lambda\sigma} s_\sigma(x/y)$, then we find that the coefficients in the supersymmetric case are exactly the same as those in the compound case. i.e. given by (3.22). The proof is almost exactly the same, except that one must use the relations

$$\begin{pmatrix} -\alpha \\ \lambda \end{pmatrix} = (-1)^{|\lambda|} \begin{pmatrix} \alpha \\ \lambda' \end{pmatrix}, \qquad \gamma^{\lambda}_{\rho'\sigma'} = \gamma^{\lambda}_{\rho\sigma}.$$

We should point out here the relationship between compound, supersymmetric and replicated S-functions. Firstly from the relation

$$\sum_\lambda s_\lambda(r^{(-\alpha)}) s_\lambda(t) = \prod_{i,j} (1 - r_i t_j)^\alpha = \sum_\lambda (-1)^{|\lambda|} s_{\lambda'}(r^{(\alpha)}) s_\lambda(t),$$

we deduce that $s_\lambda(r^{(-\alpha)}) = (-1)^{|\lambda|} s_{\lambda'}(r^{(\alpha)})$. Thus, in particular we have $h_n(x) = (-1)^n e_n(x^{(-1)})$ and vice-versa. In fact, the operation of replacing the argument $x^{(\alpha)}$ by $x^{(-\alpha)}$ in any symmetric function of weight $|\lambda|$ is equivalent to applying the involution $(-1)^{|\lambda|}\omega$ to that function. That is, on the subspace Λ^n of homogeneous symmetric functions of degree n we have,

$$\omega_{-\alpha} = (-1)^n \, \omega \, \omega_\alpha.$$

Moreover, we have

$$s_\lambda(x/y) = \sum_\rho (-1)^{|\rho|} s_{\lambda/\rho}(x) s_{\rho'}(y^{(1)}) = \sum_\rho s_{\lambda/\rho}(x) s_\rho(y^{(-1)}) = s_\lambda(x, y^{(-1)}),$$

so that supersymmetric S-functions can be viewed as compound S-functions where the second set of indeterminates is replicated -1 times.

3.3 Hall-Littlewood functions

In this section we shall extend the concept of replicated argument to Hall-Littlewood functions and consider the analogous problem of dual compound functions. It follows from (2.48) that

$$P_\lambda(\underbrace{x_1,\ldots,x_1}_{n}, \underbrace{x_2,\ldots,x_2}_{n}, \ldots; t) = \sum_{\mu\nu} \Gamma^\lambda_{\mu\nu}(t)\, P_\mu(\underbrace{1,\ldots,1}_{n}; t) P_\nu(y;t).$$

There is no succinct expression for $P_\mu(\underbrace{1,\ldots,1}_{n}; t)$ as there is in the S-function case. The best we can do is to use the inverse Kostka-Foulkes matrix, defined as the transition matrix between S- and Hall-Littlewood functions

$$P_\lambda(x;t) = \sum_\mu K^{-1}_{\lambda\mu}(t) s_\mu(x),$$

along with (3.4). This enables us to define Hall-Littlewood functions of a replicated argument, analogous to (3.6), by

$$P_\lambda(x^{(\alpha)};t) = \sum_\mu g_{\lambda\mu}(\alpha,t) P_\mu(x;t), \qquad g_{\lambda\mu}(\alpha,t) = \sum_{\rho\sigma} \Gamma^\lambda_{\mu\rho} K^{-1}_{\rho\sigma}(t) \begin{pmatrix} \alpha \\ \sigma' \end{pmatrix}, \quad (3.23)$$

so that we recover (3.6) when $t = 0$ ($K^{-1}_{\lambda\mu}(0) = \delta_{\lambda\mu}$). Similarly, we define

$$Q_\lambda(x^{(\alpha)};t) \equiv b_\lambda(t)\, P_\lambda(x^{(\alpha)};t) = \sum_\mu g_{\mu\lambda}(t) Q_\mu(x;t).$$

(Note the transposed indices in g). From the relation [33],

$$K^{-1}_{\rho\sigma}(t) = \frac{1}{b_\rho(t)} \sum_\xi \frac{1}{z_\xi(t)} X^\rho_\xi(t)\, \chi^\sigma_\xi, \quad (3.24)$$

we can rewrite the coefficients $g_{\lambda\mu}$, using (3.9) and (2.45) as

$$g_{\lambda\mu}(\alpha,t) = \frac{1}{b_\lambda(t)} \sum_\nu \frac{1}{z_\nu(t)} X^\lambda_\nu(t) X^\mu_\nu(t)\, \alpha^{\ell(\nu)}. \quad (3.25)$$

Note that, from the orthogonality relations (2.44), this transition matrix reduces to the identity matrix as $\alpha \to 1$.

Before embarking on a study of the dual compound Hall-Littlewood functions, let us try to glean some identities involving $z_\lambda(t)$, as we did in the $t = 0$ case (see (3.9) and (3.10)). Define $q_n(x^{(\alpha)};t) \equiv Q_{(n)}(x^{(\alpha)};t)$. The functions $q_n(x^{(\alpha)};t)$ have the generating function

$$\sum_{n=0}^{\infty} q_n(x^{(\alpha)};t)z^n = \prod_i \left(\frac{1-tx_iz}{1-x_iz}\right)^\alpha = \exp\left(\alpha \sum_{n\geq 1} \frac{1-t^n}{n}\, p_n(x)\right), \tag{3.26}$$

from which it follows that

$$q_n(x^{(\alpha)};t) = \sum_{\lambda\vdash n} z_\lambda^{-1}(t)\alpha^{\ell(\lambda)}p_\lambda(x). \tag{3.27}$$

These functions can be considered to be a special case of a set of generalized symmetric functions considered by Morris [203] where the single parameter t in (3.26) is replaced by a set of parameters $t_1, \ldots, t_m$. For a single variable x, if we set $x = 1$, then $q_n(1^{(\alpha)};t) = g_n^{(-\alpha,\alpha)}(t,1)$, where $g_n^{(\alpha,\beta)}(x,y)$ denotes the Lagrange polynomial with generating function [204]

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)z^n = (1-xz)^{-\alpha}(1-yz)^{-\beta}.$$

They have the explicit form

$$g_n^{(\alpha,\beta)}(x,y) = \sum_{r=0}^{n} \frac{(\alpha)_r(\beta)_{n-r}}{r!(n-r)!}\, x^r y^{n-r},$$

where $(\alpha)_r = \alpha(\alpha+1)\cdots(\alpha+r-1)$ denotes the Pochhammer symbol. If we now put $x = 1$ (one variable) in (3.27), we obtain the interesting identity

$$\sum_{\lambda\vdash n} z_\lambda^{-1}(t)\alpha^{\ell(\lambda)} = q_n(1^{(\alpha)};t) = g_n^{(-\alpha,\alpha)}(t,1) = \sum_{r=0}^{n} \frac{(-\alpha)_r(\alpha)_{n-r}}{r!(n-r)!}\, t^r. \tag{3.28}$$

In particular, setting $\alpha = 1$, -1 in the above equation, we have

$$\sum_{\lambda\vdash n} \frac{1}{z_\lambda(t)} = 1-t, \qquad \sum_{\lambda\vdash n} \frac{(-1)^{\ell(\lambda)}}{z_\lambda(t)} = t^n - t^{n-1}. \tag{3.29}$$

Note that the first of these relations can also be deduced from the orthogonality relation (2.44) upon setting $\mu = \nu = (n)$. (See also reference [203]). The identity (3.28) is a special case of the more general identity

$$b_\mu(t)\sum_\sigma K^{-1}_{\mu\sigma}(t)\binom{\alpha}{\sigma'} = \sum_\lambda \frac{\alpha^{\ell(\lambda)}}{z_\lambda(t)} X^\mu_\lambda(t), \tag{3.30}$$

which is proved in the same way as (3.9) and (3.10) were. To see that (3.28) is just a special case, put $\mu = (n)$ in (3.30) and use the fact that (see (2.41))

$$K^{-1}_{(n)\sigma}(t) = \begin{cases} (-t)^r & \text{if } \sigma = (n-r, 1^r) \\ 0 & \text{otherwise} \end{cases}.$$

3.3.1 Dual compound Hall-Littlewood functions

Similar arguments to before show us that

$$\sum_{\lambda} z_\lambda(t)^{-1} p_\lambda(x^{(\alpha)}) p_\lambda(y^{(\beta)}) = \prod_{i,j} \left(\frac{1 - t x_i y_j}{1 - x_i y_j} \right)^{\alpha\beta} = \sum_{\lambda} Q_\lambda(x^{(\alpha)}; t) P_\lambda(y^{(\beta)}; t),$$

so that under the inner product defined by (2.37), the functions $P_\lambda(x^{(\alpha)}; t)$ and $Q_\lambda(x^{(\alpha^{-1})}; t)$ are dual. As in the $t = 0$ case, this result can also be deduced using the the fact that

$$Q_\lambda(x^{(\alpha)}; t) = \omega_\alpha \omega(Q_\lambda(x; t)), \tag{3.31}$$

and using the various properties of the endomorphisms ω_α and ω.

Again, the question may be asked as to the nature of the functions dual to these compound functions under the inner product of $\Lambda(x)[t] \times \Lambda(y)[t]$. A similar analysis to that undertaken in the previous section shows that if

$$\tilde{Q}_\lambda(x, y; t) = \sum_{\mu} a_{\lambda\mu}(t)\, Q_\mu(x, y; t),$$

where $a_{\lambda\mu}(t) = g_{\lambda\mu}(\frac{1}{2}, t)$, then

$$\langle\, \tilde{Q}_\lambda(x, y; t), P_\mu(x, y; t)\, \rangle_{\Lambda(x)[t] \times \Lambda(y)[t]} = \delta_{\lambda\mu}.$$

Example

Given the weight 3 compound functions

$$\begin{aligned}
P_{(3)}(x, y; t) &= P_{(3)}(x; t) + (1 - t) P_{(2)}(x; t) P_{(1)}(y; t) \\
&+ (1 - t) P_{(1)}(x; t) P_{(2)}(y; t) + P_{(3)}(y; t), \\
P_{(21)}(x, y; t) &= P_{(21)}(x; t) + P_{(2)}(x; t) P_{(1)}(y; t) + (1 - t^2) P_{(1^2)}(x; t) P_{(1)}(y; t) \\
&+ (1 - t^2) P_{(1)}(x; t) P_{(1^2)}(y; t) + P_{(1)}(x; t) P_{(2)}(y; t) + P_{(21)}(y; t), \\
P_{(1^3)}(x, y; t) &= P_{(1^3)}(x; t) + P_{(1^2)}(x; t) P_{(1)}(y; t) + P_{(1)}(x; t) P_{(1^2)}(y; t) \\
&+ P_{(1^3)}(y; t),
\end{aligned}$$

we have the dual functions $\tilde{Q}_\lambda(x, y; t)$ given by

$$\begin{pmatrix} \tilde{Q}_{(3)}(x, y; t) \\ \tilde{Q}_{(21)}(x, y; t) \\ \tilde{Q}_{(1^3)}(x, y; t) \end{pmatrix} = \mathbf{M} \begin{pmatrix} Q_{(3)}(x, y; t) \\ Q_{(21)}(x, y; t) \\ Q_{(1^3)}(x, y; t) \end{pmatrix},$$

where

$$\mathbf{M} = \frac{1}{16} \begin{pmatrix} 5 + 2t + t^2 & -2 - t - t^2 & 1 \\ t^3 + t - 2 & 4 + t^2 - t^3 & t - 2 \\ (1 - t^2)(1 - t^3) & (t + 1)(t - 2)(1 - t^3) & 5 - 2t - 2t^2 + t^3 \end{pmatrix}.$$

3.3.2 Dual supersymmetric Hall-Littlewood functions

Let us extend the results of the previous section to the supersymmetric case. Before we do this, we first have to define what we mean by supersymmetric Hall-Littlewood functions. We define $P_\lambda(x/y;t)$ to be compound Hall-Littlewood functions in the variables x and $y^{(-1)}$. That is

$$P_\lambda(x/y;t) \equiv \sum_{\rho\sigma} \bar{B}^\lambda_{\rho\sigma} P_\rho(x;t) P_\sigma(y;t), \qquad Q_\lambda(x/y;t) \equiv \sum_{\rho\sigma} B^\lambda_{\rho\sigma} Q_\rho(x;t) Q_\sigma(y;t),$$

where

$$B^\lambda_{\rho\sigma} = \sum_\nu f^\lambda_{\rho\nu}(t) g_{\nu\sigma}(-1,t), \qquad \bar{B}^\lambda_{\rho\sigma} = \sum_\nu \bar{f}^\lambda_{\rho\nu}(t) g_{\sigma\nu}(-1,t).$$

There is no simple relation between $P_\lambda(x/y;t)$ and $P_\lambda(y/x;t)$ as there is in the S-function case. All one can say is that $P_\lambda(x/y;t) = \sum_\mu g_{\lambda\mu}(-1,t) P_\mu(y/x;t)$ and the usual S-function relationship (when $t=0$) is recovered by noticing that $g_{\lambda\mu}(-\alpha,0) = (-1)^{|\lambda|} g_{\lambda\mu'}(\alpha,0)$. Thus, for example, we have [33]

$$P_{(1^r)}(x/y;t) = \sum_{\lambda\vdash r} t^{n(\lambda)} P_\lambda(y/x;t), \qquad \text{where} \quad n(\lambda) = \sum_i (i-1)\lambda_i.$$

One more interesting observation concerns the elementary supersymmetric functions $q_n(x/y;t)$, which have the generating function

$$\sum_{n=1}^\infty q_n(x/y;t) z^n = \prod_i \left(\frac{1-tx_iz}{1-x_iz}\right)\left(\frac{1-y_iz}{1-ty_iz}\right). \tag{3.32}$$

From the generating function (2.39), we see that $q_n(x^{(-1)};t) = t^n\, q_n(x;t^{-1})$, so that

$$\begin{aligned} q_n(x/y;t) &= \sum_{p=0}^n q_p(x;t) q_{n-p}(y^{(-1)};t) = \sum_{p=0}^n t^{n-p} q_p(x;t) q_{n-p}(y;t^{-1}) \\ &= \sum_{p=0}^n (-1)^{n-p} S_{(p)}(x;t) S_{(1^{n-p})}(y;t), \end{aligned} \tag{3.33}$$

where $S_\lambda(x;t)$ denotes the functions dual to the S-functions under the inner product (2.37)

$$S_\lambda(x;t) = \det(q_{\lambda_i-i+j}(x;t)),$$

and we have used the result [40] $S_\lambda(x;t^{-1}) = (-t)^{-|\lambda|} S_{\lambda'}(x;t)$. Thus setting $x=0$ in (3.33), we get

$$q_n(0/y;t) = (-1)^n S_{(1^n)}(y;t). \tag{3.34}$$

This result (3.34) can be obtained in a different way: recall that $\Gamma^\lambda_{\mu(p)} = \delta_{\lambda\mu}$ and hence

$$q_n(x^{(\alpha)};t) \equiv (1-t) P_{(n)}(x^{(\alpha)};t) = \sum_{\mu\sigma} K^{-1}_{\mu\sigma}(t) \begin{pmatrix} \alpha \\ \sigma' \end{pmatrix} Q_\mu(x;t) = \sum_\sigma \begin{pmatrix} \alpha \\ \sigma' \end{pmatrix} S_\sigma(x;t). \tag{3.35}$$

In particular, since

$$\left(\begin{array}{c} -1 \\ \sigma' \end{array}\right) = \left\{\begin{array}{cc} (-1)^n & \text{if } \sigma = (1^n) \\ 0 & \text{otherwise} \end{array}\right. ,$$

then setting $\alpha = -1$ in (3.35) we recover (3.34).

Returning to the dual function problem, suppose we need to find functions

$$\tilde{Q}_\lambda(x/y;t) = \sum_{\rho\sigma} D^\lambda_{\rho\sigma} Q_\rho(x;t) Q_\sigma(y;t),$$

such that

$$\langle\, \tilde{Q}_\lambda(x/y;t), P_\mu(x/y;t)\,\rangle = \delta_{\lambda\mu}. \tag{3.36}$$

That is, we need $\sum_{\rho\sigma} D^\lambda_{\rho\sigma} \bar{B}^\mu_{\rho\sigma} = \delta_{\lambda\mu}$. A similar calculation to that done in the previous section, yields

$$D^\lambda_{\rho\sigma} = \sum_{\alpha\beta} f^\beta_{\alpha\sigma} g_{\rho\alpha}(-1,t) g_{\beta\lambda}(-1/2,t).$$

Thus if we write $\tilde{Q}_\lambda(x/y;t) = \sum_\mu a_{\lambda\mu} Q_\mu(y/x;t)$, we have

$$D^\lambda_{\rho\sigma} = \sum_{\mu\nu} a_{\mu\nu} f^\mu_{\rho\nu} g_{\sigma\nu}(-1,t),$$

and hence if $a_{\lambda\mu} = g_{\mu\lambda}(-1/2,t)$, then (3.36) will hold.

3.4 q–replicated symmetric functions

We shall now generalize some of the results of section 3.2 by introducing q-replicated symmetric functions whose $q \to 1$ limits are the "ordinary" replicated symmetric functions. We begin by recalling the formula for S-function of a finite number of variables evaluated at $x = (1, q, q^2, \ldots, q^{N-1})$, which takes the form [33]

$$s_\lambda(1, q, q^2, \ldots, q^{N-1}) = q^{n(\lambda)} \left[\begin{array}{c} N \\ \lambda' \end{array}\right]_q , \tag{3.37}$$

where

$$n(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i, \qquad \left[\begin{array}{c} \alpha \\ \lambda \end{array}\right]_q = \prod_{x\in\lambda} \frac{1-q^{\alpha - c(x)}}{1-q^{h(x)}},$$

with $c(x)$ and $h(x)$ denoting the content and hook length respectively at the node $x \in \lambda$. Using this we define q-replicated S-functions for any α, $q \in \mathbb{R}$ by

$$s_\lambda(x;q,\alpha) = \sum_{\mu\nu} \gamma^\lambda_{\mu\nu} q^{n(\mu)} \left[\begin{array}{c} \alpha \\ \mu' \end{array}\right]_q s_\nu(x). \tag{3.38}$$

If we again define power sums via the Frobenius formula (2.16), then

$$p_\lambda(x;q,\alpha) = \frac{\xi_\lambda(q^\alpha)}{\xi_\lambda(q)} p_\lambda(x),$$

with $\xi_\lambda(q) = (1-q^{\lambda_1})\cdots(1-q^{\lambda_m})$ where $m = \ell(\lambda)$. This allows us to derive the following identities

$$\frac{\xi_\lambda(q^\alpha)}{\xi_\lambda(q)} = \sum_\mu \chi^\mu_\lambda q^{n(\mu)} \begin{bmatrix} \alpha \\ \mu' \end{bmatrix}_q, \qquad \begin{bmatrix} \alpha \\ \lambda' \end{bmatrix}_q = q^{-n(\lambda)} \sum_\mu \frac{1}{z_\mu(q,q^\alpha)} \chi^\lambda_\mu, \tag{3.39}$$

where $z_\lambda(q,t)$ is defined in (2.51). Let us pause for a moment to examine the second of these identities. By writing $u = q^\alpha$ as an independent variable, we can rewrite this as

$$q^{n(\lambda)} \prod_{x\in\lambda} \frac{1-uq^{-c(x)}}{1-q^{h(x)}} = \sum_\mu \frac{1}{z_\mu(q,u)}\, \chi^\lambda_\mu, \tag{3.40}$$

This equation yields a wealth of non-trivial identities involving the functions $z_\lambda(q,u)$.

Examples

(a) With the choice $\lambda = (m)$, we have $\chi^{(m)}_\mu = 1$, and so

$$\sum_{\mu\vdash m} \frac{1}{z_\mu(q,u)} = \frac{(u;q)_m}{(q;q)_m}.$$

(b) With the choice $\lambda = (1^m)$, we have $\chi^{(1^m)}_\mu = (-1)^{\ell(\mu)-|\mu|}$, and hence

$$\sum_{\mu\vdash m} \frac{(-1)^{\ell(\mu)}}{z_\mu(q,u)} = (-1)^m q^{m(m-1)/2} \frac{(q^{1-m}u;q)_m}{(q;q)_m}. \tag{3.41}$$

(c) For $\lambda = (m-1,1)$ we have [7] $\chi^{(m-1,1)}_\mu = m^\mu_1 - 1$ where m^μ_1 is the number of parts of μ equal to 1. Hence

$$\sum_{\mu\vdash m} \frac{(m^\mu_1-1)}{z_\mu(q,u)} = q\left(\frac{1-q^{m-1}}{1-q}\right) \frac{(uq^{-1};q)_m}{(q;q)_m}.$$

(d) For $\lambda = (m-2,2)$ we have

$$\sum_{\mu\vdash m} \frac{(m^\mu_1(m^\mu_1-3)/2 + m^\mu_2)}{z_\mu(q,u)} = q^2 \frac{(1-u)(1-q^{m-2})}{(1-q)(1-q^2)} \frac{(uq^{-1};q)_{m-1}}{(q;q)_{m-1}}.$$

(e) For $\lambda = (m-2,1^2)$ we obtain

$$\sum_{\mu\vdash m} \frac{(m^\mu_1(m^\mu_1-3)/2 - m^\mu_2 + 1)}{z_\mu(q,u)} = q^3 \frac{(1-q^{m-1})(1-q^{m-2})}{(1-q)(1-q^2)} \frac{(uq^{-2};q)_m}{(q;q)_m}.$$

Note that letting $q\to 0$ in examples (a) and (b) reproduces the identities in (3.29).

Using (3.24), (3.37) and (3.39), it follows that

$$Q_\lambda(1,q,\ldots,q^{\alpha-1};t) = \sum_\rho \frac{\xi_\lambda(q^\alpha)}{z_\rho(q,t)}\, X^\lambda_\rho(t). \tag{3.42}$$

Note that for $0 < |q| < 1$, $\lim_{\alpha\to\infty} \xi_\lambda(q^\alpha) = 1$. In the limit $\alpha \to \infty$, equation (3.42) has two limits which are readily computable. Firstly, when $q = t$, we have

$$Q_\lambda(1, t, t^2, \ldots; t) = \sum_\rho \frac{1}{z_\rho} X_\rho^\lambda(t) = t^{n(\lambda)},$$

recovering the well known result in [33, p. 109, Ex. 1]. Similarly, when $q = t^{-1}$,

$$Q_\lambda(1, t^{-1}, t^{-2}, \ldots; t) = t^{|\lambda|} \sum_\rho \frac{(-1)^{\ell(\rho)}}{z_\rho} X_\rho^\lambda(t) = (-t)^{|\lambda|} K_{(1^{|\lambda|}),\lambda}(t) = (-t)^{|\lambda|}\, \delta_{\lambda,(1^{|\lambda|})},$$

where we have used the expression for the inverse Kostka-Foulkes matrices occuring in (3.48) for $q = 0$.

In a similar manner as was done above, we can define q-replicated Hall-Littlewood functions

$$P_\lambda(x; q, \alpha, t) = \sum_{\mu\nu} \Gamma^\lambda_{\mu\nu}(t) P_\mu(1, q, \ldots, q^{\alpha-1}; t) P_\nu(x; t). \tag{3.43}$$

Again from this, we can derive the identity

$$\sum_{\lambda\vdash n} \frac{\xi_\lambda(t)}{z_\lambda(q, q^\alpha)}\, X_\lambda^\sigma(t) = b_\sigma(t) \sum_{\mu\vdash n} K^{-1}_{\sigma\mu}(t) q^{n(\mu)} \begin{bmatrix} \alpha \\ \mu' \end{bmatrix}_q, \qquad \forall \sigma \vdash n. \tag{3.44}$$

Before examining this identity for generic values of q, let us first take the special case $q = t$. First note that the right-hand side of the above equation is just $b_\sigma(t)\, \varepsilon_{q^\alpha,q}(P_\sigma(x; t))$ where $\varepsilon_{q^\alpha,q}$ is the specialization endomorphism defined by (2.56). Setting $u = q^\alpha = t^\alpha$ we have from (2.57)

$$\sum_{\lambda\vdash n} \frac{1}{z_\lambda(u)}\, X_\lambda^\sigma(t) = b_\sigma(t) \prod_{x\in\sigma} \left. \frac{s^{a'(x)} u - t^{l'(x)}}{s^{a(x)} t^{l(x)+1} - 1} \right|_{s=0}. \tag{3.45}$$

Let us illustrate some of the two-variable identities in the above equation by making particular choices for σ. It can be checked that when $\sigma = (n)$ (respectively (1^n)) is substituted into (3.45), we recover the first identity in (3.29) (respectively (3.41)). For the case $\sigma = (n-1, 1)$, we know [28] that $X_\lambda^{(n-1,1)}(t) = m_1^\lambda + t - 1$, so that

$$\sum_{\lambda\vdash n} \frac{(m_1^\lambda + t - 1)}{z_\lambda(u)} = (u-1)(u-t).$$

Similarly, using Morris' result [28]

$$X_\lambda^{(2,1^{n-2})}(t) = (-1)^{n-\ell(\lambda)} \frac{\gamma_{n-2}(t)}{\xi_\lambda(t)} \left(m_1^\lambda (1-t) - 1 + t^n \right),$$

it follows that

$$\sum_{\lambda\vdash n} \frac{(-1)^{\ell(\lambda)}}{z_\lambda(t, u)} \left(m_1^\lambda (1-t) - 1 + t^n \right) = (-1)^n t^{(n-1)(n-2)/2}\, \frac{(t^{2-n} u; t)_{n-1}}{(t; t)_{n-2}}.$$

Let us return to (3.44) for the case of generic q. Inserting $\sigma = (n)$ we obtain, after some algebra, the q-analogue of (3.28):

$$\sum_{\lambda\vdash n} \frac{\xi_\lambda(u)}{z_\lambda(q,t)} = \sum_{j=0}^{n} \frac{(u^{-1};q)_j (u;q)_{n-j}}{(q;q)_j (q;q)_{n-j}} \, (ut)^j .$$

The particular choices of $u = q,\ q^{-1}$ reproduce (3.29), while if $u = q^2$, we see that

$$\sum_{\lambda\vdash n} \frac{\widehat{\xi}_\lambda(u)}{z_\lambda(t)} = (1-q)^{-1}\left(1 - q^{n+1} - (1+q)(1-q^n)t + q(1-q^{n-1})t^2\right),$$

where $\widehat{\xi}_\lambda(u) = \prod_{i=1}^{\ell(\lambda)}(1+q^{\lambda_i})$.

3.4.1 Relation to Macdonald functions

Let us now show that the q-replicated S- and Hall-Littlewood functions introduced in the previous section can be related to Macdonald functions. From the generating function

$$\sum_{n=0}^{\infty} s_{(n)}(x;q,\alpha) z^n = \prod_i \frac{(q^\alpha x_i z;q)_\infty}{(x_i z;q)_\infty},$$

we deduce from (2.66) that $s_{(n)}(x;q,\alpha) = Q_{(n)}(x;q,q^\alpha)$ a one-part Macdonald function. In fact, we have

$$\sum_\lambda s_\lambda(x;q,\alpha) s_\lambda(y) = \prod_{i,j} \frac{(q^\alpha x_i y_j;q)_\infty}{(x_i y_j;q)_\infty}, \tag{3.46}$$

so that the functions $s_\lambda(x;q,\alpha)$ form a basis for Λ_F, $F = \mathbb{Q}(q,q^\alpha)$, dual to the S-functions under the inner product (2.50) (with $t = q^\alpha$). We shall return to this point presently.

We can ask ourselves, once again, what are the functions dual to $s_\lambda(x;q,\alpha)$ under the normal S-function innner product (2.19) ? By inspecting the generating function (3.46), we see that

$$\langle\, s_\lambda(x;q,\alpha), s_\mu(x;q^\alpha,\alpha^{-1}) \,\rangle = \delta_{\lambda\mu}. \tag{3.47}$$

For example,

$$\begin{aligned}
s_{(3)}(x;q,\alpha) &= \frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^{\alpha+2})}{(1-q)(1-q^2)(1-q^3)}\, s_{(3)}(x) \\
&\quad + q\,\frac{(1-q^{\alpha-1})(1-q^\alpha)(1-q^{\alpha+1})}{(1-q)^2(1-q^3)}\, s_{(21)}(x) \\
&\quad + q^3\,\frac{(1-q^\alpha)(1-q^{\alpha-1})(1-q^{\alpha-2})}{(1-q)(1-q^2)(1-q^3)}\, s_{(1^3)}(x), \\
s_{(3)}(x;q^\alpha,\alpha^{-1}) &= \frac{(1-q)(1-q^{\alpha+1})(1-q^{2\alpha+1})}{(1-q^\alpha)(1-q^{2\alpha})(1-q^{3\alpha})}\, s_{(3)}(x) \\
&\quad + q^\alpha\,\frac{(1-q)(1-q^{1-\alpha})(1-q^{1+\alpha})}{(1-q^\alpha)^2(1-q^{3\alpha})}\, s_{(21)}(x) \\
&\quad + q^{3\alpha}\,\frac{(1-q)(1-q^{1-\alpha})(1-q^{1-2\alpha})}{(1-q^\alpha)(1-q^{2\alpha})(1-q^{3\alpha})}\, s_{(1^3)}(x).
\end{aligned}$$

With a bit of algebra, one can check that (3.47) is indeed satisfied by these two functions.

The q-replicated Hall-Littlewood functions (3.43) can also be related to Macdonald functions. For $|q| < 1$, let $R_\lambda(x;q,t) = \lim_{\alpha\to\infty} P_\lambda(x;q,\alpha,t)$. These functions also form a basis for Λ_F, this time dual to the Hall-Littlewood functions $Q_\lambda(x;t)$. They are related by

$$\begin{aligned} R_\lambda(x;q,t) &= \sum_{\mu\sigma} \frac{1}{z_\sigma(q,t)} X^\lambda_\sigma(t) X^\mu_\sigma(t)\, Q_\mu(x;t), \\ Q_\mu(x;t) &= \sum_{\lambda\sigma} \frac{1}{\zeta_\sigma(q,t)} X^\mu_\sigma(t) X^\lambda_\sigma(t)\, R_\lambda(x;q,t), \end{aligned}$$

where $\zeta_\sigma(q,t) = z_\sigma\, \xi_\sigma^{-1}(q)\, \xi_\sigma^{-1}(t)$. There are several distinguished bases for Λ_F, which we shall now list. Let

$$T_\lambda(x;q,t) = \det(g_{\lambda_i - i + j}).$$

where $g_n(x;q,t) \equiv Q_{(n)}(x;q,t)$ is the elementary Macdonald function whose generating function is given by (2.66). Then the $T_\lambda(x;q,t)$ form a basis of Λ_F dual to the Schur functions $s_\lambda(x)$. Thus we have the following dual bases of Λ_F under the inner product (2.50):

$$\begin{gathered} (P_\lambda(x;q,t), Q_\lambda(x;q,t)), \quad (R_\lambda(x;q,t), Q_\lambda(x;t)), \quad (T_\lambda(x;q,t), s_\lambda(x)), \\ (\Sigma_\lambda(x;q), S_\lambda(x;t)), \qquad (g_\lambda(x;q,t), m_\lambda(x)), \end{gathered}$$

where $g_\lambda = g_{\lambda_1} g_{\lambda_2} \cdots$, and the functions $\Sigma_\lambda \equiv \lim_{\alpha\to\infty} s_\lambda(x;q,\alpha)$ were introduced in reference [40, Sec. 8 Ex. 10].

The transformations between these various bases are shown in Appendix F, in terms of $X^\lambda_\mu(q,t)$, $X^\lambda_\mu(t)$, χ^λ_μ and the Kostka matrix $K_{\lambda\mu} \equiv \sum_\sigma z_\sigma^{-1} \chi^\lambda_\sigma X^\mu_\sigma(1)$. Some (but not all) of these transformations can alternatively be described in terms of the matrices $K_{\lambda\mu}$, $K_{\lambda\mu}(t)$ and $K_{\lambda\mu}(q,t)$, by using the relations [40]

$$\begin{aligned} K_{\lambda\mu}(q,t) &= \sum_\rho \frac{1}{z_\rho} \chi^\lambda_\rho X^\mu_\rho(q,t), & X^\lambda_\rho(q,t) &= \sum_\mu \chi^\mu_\rho K_{\mu\lambda}(q,t), \\ K^{-1}_{\lambda\mu}(q,t) &= \frac{1}{c_\lambda(q,t) c_{\lambda'}(q,t)} \sum_\rho \frac{1}{\zeta_\rho(q,t)} X^\lambda_\rho(q,t) \chi^\mu_\rho, & \chi^\lambda_\mu &= \sum_\rho K^{-1}_{\rho\lambda}(q,t) X^\rho_\mu(q,t). \end{aligned} \tag{3.48}$$

Chapter 4

Superconformal Algebras

The $N = 1$ and $N = 2$ superconformal algebras are the objects of study in this chapter. We derive branching rules governing the reduction of certain irreducible representations of the Ramond superalgebra into its Ramond and Neveu-Schwarz winding subalgebras. Some tensor product decompositions of the $N = 2$ superconformal algebras are then investigated, along with their winding subalgebra branching rules.

Unlike investigations of semi-simple Lie algebras, where symmetric function techniques can be applied to the branching rule problem, we must resort here to the "brute force" decomposition of the characters of the appropriate irreducible representations. As a result, the calculations of these superconformal algebra branching rules may seem to proceed in a rather *ad hoc* manner. Nevertheless, it is instructive to demonstrate some of the techniques needed to carry out such calculations. The key to these results are identities such as the Jacobi triple product, and Watson's quintuple product identities, and some new "two-variable" identities which convert infinite products to infinite sums.

4.1 Branching rules for $N = 1$ superconformal algebras

We begin by recalling the definitions of the Neveu-Schwarz and Ramond ($N = 1$) superconformal algebras. These are infinite-dimensional superalgebras with even generators $\{L_n \;:\; n \in \mathbb{Z}\}$ and odd generators $\{G_r \;:\; r \in \mathbb{Z} + \frac{1}{2} - \epsilon\}$ satisfying the equations

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{\hat{c}}{8}(n^3 - n)\delta_{n+m,0} \\
[L_n, G_r] &= (\tfrac{n}{2} - r)G_{n+r} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{\hat{c}}{2}(r^2 - \tfrac{1}{4})\delta_{r+s,0} && (4.1) \\
[\hat{c}, L_n] &= 0 = [\hat{c}, G_r] && (4.2)
\end{aligned}$$

where $n, m \in \mathbb{Z}$ for both algebras and $r, s \in \mathbb{Z}, \epsilon = \frac{1}{2}$ or $r, s \in \mathbb{Z} + \frac{1}{2}, \epsilon = 0$ for the Ramond (which we shall denote by R) and the Neveu-Schwarz (denoted by NS) algebras respectively. Obviously they both contain the Virasoro algebra as a subalgebra with central charge $c = \frac{3}{2}\hat{c}$. Again due to (4.2) $\hat{c}$ takes on a constant value on any irreducible representation. These representations $V(\hat{c}, h)$ are generated by a highest weight vector $| h\rangle$ which satisfies

$$\begin{aligned} L_n| h\rangle = G_r|h\rangle &= 0 \quad \forall n > 0 \quad \forall r > 0 \\ L_0| h\rangle &= h| h\rangle \end{aligned}$$

In reference [124] the conditions on h and $\hat{c}$ for the representation to be unitary were found to be either that $\hat{c} \geq 1$ and $h \geq 0$, or that

$$\hat{c} = 1 - \frac{8}{m(m+2)} \qquad m \geq 2 \tag{4.3}$$

$$h = h^m_{p,q} = \frac{[(m+2)p - mq]^2 - 4}{8m(m+2)} + \frac{\epsilon}{8} \tag{4.4}$$

where $p = 1, \ldots, m-1, \quad q = 1, \ldots, m+1$, with $p - q$ even for the Neveu-Schwarz algebra and odd for the Ramond algebra.

The characters of the unitary irreducible modules of the Ramond and Neveu-Schwarz algebras are defined as

$$\chi_{\hat{c},h}(z) = \mathrm{tr}(z^{L_0}),$$

where the trace is taken over the module $V(\hat{c}, h)$. They were calculated in [145] with the result that

$$\chi_{\hat{c},h}(z) = \frac{z^h}{1-\epsilon} \prod_{n=1}^{\infty} \left(\frac{1 + z^{n+\epsilon-\frac{1}{2}}}{1 - z^n} \right),$$

when $\hat{c} \geq 1$ and $h \geq 0$, or

$$\chi_{\hat{c},h}(z) = \frac{1}{1-\epsilon} z^{\epsilon/8} \Psi^m_{p,q}(z) \prod_{n=1}^{\infty} \left(\frac{1 + z^{n+\epsilon-\frac{1}{2}}}{1 - z^n} \right) \tag{4.5}$$

$$\text{where} \quad \Psi^m_{p,q}(z) = \sum_{n\in\mathbb{Z}} \left\{ z^{\alpha_n} - z^{\beta_n} \right\}$$

$$\text{and} \quad \alpha_n = \frac{[2m(m+2)n - p(m+2) + mq]^2 - 4}{8m(m+2)} \tag{4.6}$$

$$\beta_n = \frac{[2m(m+2)n + p(m+2) + mq]^2 - 4}{8m(m+2)}$$

when $\hat{c}$, h belong to the discrete series (4.3). The factor of 2 which appears in the Ramond characters is a result of the fact that when $h \neq \hat{c}/16$ the highest weight is two-fold degenerate [145]. [1]

[1] This is due to the fact that $G_0^2 = L_0 - \hat{c}/16$ and so when $h \neq \hat{c}/16$ the odd vector $G_0| h\rangle$ is also a highest weight vector.

We now turn our attention to the branching of irreducible representations of the $N=1$ superconformal algebras, induced by what are known as their *winding* subalgebras. Winding subalgebra branching rules for Kac-Moody algebras have been investigated by Kac and Wakimoto [205], while those for the Virasoro algebra have been studied by Baake [206]. We will not be able to use symmetric function techniques, and we will have to appeal directly to the decomposition of various characters. The p'th winding subalgebra of the $N=1$ superconformal algebra is generated by the operators $\widehat{L}_n$, $\widehat{G}_r$ defined by

$$\widehat{L}_n = \frac{1}{p}L_{pn} + \frac{c}{24}\left(p - \frac{1}{p}\right)\delta_{n,0} \tag{4.7}$$

$$\widehat{G}_r = \frac{1}{\sqrt{p}}G_{pr} \qquad p \in \mathbb{Z}_+ \tag{4.8}$$

If the original $N=1$ superalgebra has central charge c, then the p'th winding subalgebra is an $N=1$ superconformal algebra with central charge pc. Moreover if a module of the original $N=1$ superalgebra has a highest weight h, then this same module has a highest weight $h+\frac{c}{24}(p-\frac{1}{p})$ with respect to its p-th winding subalgebra.

The question as to what type (i.e. Ramond or Neveu-Schwarz) of subalgebra these winding subalgebras are, depends on the type of algebra being considered, and also upon whether p is even or odd. Consider the case when the original algebra is the Ramond algebra. This has generators G_r with $r \in \mathbb{Z}$. Certainly the generators $\widehat{G}_r$ defined by (4.8) will generate a Ramond subalgebra regardless of whether p is odd or even. However these generators can also generate a Neveu-Schwarz subalgebra if p is an even integer but not when p is odd.

When the original algebra is a Neveu-Schwarz algebra, things are different. This algebra has generators G_r, $r \in \mathbb{Z}+\frac{1}{2}$. The generators $\widehat{G}_r$ will thus generate a Neveu-Schwarz subalgebra if p is odd but not if p is even. In a similar manner, it can be seen that these generators will not define a Ramond winding subalgebra for any integer p.

In reference [206], the branching rules $Vir_{c=1/2} \supset Vir_{c=1}$ of the $p=2$ subalgebra of the ordinary Virasoro algebra were calculated to be the following

$$\begin{aligned}
V(\tfrac{1}{2},0)\downarrow Vir_{c=1} &= \bigoplus_{n\in\mathbb{Z}} V(1,2n^2+\tfrac{1}{2}n+\tfrac{1}{32}) \\
V(\tfrac{1}{2},\tfrac{1}{2})\downarrow Vir_{c=1} &= \bigoplus_{n\in\mathbb{Z}} V(1,2n^2+\tfrac{3}{2}n+\tfrac{9}{32}) \\
V(\tfrac{1}{2},\tfrac{1}{16})\downarrow Vir_{c=1} &= \bigoplus_{n\in\mathbb{Z}} V(1,\tfrac{1}{16}(4n+1)^2).
\end{aligned}$$

These branching rules have applications in deducing the spectrum of the 2-D Ising model with a defect [207]. It was observed that the property of these branching rules being multiplicity free, was characteristic of the $p=2$ case only. We will now see that this is also true in the super case, by computing the $SVir_{c=7/10} \supset SVir_{c=7/5}$ branching rules of the $p=2$ winding subalgebras and that not only are they multiplicity-free, but finite as well.

Since $p=2$, then from the above comments, we can look for the branching of $c=\frac{7}{10}$ Ramond modules into both Ramond and Neveu-Schwarz submodules. In this case (4.7) implies that the relationship between the characters is

$$\chi_{7/10,h}(z) = \mathrm{tr}\left(z^{L_0}\right) = z^{-7/80}\mathrm{tr}\left(\left(z^2\right)^{L_0/2+7/160}\right) = z^{-7/80}\sum_i \chi_{7/5,h_i}(z^2). \tag{4.9}$$

Let us introduce some identities which will be of use in the calculations. Firstly, there is the Jacobi Triple product identity (see for example [208])

$$\prod_{n=1}^{\infty}(1-z^{2n})(1+xz^{2n-1})(1+x^{-1}z^{2n-1}) = \sum_{n\in\mathbb{Z}} z^{n^2}x^n, \qquad x\neq 0, \quad |\, z \,|< 1, \tag{4.10}$$

and Watson's quintuple identity (see for example [209])

$$\begin{aligned}\prod_{n=1}^{\infty}&(1-z^{2n})(1-xz^{2n})(1-x^{-1}z^{2n-2})(1-x^2z^{4n-2})(1-x^{-2}z^{4n-2})\\ &= \sum_{n\in\mathbb{Z}} z^{3n^2+n}(x^{3n}-x^{3n-1}) \qquad x\neq 0, \quad |\, z \,|< 1,\end{aligned} \tag{4.11}$$

or equivalently,

$$\prod_{n=1}^{\infty}\frac{(1-x^{2n})(1-a^2x^{2n-2})(1-a^{-2}x^{2n})}{(1+ax^{2n-1})(1+a^{-1}x^{2n-1})} = \sum_{n\in\mathbb{Z}} x^{3n^2+2n}(a^{-3n}-a^{3n+2}). \tag{4.12}$$

4.1.1 Ramond subalgebra

First consider the Ramond module $V(\frac{7}{10},\frac{7}{16})$. We will now calculate how the $c=\frac{7}{10}$ representations of the Ramond algebra decompose into irreducible representations of its $c=\frac{7}{5}$ Ramond winding subalgebra. From the character formulae (4.5) and the identity (4.12)

$$\begin{aligned}\chi_{7/10,7/16} &= 2z^{7/16}\prod_{n=1}^{\infty}\frac{(1+z^n)}{(1-z^n)}\prod_{n=1}^{\infty}\frac{(1-z^{5n})(1-z^{5n-1})(1-z^{5n-4})}{(1+z^{5n+2})(1+z^{5n+3})}\\ &= 2z^{-7/80}z^{2/16}z^{2/5}\prod_{n=1}^{\infty}\frac{1+z^{2n}}{1-z^{2n}}\prod_{n=1}^{\infty}\frac{(1+z^n)^2}{1+z^{2n}}\prod_{n=1}^{\infty}\frac{(1-z^{5n})(1-z^{5n-1})(1-z^{5n-4})}{(1+z^{5n+2})(1+z^{5n+3})}\\ &= 2z^{-7/80}z^{2/16}z^{2/5}\prod_{n=1}^{\infty}\frac{1+z^{2n}}{1-z^{2n}}\prod_{n=1}^{\infty}(1+z^{2n-1})(1-z^{10n})(1-z^{10n-2})(1-z^{10n-8}).\end{aligned}$$

But the last four products appearing in the above equation can be rewritten in the form

$$\begin{aligned}&\prod_{n=1}^{\infty}(1-z^{10n})(1+z^{10n-1})(1-z^{10n-2})(1+z^{10n-3})\times\\ &\qquad\times(1+z^{10n-5})(1+z^{10n-7})(1-z^{10n-8})(1+z^{10n-9})\\ &= \prod_{n=1}^{\infty}(1-(iz^{5/2})^{2n})(1-z^{-2}(iz^{5/2})^{2n})(1-z^2(iz^{5/2})^{2n-2})\times\\ &\qquad\times(1-z^{-4}(iz^{5/2})^{4n-2})(1-z^4(iz^{5/2})^{4n-2}).\end{aligned}$$

This last equation can be simplified, by setting $q = z^2$ and using the quintuple identity, to

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} (iq^{5/4})^{3n^2+n} \left(q^{-3n} - q^{3n+1}\right) \\
&= \sum_{n\in\mathbb{Z}} \left(q^{60n^2-7n} - q^{60n^2+17n+1}\right) + \sum_{n\in\mathbb{Z}} \left(q^{60n^2-43n+7\frac{1}{2}} - q^{60n^2-67n+18\frac{1}{2}}\right) \\
&+ \sum_{n\in\mathbb{Z}} \left(q^{60n^2-13n+\frac{1}{2}} - q^{60n^2-37n+5\frac{1}{2}}\right) + \sum_{n\in\mathbb{Z}} \left(q^{60n^2-97n+39} - q^{60n^2-73n+22}\right).
\end{aligned}$$

Thus we have the identity

$$\chi^3_{1,4}(z) = z^{-7/80} \left(\chi^{10}_{2,1}(z^2) + \chi^{10}_{2,11}(z^2) + \chi^{10}_{2,5}(z^2) + \chi^{10}_{2,7}(z^2)\right),$$

from which it follows that we have the decomposition

$$V(\tfrac{7}{10}, \tfrac{7}{16}) \downarrow R_{c=7/5} = V(\tfrac{7}{5}, \tfrac{21}{80}) \oplus V(\tfrac{7}{5}, \tfrac{61}{80}) \oplus V(\tfrac{7}{5}, 2\tfrac{21}{80}) \oplus V(\tfrac{7}{5}, 7\tfrac{61}{80}). \tag{4.13}$$

Similarly, one can show that

$$V(\tfrac{7}{10}, \tfrac{3}{80}) \downarrow R_{c=7/5} = V(\tfrac{7}{5}, \tfrac{1}{16}) \oplus V(\tfrac{7}{5}, \tfrac{9}{16}) \oplus V(\tfrac{7}{5}, 1\tfrac{9}{16}) \oplus V(\tfrac{7}{5}, 4\tfrac{1}{16}) \tag{4.14}$$

Note that these branching rules are multiplicity-free and finite.

4.1.2 Neveu-Schwarz subalgebra

In this subsection we will see how the $c = \frac{7}{10}$ Ramond modules split up into the direct sum of irreducible modules of its $c = \frac{7}{5}$ Neveu-Schwarz subalgebra. For example, consider

$$\begin{aligned}
\chi_{7/10,3/80} &= 2z^{-7/80} z^{1/8} \prod_{n=1}^{\infty} \frac{1+z^{2n-1}}{1-z^{2n}} \times \\
&\times \prod_{n=1}^{\infty} \frac{(1-z^{2n})}{(1+z^{2n-1})} \frac{(1+z^{5n})(1+z^{5n-2})(1+z^{5n-3})}{(1-z^{5n-1})(1-z^{5n-4})} \\
&= 2z^{-7/80} z^{1/8} \prod_{n=1}^{\infty} \frac{1+z^{2n-1}}{1-z^{2n}} \frac{(1-z^{20n})(1-z^{20n-4})(1-z^{20n-16})}{(1-z^{20n-2})(1-z^{20n-18})}.
\end{aligned} \tag{4.15}$$

If we now use the quintuple identity in its ratio form (4.12) on the last product in (4.15), we obtain (upon letting $q = z^2$)

$$\begin{aligned}
\chi_{7/10,3/80} &= 2z^{-7/80} q^{1/16} \prod_{n=1}^{\infty} \frac{1+q^{n-1/2}}{1-q^n} \sum_{n\in\mathbb{Z}} q^{15n^2+10n} \left((-q^4)^{-3n} - (-q^4)^{3n+2}\right) \\
&= 2z^{-7/80} q^{1/16} \prod_{n=1}^{\infty} \frac{1+q^{n-1/2}}{1-q^n} \left\{ \sum_{n\in\mathbb{Z}} \left(q^{60n^2-4n} - q^{60n^2+44n+8}\right) \right. \\
&+ \left. \sum_{n\in\mathbb{Z}} \left(q^{60n^2-16n+1} - q^{60n^2-64n+17}\right) \right\}.
\end{aligned}$$

That is,

$$\chi^3_{1,2}(z) = 2z^{-7/80}\left(\chi^{10}_{4,4}(z^2) + \chi^{10}_{4,8}(z^2)\right),$$

which yields the branching rule

$$V(\tfrac{7}{10},\tfrac{3}{80}) \downarrow NS_{c=7/5} = 2\left(V(\tfrac{7}{5},\tfrac{1}{16}) \oplus V(\tfrac{7}{5},1\tfrac{1}{16})\right). \tag{4.16}$$

In a similar manner, one gets

$$V(\tfrac{7}{10},\tfrac{7}{16}) \downarrow NS_{c=7/5} = 2\left(V(\tfrac{7}{5},\tfrac{21}{80}) \oplus V(\tfrac{7}{5},3\tfrac{21}{80})\right). \tag{4.17}$$

Thus we see that these branching rules are finite, and again multiplicity-free, up to the overall doubling associated with the two-fold degeneracy of the Ramond highest weight.

For generic $p > 2$ the central charge pc, when c is any member of the discrete series of central charges (4.3), is no longer a member of this discrete series. Hence by looking at the character formulae for irreducible representations where $c > \frac{3}{2}$, it is clear that the multiplicities occurring in the $p > 2$ branching rules explode. To solve this "missing label" problem, it would be necessary to construct additional invariants for the winding subalgebras in much the same way as it is done in the ordinary Lie algebra case (see for example [210–213]).

4.2 Branching rules for $N = 2$ superconformal algebras

We shall now extend the results of the previous section to the case of the $N = 2$ superconformal algebras. In addition to investigating the winding subalgebra branching rules, we shall examine the decomposition of the tensor product of certain irreducible representations of these algebras, which turn out to be finite and multiplicity-free. These tensor product decompositions have been studied in the case of the Virasoro ($N = 0$) algebra [206,214], where the product of two irreducible $c = \frac{1}{2}$ representations is multiplicity-free, and in the case of the two $N = 1$ superconformal algebras [215], where the product of two irreducible $c = \frac{7}{10}$ representations is *finite* and multiplicity-free.

The $N = 2$ superconformal algebra is an infinite-dimensional algebra with even generators $\{L_n, J_n\}$, odd generators $\{G^1_r, G^2_r\}$ and non-zero (graded) commutation relations

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{\tilde{c}}{4}(n^3-n)\delta_{n+m,0} \\
\left[L_n, G^j_r\right] &= (\tfrac{1}{2}n - r)G^j_{n+r}, \quad j = 1,2. \\
[L_n, J_m] &= -mJ_{n+m} \\
[J_n, J_m] &= \tilde{c}n\delta_{n+m,0} \\
\left[J_n, G^j_r\right] &= i\epsilon_{jk}G^k_{n+r} \\
\left\{G^j_r, G^k_s\right\} &= 2\delta_{jk}L_{r+s} + i\epsilon_{jk}(r-s)J_{r+s} + \tilde{c}(r^2-\tfrac{1}{4})\delta_{jk}\delta_{r+s,0}
\end{aligned}$$

where ϵ_{jk} is the antisymmetric symbol with $\epsilon_{12} = +1$. Clearly the generators L_n form a Virasoro algebra with central charge $c = 3\tilde{c}$, while the generators J_n form a $\widehat{U(1)}$ Kač-Moody subalgebra with central term $\tilde{c}$.

There are three different types of algebras called the P, A and T algebras, depending upon how the various generators are moded [158]. For all three algebras the elements L_n are moded by the integers $\mathbb{Z}$. For the P algebra we have generators $J_n, n \in \mathbb{Z}$ and $G^j_r, r \in \mathbb{Z}$. For the A algebra, we have $J_n, n \in \mathbb{Z}$ and $G^j_r, r \in \mathbb{Z} + \frac{1}{2}$. Finally, for the T algebra, the generators G^1_n are moded by $\mathbb{Z}$, while for the generators J_n and G^2_n, $n \in \mathbb{Z} + \frac{1}{2}$.

Highest weight representations of the A algebra are labelled by real numbers $\tilde{c}, h, q$ corresponding to the eigenvalues of the maximal abelian subalgebra of the A algebra, which is generated by $\{\tilde{c}, L_0, J_0\}$. The module $V_A(\tilde{c}, h, q)$ is generated by a (unique up to scalar multiple) highest weight vector $|\, h, q\rangle$, satisfying

$$\begin{aligned} X_n|\, h, q\rangle &= 0 \qquad \forall \quad n > 0 \\ L_0|\, h, q\rangle &= h|\, h, q\rangle \\ J_0|\, h, q\rangle &= q|\, h, q\rangle \end{aligned} \tag{4.18}$$

where $X_n \in \{L_n, n > 0, J_m, m > 0, G^j_r, r > 0\}$ are the raising generators of the A algebra.

Highest weight representations of the P algebra are also labelled by real numbers $\tilde{c}, h, q$. However there are two types of highest weight representations $P^{\pm}$ depending upon the conditions satisfied by the highest weight vector $\left|\, h, q \mp \frac{1}{2}\right\rangle$. In both types of modules, which we will denote by $V_{P^{\pm}}(\tilde{c}, h, q)$, the highest weight vector satisfies

$$\begin{aligned} X_n\left|\, h, q \mp \tfrac{1}{2}\right\rangle &= 0 \qquad \forall \quad n > 0 \\ L_0\left|\, h, q \mp \tfrac{1}{2}\right\rangle &= h\left|\, h, q \mp \tfrac{1}{2}\right\rangle \\ J_0\left|\, h, q \mp \tfrac{1}{2}\right\rangle &= (q \mp \tfrac{1}{2})\left|\, h, q \mp \tfrac{1}{2}\right\rangle \end{aligned}$$

with $X_n \in \{L_n, n > 0, J_m, m > 0, G^j_r, r > 0\}$. Moreover for the P^+ modules the highest weight state obeys the additional relation

$$G_0\left|\, h, q - \tfrac{1}{2}\right\rangle = 0,$$

while for the P^- modules the maximal weight state satisfies

$$\bar{G}_0\left|\, h, q + \tfrac{1}{2}\right\rangle = 0.$$

Here $G_n = \frac{1}{\sqrt{2}}(G^1_n + iG^2_n)$, $\bar{G}_n = \frac{1}{\sqrt{2}}(G^1_n - iG^2_n)$. It was shown in reference [158] that the P^+ and P^- representations are actually isomorphic.

For the T algebra, the generators J_n are moded by $\mathbb{Z} + \frac{1}{2}$ so that the maximal abelian subalgebra is generated by $\{\tilde{c}, L_0\}$. Therefore highest weight representations $V_T(\tilde{c}, h)$ are labelled by the eigenvalue h of L_0 and the central charge $\tilde{c}$. The highest weight state $|\, h\rangle$ satisfies

$$\begin{aligned} X_n|\, h\rangle &= 0 \quad \forall \quad n > 0 \\ L_0|\, h\rangle &= h|\, h\rangle \end{aligned}$$

where $X_n \in \{L_n, n > 0, J_m, m > 0, G_r^j, r > 0\}$.

The conditions for a representation of the $N = 2$ SCA's to be unitary were found by Boucher, Friedan and Kent [158]. Restricting ourselves to the case when $\tilde{c} < 1$, they found that unitary representations exist only when

$$\tilde{c} = 1 - \frac{2}{m} \qquad m = 2, 3, 4, \ldots \tag{4.19}$$

Moreover, for each integer m, the corresponding eigenvalues h and q must assume the following form:

$$\begin{aligned} A \quad \text{modules}: \qquad h_{j,k}^A &= \frac{jk - \frac{1}{4}}{m}, \quad q_{j,k}^A = \frac{j-k}{m}, \\ j, k \in \mathbb{Z} + \tfrac{1}{2}, \qquad & 0 < j, k, j+k < m-1 \end{aligned} \tag{4.20}$$

$$\begin{aligned} P^{\pm} \quad \text{modules}: \qquad h_{j,k}^{P^{\pm}} &= \frac{jk}{m} + \frac{\tilde{c}}{8}, \quad q_{j,k}^{P^{\pm}} = \pm\frac{(j-k)}{m}, \\ j, k \in \mathbb{Z}, \qquad & 0 \le j-1, k, j+k \le m-1 \end{aligned} \tag{4.21}$$

$$\begin{aligned} T \quad \text{modules}: \quad h_r^T &= \frac{(m-2r)^2}{16m} + \frac{\tilde{c}}{8}, \\ r \in \mathbb{Z}, \qquad & 1 \le r \le \tfrac{m}{2}. \end{aligned} \tag{4.22}$$

The characters of the A and $P^{\pm}$ modules are defined by

$$\chi^{A,P^{\pm}}(x, y) = \mathrm{tr}(x^{L_0} y^{J_0}),$$

where the trace is taken over the module $V_{A,P^{\pm}}(\tilde{c}, h, q)$. Similarly, the character of the T modules is defined via

$$\chi^T(x) = \mathrm{tr}(x^{L_0}),$$

with the trace taken over the module $V_T(\tilde{c}, h)$.

In references [162–164] the characters of the irreducible unitary representations with $\tilde{c} < 1$ were computed. For the $A, P^{\pm}$ algebras they were found to be given by

$$\begin{aligned} \chi_{j,k}^{(m),A}(x, y) &= x^{h_{j,k}^A} y^{q_{j,k}^A} \Phi_A(x, y) \Gamma_{j,k}^{(m)}(x, y), \\ \chi_{j,k}^{(m),P^{\pm}}(x, y) &= x^{h_{j,k}^{P^{\pm}}} y^{q_{j,k}^{P^{\pm}}} \Phi_P(x, y) \Gamma_{j,k}^{(m)}(x, y^{\pm 1}), \end{aligned}$$

where

$$\Phi_A(x, y) = \prod_{n=1}^{\infty} \frac{(1 + y x^{n-1/2})(1 + y^{-1} x^{n-1/2})}{(1 - x^n)^2}, \tag{4.23}$$

$$\Phi_P(x, y) = (y^{1/2} + y^{-1/2}) \prod_{n=1}^{\infty} \frac{(1 + y x^n)(1 + y^{-1} x^n)}{(1 - x^n)^2}, \tag{4.24}$$

$$\Gamma_{j,k}^{(m)}(x, y) = \sum_{n \in \mathbb{Z}} x^{mn^2 + (j+k)n} \left(1 - \frac{x^{mn+j} y^{-1}}{1 + x^{mn+j} y^{-1}} - \frac{x^{mn+k} y}{1 + x^{mn+k} y} \right). \tag{4.25}$$

Note that we can rewrite the expression in (4.24) in the form

$$\begin{aligned}\Gamma^{(m)}_{j,k}(x,y) &= \sum_{n\in\mathbb{Z}} x^{mn^2+(j+k)n}\left(-\frac{x^{mn+j}}{y+x^{mn+j}}+\frac{x^{-mn-k}}{y+x^{-mn-k}}\right)\\ &= y\sum_{n\in\mathbb{Z}} x^{mn^2+(j+k)n}\left(\frac{1}{y+x^{mn+j}}-\frac{1}{y+x^{-mn-k}}\right)\end{aligned}$$

which we shall find useful. Matsuo [162] has given the these characters in a factorized form with

$$\Gamma^{(m)}_{j,k} = \prod_{n=1}^{\infty}\frac{(1-x^{mn})^2(1-x^{mn-j-k})(1-x^{mn+j+k-m})}{(1+yx^{mn-j})(1+y^{-1}x^{mn+j-m})(1+y^{-1}x^{mn-k})(1+yx^{mn+k-m})}.$$

This factorized form will also be very useful later on. Note also the symmetries of the A and $P^{\pm}$ characters

$$\chi^{(m)}_{p,q} = \chi^{(m)}_{p+m,q+m} = \chi^{(m)}_{p+m,q-m}$$

and also

$$\chi^{(m)}_{p,q} = \chi^{(m)}_{m-q,-p}$$

For the T algebra the characters take the form

$$\chi^{(m),T}_r(x) = 2x^{h^T_r}\Phi_T(x)\Lambda^{(m)}_r(x),$$

with

$$\begin{aligned}\Phi_T(x) &= \prod_{n=1}^{\infty}\frac{1+x^{n/2}}{1-x^{n/2}},\\ \Lambda^{(m)}_r(x) &= \sum_{n\in\mathbb{Z}}(-1)^n x^{(mn^2+(m-2r)n)/4}\\ &= \prod_{n=1}^{\infty}(1-x^{mn/2})(1-x^{(mn-r)/2})(1-x^{(mn+r-m)/2}).\end{aligned}$$

However, when $h=\tilde{c}/8$ there is no factor of 2 appearing in the above character formulae for the T algebra. This is due to the fact that when $h\neq\tilde{c}/8$ the highest weight is two-fold degenerate, spanned by the linearly independant vectors $\mid h\rangle$ and $G^1_0\mid h\rangle$. But when $h=\tilde{c}/8$, the latter vector is singular (i.e. it is annihilated by the raising generators and has zero norm) and hence the submodule with this vector as its highest weight vector is factored out when creating the irreducible module with these particular values of highest weight h and central charge $\tilde{c}$. Thus there are only "half" the number of states in the irreducible module with the consequence that the degeneracy of each eigenvalue of the L_0 operator is half what it usually is.

If we take two modules V_1 and V_2 then we can form the tensor product module $V_1\otimes V_2$ which is also a module of the SCA under the diagonal action $\Delta(X_n) = 1\otimes X_n + X_n\otimes 1$. Certainly, if V_k has central charge $\tilde{c}_k$, $k=1,2$, then the tensor product module has central charge $\tilde{c}_1 + \tilde{c}_2$. Now it may happen that V_1 and V_2 are irreducible but that $V_1 \otimes V_2$ is not, and that it decomposes into a direct sum (which may be finite or infinite) of irreducible submodules. In reference [215] it was found that a sufficient condition for the tensor product decomposition of two unitary, irreducible $N = 1$ modules with central charges $\hat{c}_1$ and $\hat{c}_2$ to be finite, was that

$$\hat{c}_3 = \hat{c}_1 + \hat{c}_2 < 1 \tag{4.26}$$

where $\hat{c}_1, \hat{c}_2$ and $\hat{c}_3$ belonged to the unitary discrete series (4.3). Moreover, these decompositions were multiplicity-free for the Neveu-Schwarz algebra. For the $N = 1$ SCA the only pair of central charges$(\hat{c}_1, \hat{c}_2)$ belonging to the series (4.3) which satisfy the condition (4.26) is $(\frac{7}{15}, \frac{7}{15})$.

In the case of the $N = 2$ SCA we conjecture that the same condition applies. Namely, given two $N = 2$ SCA modules with central charges $\tilde{c}_1$ and $\tilde{c}_2$ a necessary and sufficient condition for the tensor product decomposition of these two modules to be finite, is that

$$\tilde{c}_3 = \tilde{c}_1 + \tilde{c}_2 < 1$$

where $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{c}_3$ belong to the discrete series (4.19). Specifically, the pairs of central charges $(\tilde{c}_1, \tilde{c}_2)$ which satisfy $\tilde{c}_3 = \tilde{c}_1 + \tilde{c}_2 < 1$ with $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ belonging to the $N = 2$ discrete series of central charges (4.19) are

$$(\tfrac{1}{3}, \tfrac{1}{3}), \quad (\tfrac{1}{3}, \tfrac{1}{2}), \quad (\tfrac{1}{3}, \tfrac{3}{5}) \tag{4.27}$$

In this section we shall be calculating explicitly some of the tensor product decompositions of unitary, irreducible $N = 2$ modules with these central charges, and exhibit their finite nature. As we also observed with the $N = 1$ winding subalgebra branchings, there appears to be no systematic way of examining all cases together. We therefore examine in detail some selected decompositions which exhibit some of the techniques required to tackle such calculations, and list the rest of the results in Appendix G.

For the T algebra, in addition to examining the finite case, we shall also be calculating the tensor product decomposition of two $N = 2$ modules whose central charges satisfy

$$\tilde{c}_1 + \tilde{c}_2 = 1. \tag{4.28}$$

It was shown in [206, 214] that tensor product decompositions of two unitary, irreducible Virasoro algebra ($N = 0$ SCA) modules was multiplicity-free (but infinite) when their respective central charges c_1, c_2 satisfied $c_1 + c_2 = 1$ (in fact for the discrete unitary series $c = 1 - 6/(m(m+1))$ only the pair $(\frac{1}{2}, \frac{1}{2})$ has this property). Unfortunately for the $N = 1$ superalgebras there are no pairs of central charges with this property. But for the $N = 2$ SCA's the pairs

$$(\tilde{c}_1, \tilde{c}_2) = (\tfrac{1}{3}, \tfrac{2}{3}), \quad (\tfrac{1}{2}, \tfrac{1}{2}) \tag{4.29}$$

with $\tilde{c}_1, \tilde{c}_2$ belonging to the series (4.19), satisfy (4.28). For these cases we shall find that infinite, multiplicity-free (up to an overall factor of 2 which may be induced by the double degeneracy of the highest weight) tensor product decompositions of T algebra modules exist for these values of central charges too.

4.2.1 The T algebra

In this subsection we will restrict our attention to the T algebra. The techniques that one must use are very similar to those used in reference [215] involving the use of the Euler pentagonal identity

$$\prod_{n=1}^{\infty}(1-x^{2n}) = \sum_{n\in\mathbb{Z}}(-1)^n x^{3n^2-n}, \tag{4.30}$$

along with the Jacobi triple product identity (4.10) and Watson's quintuple identity (4.11).

Let us first consider T algebra representations with $\tilde{c}=\frac{1}{3}$. From the unitary constraints (4.22) the only module with $\tilde{c}=\frac{1}{3}$ has highest weight $h_1^{(3),T}=\frac{1}{16}$ and the character

$$\chi_1^{(3),T} = 2x^{1/16}\prod_{n=1}^{\infty}(1+x^{n/2}).$$

From this, one deduces that

$$\begin{aligned}\left[\chi_1^{(3),T}(x)\right]^2 &= 4x^{1/8}\Phi_T(x)\prod_{n=1}^{\infty}(1-x^n)^2 \\ &= 2\chi_2^{(6),T}(x),\end{aligned}$$

and hence

$$V_T(\tfrac{1}{3},\tfrac{1}{16})\otimes V_T(\tfrac{1}{3},\tfrac{1}{16}) = 2V_T(\tfrac{2}{3},\tfrac{1}{8}).$$

We can also compute

$$\begin{aligned}\chi_1^{(3),T}(x)\chi_1^{(4),T}(x) &= 4x^{3/16}\Phi_T(x)\prod_{n=1}^{\infty}(1+x^{n/2})(1-x^{2n})(1-x^{2n-1/2})(1-x^{2n-3/2}) \\ &= 4x^{3/16}\Phi_T(x)\prod_{n=1}^{\infty}(1-x^{2n}) \\ &= 2\chi_4^{(12),T}(x),\end{aligned}$$

where we have used the fact that $\prod_{n=1}^{\infty}(1+x^n)(1-x^{2n-1})=1$. This gives us

$$V_T(\tfrac{1}{3},\tfrac{1}{16})\otimes V_T(\tfrac{1}{2},\tfrac{1}{8}) = 2V_T(\tfrac{5}{6},\tfrac{3}{16}).$$

The other tensor product decomposition we can examine for this pair of central charges is

$$\begin{aligned}\chi_1^{(3),T}(x)\chi_2^{(4),T}(x) &= 2x^{1/8}\Phi_T(x)\prod_{n=1}^{\infty}(1+x^{n/2})(1-x^{2n})(1-x^{2n-1})^2 \\ &= 2x^{1/8}\Phi_T(x)\prod_{n=1}^{\infty}(1-(-x^{1/2})^n) \\ &= 2\Phi_T(x)\left(\sum_{n\in\mathbb{Z}}(-1)^n x^{3n^2-n/2+1/8}+\sum_{n\in\mathbb{Z}}(-1)^n x^{3n^2+5n/2+5/8}\right) \\ &= \chi_5^{(12),T}(x)+\chi_1^{(12),T}(x),\end{aligned}$$

i.e.

$$V_T(\tfrac{1}{3},\tfrac{1}{16})\otimes V_T(\tfrac{1}{2},\tfrac{1}{16}) = V_T(\tfrac{5}{6},\tfrac{1}{8})\oplus V_T(\tfrac{5}{6},\tfrac{5}{8}).$$

Note that this decomposition is multiplicity-free, which is due to the fact that the highest weight in the module $V_T(\frac{1}{2},\frac{1}{16})$ is non-degenerate.

Finally, for the pair of central charges $(\frac{1}{3},\frac{3}{5})$ we have another two decompositions. Firstly,

$$\begin{aligned}
\chi_1^{(3),T}(x)\chi_1^{(5),T}(x) &= 4x^{1/4}\Phi_T(x)\prod_{n=1}^{\infty}(1+x^{n/2})(1-x^{5n/2})(1-x^{5n/2-1/2})(1-x^{5n/2-2}) \\
&= 4x^{1/4}\Phi_T(x)\prod_{n=1}^{\infty}(1-x^{5n})(1+x^{5n-3/2})(1+x^{5n-7/2})(1-x^{10n-2})(1-x^{10n-8}) \\
&= 4\Phi_T(x)\left(\sum_{n\in\mathbb{Z}}x^{15n^2/2+13n+23/4}+\sum_{n\in\mathbb{Z}}x^{15n^2/2+7n+7/4}\right) \\
&= 2\left(\chi_{11}^{(12),T}(x)+\chi_1^{(12),T}(x)\right) \\
\therefore \quad V_T(\tfrac{1}{3},\tfrac{1}{16})\otimes V_T(\tfrac{3}{5},\tfrac{3}{16}) &= 2\left(V_T(\tfrac{5}{6},\tfrac{1}{4})\oplus V_T(\tfrac{5}{6},\tfrac{7}{4})\right).
\end{aligned}$$

A similar calculation of $\chi_1^{(3),T}\chi_2^{(5),T}$ shows that

$$V_T(\tfrac{1}{3},\tfrac{1}{16})\otimes V_T(\tfrac{3}{5},\tfrac{7}{80}) = 2\left(V_T(\tfrac{5}{6},\tfrac{3}{20})\oplus V_T(\tfrac{5}{6},\tfrac{13}{20})\right).$$

In Appendix G we have summarized the finite tensor product decompositions which have been obtained.

Let us now turn our attention to the pairs of central charges in (4.29) whose decompositions turn out to be infinite, and consider the product of two $\tilde{c}=\frac{1}{2}$ T module characters. For example,

$$\begin{aligned}
\left(\chi_1^{(4),T}(x)\right)^2 &= 4x^{1/4}\left[\Phi_T(x)\right]^2\prod_{n=1}^{\infty}(1-x^{2n})^2(1-x^{2n-1/2})^2(1-x^{2n-3/2})^2 \\
&= 4x^{1/4}\Phi_T(x)\prod_{n=1}^{\infty}(1-x^{4n})(1+x^{4n-1})(1+x^{4n-3}) \\
&= 4\Phi_T(x)\sum_{n\in\mathbb{Z}}x^{2n^2+n+1/4} \\
\therefore V(\tfrac{1}{2},\tfrac{1}{8})\otimes V(\tfrac{1}{2},\tfrac{1}{8}) &= \bigoplus_{n\in\mathbb{Z}}2V(1,2n^2+n+\tfrac{1}{4}).
\end{aligned}$$

Similar calculations reveal that

$$\begin{aligned}
V(\tfrac{1}{2},\tfrac{1}{16})\otimes V(\tfrac{1}{2},\tfrac{1}{16}) &= \bigoplus_{n\geq 0}V(1,\tfrac{1}{2}n^2+\tfrac{1}{8}), \\
V(\tfrac{1}{2},\tfrac{1}{16})\otimes V(\tfrac{1}{2},\tfrac{1}{8}) &= \bigoplus_{n\in\mathbb{Z}}V(1,\tfrac{1}{2}n^2+\tfrac{1}{2}n+\tfrac{3}{16}).
\end{aligned}$$

Note that the above two decompositions are multiplicity-free, coming from the fact that they involve the module $V(\frac{1}{2},\frac{1}{16})$ which has a non-degenerate highest weight.

We can do the same thing with a $\tilde{c}=\frac{1}{3}$ T character and a $\tilde{c}=\frac{2}{3}$ T character and find that

$$\begin{aligned}
V(\tfrac{1}{3},\tfrac{1}{16})\otimes V(\tfrac{2}{3},\tfrac{1}{4}) &= \bigoplus_{n\in\mathbb{Z}} 2V(1,3n^2+\tfrac{3}{2}n+\tfrac{5}{16}),\\
V(\tfrac{1}{3},\tfrac{1}{16})\otimes V(\tfrac{2}{3},\tfrac{1}{8}) &= \bigoplus_{n\in\mathbb{Z}} 2V(1,n^2+\tfrac{1}{2}n+\tfrac{3}{16}),\\
V(\tfrac{1}{3},\tfrac{1}{16})\otimes V(\tfrac{2}{3},0) &= \bigoplus_{n\in\mathbb{Z}} 2V(1,\tfrac{3}{4}n^2+\tfrac{1}{4}n+\tfrac{1}{16}).
\end{aligned}$$

4.2.2 The A algebra

The calculation of tensor product decompositions of A algebra modules is somewhat more involved due to the fact that the vectors in these modules are indexed by two labels, and so the characters are functions of two indeterminates. This means we can no longer use the one-variable identities such as the triple product and quintuple product identities.

One of the two-variable identities which we shall find use for is the one found in [162, 163, 216]

$$\prod_{n=1}^{\infty}\frac{(1-x^n)^2}{(1+yx^{n-1/2})(1+y^{-1}x^{n-1/2})}=\sum_{n\in\mathbb{Z}} x^{2n^2+n}\left(-1+\frac{1}{1+yx^{2n+1/2}}+\frac{1}{1+y^{-1}x^{2n+1/2}}\right) \tag{4.31}$$

where $y\neq 0$ and $\mid x\mid<1$. This identity arose in references [162, 163] from the character formulae and the fact that the trivial one dimensional representation of the A algebra has unit character. One finds the same (up to a change of variables) identity from examination of the character of the trivial representation of the $P^{\pm}$ algebras. For a recent discussion of these types of identities which arise from considering various character identities of affine Lie superalgebras, see reference [217].

When $\tilde{c}=\frac{1}{3}$ $(m=3)$, unitarity restricts the values of j and k (i.e. h and q) to the values $\frac{1}{2},\frac{3}{2}$ and hence there are only three unitary simple modules with this value of central charge with characters $\chi^{(3)}_{1/2,1/2}$, $\chi^{(3)}_{1/2,3/2}$, $\chi^{(3)}_{3/2,1/2}$ (we drop the superscripts A and $P^{\pm}$ for convenience). Thus there are only six distinct tensor products we can consider and these fall into two different types. One set does not reduce (that is, the tensor product modules remain irreducible) and the other set decompose into two irreducible modules. As an example of the former, let us consider the tensor product decomposition of the module $V(\frac{1}{3},0,0)$ with the module $V(\frac{1}{3},\frac{1}{6},-\frac{1}{3})$.

$$\chi^{(3)}_{1/2,1/2}\chi^{(3)}_{1/2,3/2} = x^{1/6}y^{-1/3}\prod_{n=1}^{\infty}\frac{(1+yx^{n-1/2})^2(1+y^{-1}x^{n-1/2})^2}{(1-x^n)^4}\times$$
$$\times\prod_{n=1}^{\infty}\frac{(1-x^{3n})^2(1-x^{3n-1})(1-x^{3n-2})}{(1+yx^{3n-1/2})(1+y^{-1}x^{3n-1/2})(1+yx^{3n-5/2})(1+y^{-1}x^{3n-5/2})}\times$$
$$\times\prod_{n=1}^{\infty}\frac{(1-x^{3n})^2(1-x^{3n-1})(1-x^{3n-2})}{(1+yx^{3n-1/2})(1+y^{-1}x^{3n-3/2})(1+yx^{3n-3/2})(1+y^{-1}x^{3n-5/2})}$$
$$= x^{1/6}y^{-1/3}\Phi_A(x,y)\prod_{n=1}^{\infty}\frac{(1-x^{3n})^2}{(1+x^{3n-1/2})(1+y^{-1}x^{3n-5/2})}$$

Here we have used the facts that $\prod_{n=1}^{\infty}(1-x^n) = \prod_{n=1}^{\infty}(1-x^{3n})(1-x^{3n-1})(1-x^{3n-2})$ and similarly $\prod_{n=1}^{\infty}(1+y^{\pm 1}x^{n-1/2}) = \prod_{n=1}^{\infty}(1+y^{\pm 1}x^{3n-1/2})(1+y^{\pm 1}x^{3n-3/2})(1+y^{\pm 1}x^{3n-5/2})$. If we now use the identity (4.31) with y replaced by yx, we obtain

$$\begin{aligned}\chi^{(3)}_{1/2,1/2}\chi^{(3)}_{1/2,3/2} &= x^{1/6}y^{-1/3}\sum_{n\in\mathbb{Z}}x^{6n^2+3n}\left(-1+\frac{1}{1+yx^{6n+5/2}}+\frac{1}{1+y^{-1}x^{6n+1/2}}\right)\\ &= \chi^{(6)}_{1/2,5/2},\end{aligned}$$

thus obtaining the relation

$$V(\tfrac{1}{3},0,0)\otimes V(\tfrac{1}{3},\tfrac{1}{6},-\tfrac{1}{3}) = V(\tfrac{2}{3},\tfrac{1}{6},-\tfrac{1}{3}).$$

As an example of the other type of decomposition between two $\tilde{c}=\frac{1}{3}$ A modules consider the product

$$\left[\chi^{(3)}_{1/2,1/2}\right]^2 = \prod_{n=1}^{\infty}\frac{(1+yx^{n-1/2})^2(1+y^{-1}x^{n-1/2})^2}{(1-x^n)^4}\times$$
$$\times\prod_{n=1}^{\infty}\frac{(1-x^{3n})^4(1-x^{3n-1})^2(1-x^{3n-2})^2}{(1+yx^{3n-1/2})^2(1+y^{-1}x^{3n-1/2})^2(1+yx^{3n-1/2})^2(1+y^{-1}x^{3n-5/2})^2}$$
$$= \Phi_A(x,y)\prod_{n=1}^{\infty}\frac{(1-x^{3n})^2(1+yx^{3n-3/2})(1+y^{-1}x^{3n-3/2})}{(1+x^{3n-1/2})(1+y^{-1}x^{3n-1/2})(1+yx^{3n-5/2})(1+y^{-1}x^{3n-5/2})}$$

Using the identity (see Appendix A for details)

$$\begin{aligned}F_1(y) &\equiv \prod_{n=1}^{\infty}\frac{(1-x^{3n})^2(1+yx^{3n-3/2})(1+y^{-1}x^{3n-3/2})}{(1+x^{3n-1/2})(1+y^{-1}x^{3n-1/2})(1+yx^{3n-5/2})(1+y^{-1}x^{3n-5/2})}\\ &= \sum_{n\in\mathbb{Z}}x^{6n^2+5n+1}\left(\frac{x^{-6n-5/2}}{y+x^{-6n-5/2}}-\frac{x^{6n+5/2}}{y+x^{6n+5/2}}\right)\\ &\quad+\sum_{n\in\mathbb{Z}}x^{6n^2+n}\left(\frac{x^{-6n-1/2}}{y+x^{-6n-1/2}}-\frac{x^{6n+1/2}}{y+x^{6n+1/2}}\right),\end{aligned}\tag{4.32}$$

we obtain

$$\left[\chi^{(3)}_{1/2,1/2}\right]^2 = \chi^{(6)}_{1/2,1/2}+\chi^{(6)}_{5/2,5/2}.$$

This is equivalent to the tensor product decomposition

$$V(\tfrac{1}{3},0,0)\otimes V(\tfrac{1}{3},0,0) = V(\tfrac{2}{3},0,0)\oplus V(\tfrac{2}{3},1,0).$$

The entire set of six decompositions between the various $\tilde{c} = \frac{1}{3}$ A modules is given in Appendix G.

We will now consider the decompositions one obtains from modules with central charge $\tilde{c} = \frac{1}{2}$ tensored together with $\tilde{c} = \frac{1}{3}$ modules. While there were only three unitary $\tilde{c} = \frac{1}{3}$ modules there are six unitary $\tilde{c} = \frac{1}{2}$ modules with characters $\chi^{(4)}_{j,k}$ where $(j,k) \in \{(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{3}{2}),(\frac{1}{2},\frac{5}{2}),(\frac{3}{2},\frac{1}{2}),(\frac{3}{2},\frac{3}{2}),(\frac{5}{2},\frac{1}{2})\}$. This gives total of 18 different tensor products of this type. In this section we will consider a selection of them. Consider, for example, the following product of characters

$$\begin{aligned}
&\chi^{(3)}_{1/2,1/2}\chi^{(4)}_{1/2,1/2} = \Phi_A(x,y) \times \\
&\times \prod_{n=1}^{\infty} \frac{(1-x^{3n})(1-x^{4n})(1+yx^{3n-3/2})(1+y^{-1}x^{3n-3/2})}{(1-x^{4n-2})(1+yx^{4n-1/2})(1+y^{-1}x^{4n-1/2})(1+yx^{4n-7/2})(1+y^{-1}x^{4n-7/2})} \\
&\equiv \Phi_A(x,y)F_2(y).
\end{aligned}$$

With an argument similar to that used in Appendix A we can convert the function F_2 to an infinite sum of the form

$$\begin{aligned}
F_2(y) &= \sum_{n\in\mathbb{Z}} x^{12n^2+n}\left(\frac{x^{-12n-1/2}}{y+x^{-12n-1/2}} - \frac{x^{12n+1/2}}{y+x^{12n+1/2}}\right) \\
&+ \sum_{n\in\mathbb{Z}} x^{12n^2+7n+1}\left(\frac{x^{-12n-7/2}}{y+x^{-12n-7/2}} - \frac{x^{12n+7/2}}{y+x^{12n+7/2}}\right). \qquad (4.33)
\end{aligned}$$

Thus it is easy to see that

$$\begin{aligned}
\chi^{(3)}_{1/2,1/2}\chi^{(4)}_{1/2,1/2} &= \chi^{(12)}_{1/2,1/2} + \chi^{(12)}_{7/2,7/2} \\
\therefore V(\tfrac{1}{3},0,0)\otimes V(\tfrac{1}{3},0,0) &= V(\tfrac{5}{6},0,0)\oplus V(\tfrac{5}{6},1,0).
\end{aligned}$$

With the one identity (4.33) we are able to calculate several other tensor product decompositions of this type, although sometimes it is not particularly easy to see that this identity can be used. For example, let us examine the following product of characters.

$$\begin{aligned}
&\chi^{(3)}_{1/2,3/2}\chi^{(4)}_{1/2,5/2} = x^{5/12}y^{-5/6}\Phi_A(x,y) \times \\
&\times \prod_{n=1}^{\infty} \frac{(1-x^{3n})(1-x^{4n})(1+yx^{3n-5/2})(1+y^{-1}x^{3n-1/2})}{(1-x^{4n-2})(1+yx^{4n-1/2})(1+y^{-1}x^{4n-7/2})(1+yx^{4n-5/2})(1+y^{-1}x^{4n-3/2})}, \qquad (4.34)
\end{aligned}$$

As it stands, the identity (4.33) cannot be applied to the infinite product which appears in (4.34). However let us make the substitution $y = zx^{13}$ in (4.34). Then the infinite product in that equation takes the form

$$\prod_{n=1}^{\infty} \frac{(1-x^{3n})(1-x^{4n})(1+zx^{3n+10\frac{1}{2}})(1+z^{-1}x^{3n-13\frac{1}{2}})}{(1-x^{4n-2})(1+zx^{4n+12\frac{1}{2}})(1+z^{-1}x^{4n-16\frac{1}{2}})(1+zx^{4n+11\frac{1}{2}})(1+z^{-1}x^{4n-15\frac{1}{2}})} \qquad (4.35)$$

$$= x^{-14\frac{1}{2}}y^3F_2(z) \qquad (4.36)$$

where going from equation (4.35) to (4.36) we have shifted the indices by an appropriate amount and then cancelled the resulting factors. Hence we can write the product of characters as

$$\chi^{(3)}_{1/2,3/2}\chi^{(4)}_{1/2,5/2} = x^{-1\frac{1}{2}}y^{2\frac{1}{6}}\Phi_A(x,y)\left\{\sum_{n\in\mathbb{Z}} x^{12n^2+11n-1/2}\left(\frac{1}{y+x^{12n+12\frac{1}{2}}}\right.\right.$$
$$\left.-\frac{1}{y+x^{-12n+\frac{3}{2}}}\right)+\sum_{n\in\mathbb{Z}} x^{12n^2+5n-5/2}\left(\frac{1}{y+x^{12n+9\frac{1}{2}}}-\left.\left.\frac{1}{y+x^{-12n+4\frac{1}{2}}}\right)\right\}$$
$$=\chi^{(12)}_{12\frac{1}{2},-\frac{3}{2}}+\chi^{(12)}_{9\frac{1}{2},-4\frac{1}{2}}=\chi^{(12)}_{\frac{1}{2},10\frac{1}{2}}+\chi^{(12)}_{4\frac{1}{2},2\frac{1}{2}}.$$

This provides us with the decomposition

$$V(\tfrac{1}{3},\tfrac{1}{6},-\tfrac{1}{3})\otimes V(\tfrac{1}{2},\tfrac{1}{4},-\tfrac{1}{2}) = V(\tfrac{5}{6},\tfrac{5}{12},-\tfrac{5}{6})\oplus V(\tfrac{5}{6},\tfrac{11}{12},\tfrac{1}{6}).$$

The identity (4.33) can similarly be used to compute four other tensor product decompositions between $\tilde{c}=\frac{1}{3}$ and $\tilde{c}=\frac{1}{2}$ A modules. These are listed in Appendix G.

4.2.3 The P algebra

The calculations needed to calculate the tensor product decompositions of unitary, irreducible $P^{\pm}$ modules are very similar to those in the case of the A algebra. Here we will consider the P^+ algebra only, as the P^- decompositions follow from the simple relation $\chi^{(m),P^-}_{j,k}(x,y) = \chi^{(m),P^+}_{j,k}(x,y^{-1})$, and restrict our attention to the modules with $\tilde{c}=\frac{1}{3}$. Here there are only three modules which are unitary with characters $\chi^{(3)}_{1,0}$, $\chi^{(3)}_{1,1}$, $\chi^{(3)}_{2,0}$. Hence there are only six different decompositions among these modules.

Consider now the product of characters

$$\begin{aligned}\left[\chi^{(3)}_{1,1}\right]^2 &= \left\{x^{3/8}(y^{1/2}+y^{-1/2})\right\}\prod_{n=1}^{\infty}\frac{(1+yx^{3n})(1+y^{-1}x^{3n})}{(1-x^{3n-1})(1-x^{3n-2})}\times\\ &\times\Phi_A(x,y)\prod_{n=1}^{\infty}\frac{(1-x^{3n})^2(1-x^{3n-1})(1-x^{3n-2})}{(1+yx^{3n-1})(1+y^{-1}x^{3n-1})(1+yx^{3n-2})(1+y^{-1}x^{3n-2})}\\ &= x^{3/4}(y^{1/2}+y^{-1/2})\Phi_A(x,y)\times\\ &\times y^{-1/2}\prod_{n=1}^{\infty}\frac{(1-x^{3n})^2(1+yx^{3n-3})(1+y^{-1}x^{3n})}{(1+yx^{3n-1})(1+y^{-1}x^{3n-1})(1+yx^{3n-2})(1+y^{-1}x^{3n-2})}.\end{aligned}\tag{4.37}$$

Letting $y=zx^{3/2}$ we can reexpress the latter product in (4.37) as

$$\begin{aligned}&\frac{1+zx^{1/2}}{1+z^{-1}x^{-1/2}}\prod_{n=1}^{\infty}\frac{(1-x^{3n})^2(1+zx^{3n-3/2})(1+z^{-1}x^{3n-3/2})}{(1+zx^{3n-1/2})(1+z^{-1}x^{3n-1/2})(1+zx^{3n-5/2})(1+z^{-1}x^{3n-5/2})}\\ &= yx^{-1}\left\{x\sum_{n\in\mathbb{Z}}x^{6n^2+5n}\left(\frac{x^{-6n-1}}{y+x^{-6n-1}}-\frac{x^{6n+4}}{y+x^{6n+4}}\right)\right.\\ &\qquad\left.+x\sum_{n\in\mathbb{Z}}x^{6n^2+5n}\left(\frac{1}{y+x^{6n+1}}-\frac{1}{y+x^{-6n-4}}\right)\right\}\end{aligned}$$

where we have used the identity (4.32). Hence $\left[\chi^{(3)}_{1,1}\right]^2 = \chi^{(6)}_{4,1} + \chi^{(6)}_{1,4}$, and thus

$$V(\tfrac{1}{3}, \tfrac{3}{8}, 0) \otimes V(\tfrac{1}{3}, \tfrac{3}{8}, 0) = V(\tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{2}) \oplus V(\tfrac{2}{3}, \tfrac{3}{4}, -\tfrac{1}{2}). \tag{4.38}$$

Note that in (4.38) the eigenvalues of J_0 do not seem to add up correctly. This is because the highest weight state has J_0 eigenvalue $q - \frac{1}{2}$ whereas we are labelling the modules by the quantity q.

The calculations needed to find the other five tensor product decompositions between $\tilde{c} = \frac{1}{3}$ modules of the P^+ algebra are very similar and the results are shown in Appendix G.

4.3 Winding subalgebras of the $N = 2$ superconformal algebras

We shall conclude this chapter with a study of the finite branching rules of the various winding subalgebras of the $N = 2$ superconformal algebras. As in the $N = 1$ case, these subalgebras are indexed by a (positive) integer p, and are generated by the elements (we shall ignore the modings for the moment) $\hat{L}_n$, $\hat{G}^j_r$, $\hat{J}_m$ which are defined by

$$\begin{aligned} \hat{L}_n &= \tfrac{1}{p} L_{pn} + \frac{\tilde{c}}{8}\left(p - \frac{1}{p}\right) \\ \hat{G}^j_r &= \frac{1}{\sqrt{p}} G^j_{pr} \\ \hat{J}_m &= J_{pm} \end{aligned} \tag{4.39}$$

These winding subalgebras carry a central charge of $p\tilde{c}$ as can easily be checked by direct calculation. The type (A, $P^\pm$ or T) of algebra the winding subalgebra is, depends on p and on the type of algebra the original algebra was. Suppose the original algebra was a T SCA. Then if p is odd there exists a T winding subalgebra, but if p is even then no winding subalgebras exist. Similarly the A algebra has only winding A subalgebras if p is odd but no others if p is even. The case of the $P^\pm$ algebra is more interesting. Certainly it is easy to see that for p both even and odd there are $P^\pm$ winding subalgebras. Moreover when p is even, there exist well-defined T and A winding subalgebras. We will, in fact, be able to show that when $\tilde{c} = \frac{1}{3}$ and $p = 2$, the branching rules are finite. So let us now consider the winding A, P^+, and T subalgebras of the $\tilde{c} = \frac{1}{3}$ P^+ algebra.

4.3.1 Winding A subalgebras

Let us examine the winding A subalgebras of the P^+ algebra (for definiteness). For the $\tilde{c} = \frac{1}{3}$ P^+ modules the $p = 2$ winding A subalgebra has central charge $\tilde{c} = \frac{2}{3}$ which is also a central charge which belongs to the discrete series. To compute these branching rules, note that the P^+ character can be written in terms of an A character of its A winding subalgebra by means of

$$\begin{aligned}\chi^{P+}_{\tilde{c},h}(x,y) &= \operatorname{tr}\left(x^{L_0}y^{J_0})\right) \\ &= x^{-\tilde{c}(p^2-1)/8}\operatorname{tr}\left(x^{2\hat{L}_0}y^{\hat{J}_0}\right) \\ &= x^{-\tilde{c}(p^2-1)/8}\chi^{A}_{2\tilde{c},h}(x^2,y).\end{aligned}$$

Consider, as an example, the P^+ character $\chi^{(3)}_{1,0}$ which we can write as

$$\begin{aligned}\chi^{(3)}_{1,0}(x,y) &= x^{1/24}y^{-1/6}\Phi_A(x^2,y)\prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{(1+yx^{2n-1})(1+y^{-1}x^{2n-1})}\times \\ &\quad \times\prod_{n=1}^{\infty}\frac{(1+yx^{3n-2})(1+y^{-1}x^{3n-1})}{(1-x^{3n-1})(1-x^{3n-2})} \\ &= x^{-1/8}q^{1/12}y^{-1/6}\Phi_A(q,y)F_3(q,y),\end{aligned} \tag{4.40}$$

where $q\equiv x^2$ and the function F_3 is defined by

$$\begin{aligned}F_3(q,y) = \prod_{n=1}^{\infty}&\frac{(1-q^{3n})^2(1-q^{3n-1})(1-q^{3n-2})}{(1-q^{3n-1/2})(1-q^{3n-5/2})}\times \\ \times\prod_{n=1}^{\infty}&\frac{(1+yq^{3n-1})(1+y^{-1}q^{3n-2})}{(1+yq^{3n-1/2})(1+yq^{3n-3/2})(1+y^{-1}q^{3n-3/2})(1+y^{-1}q^{3n-5/2})}.\end{aligned} \tag{4.41}$$

But we can reexpress F_3 as

$$\begin{aligned}F_3(q,y) = \sum_{n\in\mathbb{Z}}&q^{6n^2+4n-3/2}\left(\frac{1}{y+q^{6n-3/2}}-\frac{1}{y+q^{-6n-11/2}}\right) \\ &+\sum_{n\in\mathbb{Z}}q^{6n^2+2n-2}\left(\frac{1}{y+q^{6n-5/2}}-\frac{1}{y+q^{-6n-9/2}}\right),\end{aligned} \tag{4.42}$$

with the equivalence of expressions (4.41) and (4.42) being established by an argument similar to that given in Appendix A.

Using (4.40) and (4.42), it is easy to see that

$$\chi^{(3)}_{1,0} = x^{-1/8}\left(\chi^{(6)}_{1/2,3/2}(x^2,y)+\chi^{(6)}_{3/2,5/2}(x^2,y)\right),$$

and hence

$$V_{P^+}(\tfrac{1}{3},\tfrac{1}{24},\tfrac{1}{3}) = V_A(\tfrac{2}{3},\tfrac{1}{12},-\tfrac{1}{6})\oplus V_A(\tfrac{2}{3},\tfrac{7}{12},-\tfrac{1}{6}). \tag{4.43}$$

Note how the L_0 eigenvalue of the P^+ highest weight state changes when considered as a highest weight state of its winding A subalgebra. The highest weight vector of the P^+ module had an L_0 eigenvalue $\frac{1}{24}$ but the new A generator $\hat{L}_0=\frac{1}{2}L_0+\frac{1}{16}$ turns this into a highest weight vector of the first A module in (4.43) with L_0 eigenvalue $\frac{1}{2}\frac{1}{24}+\frac{1}{16}=\frac{1}{12}$. The J_0 eigenvalue of this first module however does not change (it only appears to because of the way $P^{\pm}$ modules are labelled: the J_0 eigenvalue of the highest weight vector of the module $V_{P^\pm}(\tilde{c},h,q)$ is $q\mp\frac{1}{2}$). In a similar fashion we can calculate the branching rules

$$\begin{aligned}V_{P^+}(\tfrac{1}{3},\tfrac{3}{8},0) &= V_A(\tfrac{2}{3},\tfrac{1}{4},-\tfrac{1}{2})\oplus V_A(\tfrac{2}{3},\tfrac{1}{4},\tfrac{1}{2}), \\ V_{P^+}(\tfrac{1}{3},\tfrac{1}{24},\tfrac{2}{3}) &= V_A(\tfrac{2}{3},\tfrac{1}{12},\tfrac{1}{6})\oplus V_A(\tfrac{2}{3},\tfrac{7}{12},\tfrac{1}{6}).\end{aligned}$$

4.3.2 Winding P^+ subalgebras

If we now consider the $p = 2$ winding P^+ subalgebra of the $\tilde{c} = \frac{1}{3}$ P^+ algebra we can calculate, with similar methods as were used for the winding A subalgebra, the following character decompositions,

$$\begin{aligned}
\chi^{(3)}_{1,0} &= x^{-1/8}\left(\chi^{(6)}_{2,0}(x^2,y) + \chi^{(6)}_{3,1}(x^2,y)\right), \\
\chi^{(3)}_{1,1} &= x^{-1/8}\left(\chi^{(6)}_{1,1}(x^2,y) + \chi^{(6)}_{2,2}(x^2,y)\right), \\
\chi^{(3)}_{2,0} &= x^{-1/8}\left(\chi^{(6)}_{4,0}(x^2,y) + \chi^{(6)}_{1,3}(x^2,y)\right),
\end{aligned}$$

which provides us with the branching rules

$$\begin{aligned}
V_{P^+}(\tfrac{1}{3},\tfrac{1}{24},\tfrac{1}{3}) &= V_{P^+}(\tfrac{2}{3},\tfrac{1}{12},\tfrac{1}{3}) \oplus V_{P^+}(\tfrac{2}{3},\tfrac{7}{12},\tfrac{1}{3}), \\
V_{P^+}(\tfrac{1}{3},\tfrac{3}{8},0) &= V_{P^+}(\tfrac{2}{3},\tfrac{1}{4},0) \oplus V_{P^+}(\tfrac{2}{3},\tfrac{3}{4},0), \\
V_{P^+}(\tfrac{1}{3},\tfrac{1}{24},\tfrac{2}{3}) &= V_{P^+}(\tfrac{2}{3},\tfrac{1}{12},\tfrac{2}{3}) \oplus V_{P^+}(\tfrac{2}{3},\tfrac{7}{12},-\tfrac{1}{3}).
\end{aligned}$$

4.3.3 Winding T subalgebras

The case of the winding T subalgebras is more interesting, most notably because T characters are functions of only one parameter, whereas P^+ characters are functions of two parameters. We can, however, comfortably set $y \equiv 1$ in the P^+ characters since those characters are valid for all $y \neq 0$. With this in mind we observe that

$$\begin{aligned}
\chi^{(3)}_{1,0}(x,1) &= 2x^{3/8}\prod_{n=1}^{\infty}\frac{(1+x^n)^2}{(1-x^n)^2} \times \\
&\quad \times \prod_{n=1}^{\infty}\frac{(1-x^{3n})^2(1-x^{3n-1})(1-x^{3n-2})}{(1+x^{3n-1})(1+x^{3n-2})(1+x^{3n})(1+x^{3n-3})} \\
&= x^{3/8}\Phi_T(x^2)\prod_{n=1}^{\infty}\frac{(1-x^{3n})}{(1+x^{3n})} \\
&= x^{3/8}\Phi_T(x^2)\prod_{n=1}^{\infty}(1-x^{6n})(1-x^{6n-3})^2 \\
&= x^{-1/8}\chi^{(6)}_{3}(x^2), \qquad (4.44)
\end{aligned}$$

so that

$$V_{P^+}(\tfrac{1}{3},\tfrac{1}{24},\tfrac{1}{3}) = V_T(\tfrac{2}{3},\tfrac{1}{12}).$$

Note that there is no factor of 2 occuring in (4.44). This is because $h^{(6),T}_3 = \frac{1}{12} = \frac{2}{3}\times\frac{1}{8} = \frac{\tilde{c}}{8}$ and hence the highest weight state in this irreducible T module is not degenerate. With similar calculations we find that

$$\begin{aligned}
\chi^{(3)}_{1,1}(x,1) &= 2x^{-1/8}\chi^{(6)}_{1}(x^2), \\
\chi^{(3)}_{2,0}(x,1) &= x^{-1/8}\chi^{(6)}_{3}(x^2),
\end{aligned}$$

with the result that

$$\begin{aligned}
V_{P^+}(\tfrac{1}{3},\tfrac{3}{8},0) &= 2V_T(\tfrac{2}{3},\tfrac{1}{4}) \\
V_{P^+}(\tfrac{1}{3},\tfrac{1}{24},\tfrac{2}{3}) &= V_T(\tfrac{2}{3},\tfrac{1}{12}).
\end{aligned}$$

This concludes the calculations of the branching rules of the winding subalgebras of the P^+ algebra with $\tilde{c} = \frac{1}{3}$, which we conjecture are the only finite decompositions associated with the winding subalgebras of the $N = 2$ SCA's.

We note here that, unlike the case for the $N = 1$ SCA's [215], the necessary condition for a module $V(\tilde{c}_1 + \tilde{c}_2, h, q)$ to occur in the tensor product decomposition of $V(\tilde{c}_1, h_1, q_1) \otimes V(\tilde{c}_2, h_2, q_2)$, namely

$$\begin{array}{rl} h - (h_1 + h_2) \in \mathbb{Z} \quad \text{or} \quad h - (h_1 + h_2) \in \mathbb{Z} + \frac{1}{2} & \\ \text{together with} \quad q - (q_1 + q_2) \in \mathbb{Z} \quad \text{for the } A \text{ algebra} & \\ \text{or} \quad q - (q_1 + q_2) \in \mathbb{Z} + \frac{1}{2} \text{ for the } P \text{ algebra} & \end{array} \tag{4.45}$$

is not a sufficient condition as well. As an example, consider the A algebra decomposition

$$V_A(\tfrac{1}{3}, 0, 0) \otimes V_A(\tfrac{1}{2}, \tfrac{1}{2}, 0) = V_A(\tfrac{5}{6}, \tfrac{1}{2}, 0) \oplus V_A(\tfrac{5}{6}, 2\tfrac{1}{2}, 0).$$

There is one more additional $\tilde{c} = \frac{5}{6}$ module satisfying the conditions (4.45), which is the one corresponding to $m = 12$, $r = s = 7/2$, that is, the module $V_A(\frac{5}{6}, 1, 0)$. However this module does not appear in the above tensor product decomposition and so the above conditions are only necessary, not sufficient. It is to be hoped that in the future, sufficient conditions for a module to appear in a tensor product decomposition of two irreducible modules of the $N = 2$ SCA can be found.

Chapter 5

Applications of the Boson–Fermion Correspondence

In this chapter we discuss the boson-fermion correspondence and how it can be applied to derive many explicit results concerning symmetric functions. We begin with a brief review of the boson-fermion correspondence for S-functions and Q-functions and derive some explicit results concerning the multiplication of these functions by power sums. We then proceed to apply the boson-fermion correspondence to the problem of outer multiplication (Littlewood-Richardson rule) and outer plethysm of S-functions. We then turn to Hall-Littlewood functions, deriving an algorithm for decomposing these functions in terms of S-functions. Finally we define the concept of the outer plethysm of Hall-Littlewood functions, and generalize the results developed in earlier sections to enable us to calculate some examples explicitly.

5.1 The boson–fermion correspondence for S-functions

Central to the development of the results in this chapter is the concept of the boson–fermion (B-F) correspondence [218], and its relationship to the theory of symmetric functions. There are actually three different versions which we shall be using, which can be related to S, Q and Hall-Littlewood functions. They have been used to great effect in deriving various symmetric function identities [196, 219–221], not to mention their use in studies of the KP and BKP hierarchies [113, 222, 223]. Due to its importance in what follows, we shall devote considerable attention to reviewing the classical (pertaining to S-functions) B-F correspondence which relates free fermions (which generate a Clifford algebra) to "free" bosons (which generate a Heisenberg algebra), the latter being able to be realized in terms of symmetric functions. We shall be mainly using the notation used by the Kyoto school [113].

The algebra $\mathcal{A}$ of free fermions is generated by ψ_i, ψ_i^*, $i \in \mathbb{Z}$ satisfying the anti-commutation relations

$$\{\psi_i, \psi_j\} = 0 = \{\psi_i^*, \psi_j^*\}, \qquad \{\psi_i, \psi_j^*\} = \delta_{ij}. \tag{5.1}$$

There is a Fock representation $\mathcal{F}$ of this algebra with a vacuum $|0\rangle$ which satisfies

$$\begin{array}{ll} \psi_i|0\rangle = 0 \quad (i < 0), & \psi_i^*|0\rangle = 0 \quad (i \geq 0), \\ \langle 0|\psi_i = 0 \quad (i \geq 0), & \langle 0|\psi_i^* = 0 \quad (i < 0). \end{array} \tag{5.2}$$

Using this definition of the vacuum, we can compute the vacuum expectation value $\langle a\rangle \equiv \langle 0|a|0\rangle$ for any product of free fermions. In particular we have

$$\begin{array}{ll} \langle\psi_i\psi_j\rangle = 0, & \langle\psi_i^*\psi_j^*\rangle = 0, \\ \langle\psi_i\psi_j^*\rangle = \left\{ \begin{array}{cc} \delta_{ij} & i = j < 0 \\ 0 & \text{else} \end{array} \right., & \langle\psi_i^*\psi_j\rangle = \left\{ \begin{array}{cc} \delta_{ij} & i = j \geq 0 \\ 0 & \text{else} \end{array} \right.. \end{array}$$

If we define normal ordering of a product of free fermions by $: \psi_i\psi_j^* := \psi_i\psi_j^* - \langle\psi_i\psi_j^*\rangle$ then

$$: \psi_i\psi_j^* := \left\{ \begin{array}{ll} \psi_i\psi_j^*, & j \geq 0 \\ -\psi_j^*\psi_i & j < 0, \end{array} \right. \qquad : \psi_i^*\psi_j := \left\{ \begin{array}{ll} \psi_i^*\psi_j & j < 0 \\ -\psi_j\psi_i^* & j \geq 0. \end{array} \right.$$

Let

$$H_n = \sum_{i\in\mathbb{Z}} : \psi_i\psi_{i+n}^* :, \qquad n \in \mathbb{Z}. \tag{5.3}$$

Then the operators H_n generate a Heisenberg algebra

$$[H_n, H_m] = n\,\delta_{n+m,0}. \tag{5.4}$$

Suppose we have a set of Heisenberg generators $\{\alpha_n : n \in \mathbb{Z}\}$ satisfying (5.4). For $n \neq 0$, these have a realization on the space $\Lambda(x)$ of symmetric polynomials in the indeterminates $(x_1, x_2, \ldots)$ in terms of power sum symmetric functions $p_k(x) = \sum_i x_i^k$ in the form

$$\alpha_{-k} \leftrightarrow p_k, \qquad \alpha_k \leftrightarrow k\frac{\partial}{\partial p_k} \qquad \text{for } k > 0. \tag{5.5}$$

Let us adjoin to the Heisenberg algebra an operator q satisfying

$$[q, \alpha_n] = 0 \qquad \text{for } n \neq 0, \qquad [q, \alpha_0] = i.$$

We shall be considering representations of the Heisenberg algebra on the space $\bar{\Lambda} = \Lambda(x) \otimes \left(\oplus_{k\in\mathbb{Z}}\, e^{ikq}\right)$ where the generators α_n for $n \neq 0$ act on $\Lambda(x)$ via (5.5) and α_0, q act according to

$$\alpha_0\, e^{ikq} = k\, e^{ikq}, \qquad e^{iq}e^{ikq} = e^{i(k+1)q}. \tag{5.6}$$

Define vertex operators acting on $\bar{\Lambda}$

$$\begin{aligned} \psi(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{p_n(x)}{n}\, z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{\partial}{\partial p_n(x)}\, z^{-n}\right)\; e^{iq} z^{\alpha_0}, \\ \psi^*(z) &= \exp\left(-\sum_{n=1}^{\infty} \frac{p_n(x)}{n}\, z^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{\partial}{\partial p_n(x)}\, z^{-n}\right)\; z^{-\alpha_0} e^{-iq}. \end{aligned} \tag{5.7}$$

(For notational simplicity, we drop the symbol $\otimes$ in these and subsequent formulae.) If the modes of these vertex operators are given by the expansion

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^n, \qquad \psi^*(z) = \sum_{n\in\mathbb{Z}} \psi_n^* z^{-n},$$

then it is well known that the modes ψ_n, ψ_n^* satisfy the anti-commutation relations of the free fermion algebra (5.1). Moreover, every state $a|0\rangle$, $a \in \mathcal{A}$ in the fermionic Fock space can be identified with a symmetric function as follows: define a grading (charge) on the elements of $\mathcal{A}$ by setting $\deg(\psi_i) = 1$, $\deg(\psi_i^*) = -1$, for all $i \in \mathbb{Z}$ (this can be achieved by the grading operator $\mathrm{ad}(H_0)$ with H_0 defined in (5.3)). Thus an element $\psi_{j_1}^* \cdots \psi_{j_r}^* \psi_{i_s} \cdots \psi_{i_1}|0\rangle \in \mathcal{F}$ will have a charge $l = s - r$. If we let

$$\langle l| = \begin{cases} \langle 0|\psi_{-1}\cdots\psi_l & \text{if } l < 0 \\ \langle 0| & \text{if } l = 0 \\ \langle 0|\psi_0^*\cdots\psi_{l-1}^* & \text{if } l > 0 \end{cases}, \qquad H(x) = \sum_{n=1}^{\infty} \frac{1}{n} p_n(x)\, H_n,$$

then we have the isomorphism $\varrho : \mathcal{F} \longrightarrow \bar{\Lambda}$

$$\varrho(a|0\rangle) \ = \langle l|e^{H(x)}\, a|0\rangle\,. \tag{5.8}$$

In fact each of the charge subspaces $\mathcal{F}_l$ consisting of states $a|0\rangle$ where a has charge l, is isomorphic to $\Lambda(x)$. The right-hand side of (5.8) can be evaluated explicitly in terms of S-functions for a generic state in $\mathcal{F}$ so that, if $0 \le i_s < \cdots < i_1$, $0 < j_r < \cdots < j_1$,

$$\varrho(\psi_{-j_1}^* \cdots \psi_{-j_r}^* \psi_{i_s} \cdots \psi_{i_1}|0\rangle) \ = \ (-1)^{j_1+\cdots j_r + l(l-1)/2}\ s_\lambda(x)\, e^{ilq}, \tag{5.9}$$

where λ is a partition of the form

$$\lambda = (i_1 + 1 - l, i_2 + 2 - l, \ldots, i_s + s - l, r^{j_r-1}, (r-1)^{j_{r-1}-j_r-1}, \ldots, 2^{j_2-j_3-1}, 1^{j_1-j_2-1}). \tag{5.10}$$

Note that for $l = 0$, we can write this in Frobenius notation as

$$\lambda = \begin{pmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 - 1 & j_2 - 1 & \cdots & j_r - 1 \end{pmatrix}.$$

In deriving the results in this chapter, we shall often ignore the momentum factor e^{ilq} occuring in (5.9), when no confusion arises. Observe that in the case $l = 0$, using the anti-commutation relations (5.1) and the isomorphism (5.9), one can derive the modification rules for S-functions given in Chapter 2 (see page 13).

As previously mentioned, the boson-fermion correspondence has been used to prove useful identities involving S-functions. We now intend to show that it can also be applied to decompose power sum symmetric functions in terms of S-functions. By using the vertex operators (5.7) representing the fermionic currents $\psi(z)$ and $\psi^*(z)$ it was shown [113] that under the above isomorphism, the Heisenberg generators (5.3) are mapped onto the following operators on Λ;

$$H_{-n} \longleftrightarrow p_n(x), \qquad H_n \longleftrightarrow n\frac{\partial}{\partial p_n(x)} \qquad n > 0, \tag{5.11}$$

while $H_0 \leftrightarrow 0$. Thus, acting on the vacuum, we know that $\varrho(H_{-n}|0\rangle) = p_n(x).1 = p_n(x)$. However, according to (5.3), $H_{-n}|0\rangle$ can be written as an infinite sum of bilinear fermionic modes acting on the vacuum (only a finite number of which will survive) which can then be expressed as Schur functions using the explicit correspondence (5.9). Thus,

$$\begin{aligned} p_n(x) &= \varrho(H_{-n}|0\rangle) = \varrho\left(\sum_{i\in\mathbb{Z}} :\psi_i\psi^*_{i-n}: \ |0\rangle\right) \\ &= \varrho\left(\left(\sum_{i\geq n}\psi_i\psi^*_{i-n} - (\sum_{i<0}+\sum_{i=0}^{n-1})\psi^*_{i-n}\psi_i\right)|0\rangle\right) = \sum_{i=0}^{n-1}(-1)^j\ s_{(n-1-j|j)}(x), \end{aligned}$$

which is a well-known identity between the simple power sums $p_n(x)$ and one-hook S-functions. In a similar manner we can calculate $p_{(n,m)}(x) = p_n(x)p_m(x)$, by writing out $H_{-n}H_{-m}|0\rangle$ in terms of fermionic modes, and then using the fermionic anti-commutation relations to write this as a finite sum of terms of the form $\psi^*_{-j_1}\psi^*_{-j_2}\psi_{i_2}\psi_{i_1}|0\rangle$ which can then be interpreted in terms of S-functions. In this manner, we get

$$\begin{aligned} p_{(n,m)}(x) &= \left(\sum_{k=n+1}^{n+m} - \sum_{k=1}^{m}\right)(-1)^{n+m-k}\ s_{(k-1|n+m-k)}(x) \\ &\quad + \sum_{k=1}^{n}\sum_{l=1}^{m}(-1)^{n+m-k-l}\ s_{(k-1,l-1|n-k,m-l)}(x), \qquad n\geq m\geq 0. \end{aligned}$$

In principle, one could expand the power sum $p_\lambda(x)$ in terms of S-functions by applying the operator $H_{-\lambda_1}H_{-\lambda_2}\cdots$ to the Fock space vacuum. However, it would be easier to use recursively the following result

Lemma 1 *If* $\lambda = \begin{pmatrix} i_1 & \cdots & i_r \\ j_1-1 & \cdots & j_r-1 \end{pmatrix}$, *where* $j_k \geq 1,\ i_k \geq 0$, *then*

$$\begin{aligned} p_n\ s_\lambda &= \sum_{q=1}^{r}\left(s_{\mu_q^+} - (-1)^n\ s_{\nu_q^+}\right) - \sum_{k=0}^{n-1}(-1)^{k-n}\ s_{\sigma_k}, \\ n\frac{\partial}{\partial p_n}\ s_\lambda &= \sum_{q=1}^{r}\left(s_{\mu_q^-} - (-1)^n\ s_{\nu_q^-}\right) - \sum_{p,q}(-1)^{p+q+1+j_q}\delta_{i_p+j_q,n}\ s_{\xi_{pq}}, \end{aligned} \tag{5.12}$$

where

$$\mu_q^\pm = \begin{pmatrix} i_1 & \cdots & i_q\pm n & \cdots & i_r \\ j_1-1 & \cdots & j_q-1 & \cdots & j_r-1 \end{pmatrix},\ \sigma_k = \begin{pmatrix} i_1 & \cdots & i_r & k \\ j_1-1 & \cdots & j_r-1 & n-k-1 \end{pmatrix},$$

and

$$\begin{aligned} \nu_q^\pm &= \begin{pmatrix} i_1 & \cdots & i_q & \cdots & i_r \\ j_1-1 & \cdots & j_q\pm n-1 & \cdots & j_r-1 \end{pmatrix}, \\ \xi_{pq} &= \begin{pmatrix} i_1 & \cdots & \widehat{i_p} & \cdots & i_r \\ j_1-1 & \cdots & \widehat{j_q-1} & \cdots & j_r-1 \end{pmatrix}, \end{aligned}$$

where $\widehat{i_p}$ *means that the label* i_p *is omitted.*

This can be proved by applying the Heisenberg generators $H_{\pm n}$ as given by (5.3) to the S-function given by (5.9) and utilizing formulae such as

$$\psi_j^*\psi_{i_1}\cdots\psi_{i_m}|0\rangle = \sum_{p=1}^{m}(-1)^{p-1}\,\delta_{j,i_p}\;\psi_{i_1}\cdots\widehat{\psi_{i_p}}\cdots\psi_{i_m}|0\rangle \qquad j\geq 0,$$

where $\hat{}$ denotes omission of the relevant object. Note that non-standard partitions $\mu_q^{\pm}$, $\nu_q^{\pm}$ and σ_k will arise in the above expansion. However, these are easily modified to standard partitions using the modification rules outlined on page 13. Equation (5.12) was proven by Macdonald [33, pg. 32] in the form

$$p_n\;s_\lambda = \sum_{\mu}(-1)^{\mathrm{ht}(\mu-\lambda)}\;s_\mu, \tag{5.13}$$

where the sum is over all partitions μ such that the skew diagram $\theta=\mu-\lambda$ is a border strip of length n. By this we mean that θ is a connected skew diagram which contains no 2×2 blocks, the length of θ is $\sum_i\theta_i$, and $\mathrm{ht}(\theta)$ is the one less than the number of rows θ occupies. Hence to calculate $p_n\,s_\lambda$ one can use the algebraic result (5.12), or the combinatorical result (5.13).

We have seen how certain fermionic bilinear expressions, which represent power sums, were able to be decomposed in terms of S-functions via the B-F correspondence. A question which might now be asked is, are there any other bilinear expressions in fermionic modes, like (5.3), which represent known symmetric functions. Let us try a simple generalization of the form

$$H_n(t) = \sum_{i\in\mathbb{Z}} t^{-n-i} : \psi_i\psi_{i+n}^* : . \tag{5.14}$$

These generators fulfill the commutation relations

$$[H_n(t), H_m(t)] = (t^m - t^n)H_{n+m}(t^2) + t\;\frac{t^n-t^{-n}}{t-t^{-1}}\;\delta_{n+m,0},$$

the central term arising due to normal ordering in (5.14). Note that the usual (free fermion) relations are obtained in the limit $t\to 1$. For generic t these generators do not form a closed algebra. The set $\{H_n(-1) : n \text{ even}\}$ form an ordinary Heisenberg algebra while the set $\{H_n(-1) : n \text{ odd}\}$ obey the commutation relations.

$$[H_n(-1), H_m(-1)] = -n\,\delta_{n+m,0}.$$

These operators can be used to generate the *principal* realization [108] of the level one Kac-Moody algebra $\widehat{sl(2)}$, which is generated by $\{h_n, e_n, f_n : n\in\mathbb{Z}\}$, where $e_n = \frac{1}{2}(H_{2n+1}(1) - H_{2n+1}(-1))$, $f_n = \frac{1}{2}(H_{2n-1}(1) + H_{2n-1}(-1))$, and $h_n = H_{2n}(-1)$, $n\neq 0$ ($h_0 = H_0(-1)+\frac{1}{2}$). Indeed, the operators $H_n(t)$ where t is a principal N'th root of unity are the building blocks of the principal realization of $\widehat{gl(N)}$ [106].

The elegant representation of $H_n(1)\equiv H_n$ in terms of power sums and their adjoints unfortunately does not generalize to the $t\neq 1$ case. In fact, if we have the generating function $H(w,t) = \sum_n H_{-n}(t) w^n$ then it follows from the vertex operator realization of the fermionic currents $\psi(z)$ and $\psi^*(z)$ in (5.7), that

$$H(w,t) = \,:\psi(w)\psi^*(tw):\, = Z(w,wt), \tag{5.15}$$

where [113]

$$Z(u,v) = \frac{v}{u-v}\left(\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}p_n(u^n - v^n)\right)\,\exp\left(\sum_{n=1}^{\infty}\frac{\partial}{\partial p_n}(u^{-n} - v^{-n})\right) - 1\right).$$

By comparing coefficients of w in (5.15), we see that

$$\begin{aligned} H_0(t) &= \frac{t^{1-\alpha_0}}{1-t}\sum_{n=1}^{\infty} q_n(x;t)\bar{q}_n(x;t), \\ H_p(t) &= \frac{t^{1-\alpha_0}}{1-t}\sum_{n=0}^{\infty} q_n(x;t)\bar{q}_{n+p}(x;t), \qquad p \neq 0, \end{aligned} \tag{5.16}$$

where $q_n(x;t)$ is an elementary Hall-Littlewood function (whose generating function is given by (2.39)) and the differential operator $\bar{q}_n(x;t)$ has the generating function

$$\sum_{n=1}^{\infty} \bar{q}_n(x;t)\, z^n = \exp\left(\sum_{n=1}^{\infty}(t^{-n}-1)\frac{\partial}{\partial p_n(x)}\, z^n\right).$$

Thus the operators $H_n(t)$ can be realized in terms of fermions, as in (5.14), or in terms of bosons, as in (5.16). Note that the operators given by (5.16) have the correct limits (5.11) as $t \to 1$. Using (5.16) we see that

$$H_{-n}(t)|0\rangle = \frac{t}{1-t}\, q_n(x;t) = t\, P_{(n)}(x;t), \qquad n > 0,$$

so that if we now use the definition (5.14) along with the correspondence (5.9), we obtain another proof (c.f. page 20) of the well-known expression of a one-part Hall-Littlewood function in terms of one hook S-functions

$$P_{(n)}(x;t) = \sum_{k=0}^{n-1}(-t)^k\, s_{(n-1-k|k)}(x), \tag{5.17}$$

To see the action of $H_{-n}(t)H_{-m}(r)$ on the vacuum, we first need the result

$$\begin{aligned} \bar{q}_i(x,t)\, q_j(x;r) &= \sum_{n=0}^{i}[n]_{r,t}\Big\{ q_{j-n}(x;r)\bar{q}_{i-n}(x;t) - (1+\frac{r}{t})\, q_{j-n-1}(x;r)\bar{q}_{i-n-1}(x;t) \\ &\quad + \frac{r}{t}\, q_{j-n-2}(x;r)\bar{q}_{i-n-2}(x;t)\Big\}, \end{aligned}$$

where $[n]_{r,t} = \frac{t^{-n} - tr^{n+1}}{1-tr}$, which is proved using the generating functions for q_n and $\bar{q}_n$. From this, one obtains the following two equivalent expressions for $H_{-n}(t)H_{-m}(r)|0\rangle$

$$
\begin{aligned}
&\left(tr\sum_{i=0}^{m}[i]_{r,t} - r(t+r)\sum_{i=1}^{m}[i-1]_{r,t} + r^2\sum_{i=2}^{m}[i-2]_{r,t}\right) P_{(i+n)}(x;t)\, P_{(m-i)}(x;r) = \\
&\sum_{k=0}^{m-1} t^{n+m-k}(-r)^{m-k} s_{(k|n+m-k-1)}(x) - \sum_{k=0}^{m-1} t^k(-r)^{m-k} s_{(n+k|m-k-1)}(x) \\
&+\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}(-t)^{n-k}(-r)^{m-l}\, s_{(k,l|n-k-1,m-l-1)}(x),
\end{aligned}
$$

which is an unusual identity involving elementary Hall-Littlewood and S-functions. If one were to consider a general string of operators $H_{-n_1}(t_1)\cdots H_{-n_k}(t_k)|0\rangle$ one would obtain a relation between k different elementary Hall-Littlewood functions $P_n(x;t_i)$ $i = 1,\ldots,k$ and hook S-functions of Frobenius rank $1, 2, \ldots, k$.

5.2 The boson–fermion correspondence for Q-functions

In addition to the (classical) B-F correspondence between free fermions and bosons, there is a B-F correspondence between neutral free fermions and bosons. Again, the bosons can be realized on the space of symmetric functions. In this case however, the states created out of the neutral free fermions have a nice representation, via this correspondence, in terms of Q-functions. The neutral free fermions ϕ_i, $i \in \mathbb{Z}$ are defined in terms of free fermions, by

$$\phi_j = \frac{\psi_j + (-1)^j\psi^*_{-j}}{\sqrt{2}},$$

so that they satisfy the anti-commutation relations

$$\{\phi_i, \phi_j\} = (-1)^i\delta_{i+j,0}. \tag{5.18}$$

There is a vacuum $|0\rangle$ defined by $\phi_i|0\rangle = 0$ for $i < 0$. Let

$$G_n = \frac{1}{2}\sum_{i\in\mathbb{Z}}(-1)^{i-1} :\phi_i\phi_{-i-n}:, \tag{5.19}$$

with normal ordering defined as for free fermions. That is,

$$:\phi_i\phi_j: = \left\{\begin{array}{cl} \phi_i\phi_j & \text{if } j<0 \\ -\phi_j\phi_i & \text{if } j>0 \\ (1-\delta_{i,0})\phi_i\phi_0 & \text{if } j=0 \end{array}\right. .$$

Then the generators $\{G_n : n \in 2\mathbb{Z}+1\}$ generate a Heisenberg algebra

$$[G_n, G_m] = \frac{n}{2}\,\delta_{n+m,0}.$$

Again, if one lets

$$X(z) = \frac{1}{\sqrt{2}} \exp\left(2 \sum_{n\, odd} \frac{p_n}{n} z^n\right) \exp\left(- \sum_{n\, odd} \frac{\partial}{\partial p_n} z^{-n}\right) = \sum_{j\in\mathbb{Z}} X_j z^j,$$

then there is an isomorphism $\phi_j \leftrightarrow X_j$. Moreover, there is an isomorphism of the states [117, 219]

$$\phi_{\lambda_1} \cdots \phi_{\lambda_p}|0\rangle \longleftrightarrow \begin{cases} \langle e^{H(x)} \phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_p}\rangle = 2^{-p/2}\, Q_\lambda(x) & \text{if } p \text{ is even} \\ \langle e^{H(x)} \phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_p}\phi_0\rangle = 2^{-(p+1)/2}\, Q_\lambda(x) & \text{if } p \text{ is odd} \end{cases},$$

where $G(x) = \sum_{n\, odd} \frac{2}{n} p_n(x) G_n$. In particular the state $\phi_m\phi_n|0\rangle \leftrightarrow \frac{1}{2} Q_{(m,n)}(x)$. Under this isomorphism, the Heisenberg generators have the realization

$$G_{-n} \longleftrightarrow p_n(x), \qquad G_n \longleftrightarrow \frac{n}{2}\frac{\partial}{\partial p_n(x)}, \qquad n > 0 \qquad n \text{ odd}. \tag{5.20}$$

As in the S-function case, there is a relation between (odd) power sums and Q-functions, which is derived by considering the two different ways of writing $G_{-2k-1}|0\rangle$ using the fermionic realization (5.19) and the bosonic one (5.20). Doing this, we arrive the decomposition of odd power sums in terms of two-part Q-functions

$$p_{2k+1}(x) = \frac{1}{2}\sum_{j=0}^{k} (-1)^j\, Q_{(2k+1-j,j)}(x).$$

Note that it is only possible to express the odd power sums in terms of Q-functions, because the space spanned by the Q-functions is isomorphic to $\mathbb{Q}[p_1, p_3, p_5, \ldots]$. Similarly one can consider $G_{-2m-1}G_{-2n-1}|0\rangle$ which results in

$$\begin{aligned} p_{(2m+1,2n+1)}(x) &= \frac{1}{2}\sum_{j=0}^{n}(-1)^j Q_{(2m+2n+2-j,j)}(x) - \frac{1}{2}\sum_{j=1}^{n}(-1)^j Q_{(2m+1+j,2n+1-j)}(x) \\ &+ \frac{1}{4}\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j} Q_{(2m+1-i,i,2n+1-j,j)}(x). \end{aligned}$$

Again, we can recursively decompose $p_\lambda(x)$ with each λ_i odd, with respect to Q-functions, by using

Lemma 2

$$\begin{aligned} p_{2k+1}\, Q_{(\lambda_1,\ldots,\lambda_n)} &= \sum_{j=1}^{n} Q_{(\lambda_1,\ldots,\lambda_j+2k+1,\ldots,\lambda_n)} + \frac{1}{2}\sum_{i=0}^{k}(-1)^i\, Q_{(\lambda_1,\ldots,\lambda_n,2k+1-i,i)}, \qquad (5.21) \\ \frac{n}{2}\frac{\partial}{\partial p_{2k+1}} Q_{(\lambda_1,\ldots,\lambda_n)} &= 2\sum_{p,q}(-1)^{p+q}\delta_{2k+1,\lambda_p+\lambda_q}\, Q_{(\lambda_1,\ldots,\widehat{\lambda_p},\ldots,\widehat{\lambda_q},\ldots,\lambda_n)} \\ &\quad + \sum_{j=1}^{n} Q_{(\lambda_1,\ldots,\lambda_j-2k-1,\ldots,\lambda_n)}, \end{aligned}$$

which can be proved using the anti-commutation relations of the neutral free fermions (5.18). Once again, the partitions in the above expression are non-standard and are changed to standard ones by noting that the interchange of any two consecutive partition labels introduces a minus sign in front of the Q-function. As in the S-function case, there is a combinatorial version of (5.21). We refer the reader to [197, 224] for the details.

Example
Using the elementary Q-functions

$$q_1 = 2p_1, \qquad q_2 = 2p_1^2, \qquad q_3 = \tfrac{4}{3}p_1^3 + \tfrac{2}{3}p_3, \qquad q_4 = \tfrac{2}{3}p_1^4 + \tfrac{4}{3}p_3p_1,$$
$$q_5 = \tfrac{4}{15}p_1^5 + \tfrac{4}{3}p_3p_1^2 + \tfrac{2}{5}p_5, \qquad q_6 = \tfrac{4}{45}p_1^6 + \tfrac{8}{9}p_3p_1^3 + \tfrac{2}{9}p_3^2 + \tfrac{4}{5}p_5p_1,$$

we have

$$p_3Q_{(2,1)} = \tfrac{4}{3}(p_3p_1^3 - p_3^2) = Q_{(5,1)} - Q_{(4,2)} + \tfrac{1}{2}Q_{(3,2,1)}.$$

5.3 Littlewood-Richardson rule

As another application of the boson-fermion correspondence, we shall now show how it can be utilized to derive fairly explicit formulae for the multiplication of two S-functions in terms of S-functions with non-standard partitions. Let us begin with the Pieri formula for the multiplication of an S-function by a complete symmetric function, which takes the form [33]

$$h_n \; s_\mu = \sum_\lambda s_\lambda, \tag{5.22}$$

where the sum is over all partitions λ such that $\lambda - \mu$ is a horizontal n-strip. That is, the partitions λ occuring in the above product are those obtained by adding n extra boxes to the diagram μ in any manner *provided* that the resulting diagram is a valid diagram, *and* no two of the added boxes lie in the same column.

The question we can ask ourselves is: how can we turn multiplication by h_n into an operation involving free fermions? The answer comes from the generating function for h_n, which we can write as

$$R(z) = \sum_{p=0}^{\infty} h_p \, z^p = \exp\left(\sum_{n\geq 1} \frac{p_n}{n} \, z^n\right) = \psi(z)\eta(z),$$

where

$$\eta(z) = \exp\left(\sum_{n\geq 1} \frac{\partial}{\partial p_n} \, z^{-n}\right) z^{-\alpha_0} e^{-iq}.$$

Thus, when we multiply an S-function, represented by a product of free fermionic currents, by the function h_n, represented by the current $R(z)$, we can shuffle the (annihilation) operator $\eta(z)$ through the currents $\psi(w)$ using the relation

$$\eta(z)\psi(w) = \left(\frac{w/z}{1 - w/z}\right) \psi(w)\eta(z), \tag{5.23}$$

which will then hit the vacuum, leaving us with an expression involving just free fermions. As an example, let us look at the product $h_k\, s_{(n,m)}$. We know that $s_{(n,m)} = \varrho(\psi_{n+1}\psi_m|0\rangle)$, so that (dropping the $\varrho(\cdot)$ for simplicity)

$$
\begin{aligned}
h_k\, s_{(n,m)} &= \frac{1}{2\pi i}\oint \frac{dzdw_1dw_2}{zw_1w_2} z^{-k} w_1^{-n-1} w_2^{-m} R(z)\psi(w_1)\psi(w_2)|0\rangle \\
&= \frac{1}{2\pi i}\oint \frac{dzdw_1dw_2}{zw_1w_2} z^{1-k} w_1^{-n-1} w_2^{-m} \frac{w_1/z}{1-w_1/z}\frac{w_2/z}{1-w_2/z}\psi(z)\psi(w_1)\psi(w_2)e^{-iq}|0\rangle \\
&= \sum_{i,j\geq 1} \psi_{k-1+i+j}\psi_{n+1-i}\psi_{m-j}e^{-iq}|0\rangle\,,
\end{aligned}
$$

where the contours in the above integrals circle the origin. The upper limits of this last sum are constrained by the fact that $e^{-iq}|0\rangle = \psi^*_{-1}|0\rangle$, so that $\psi_p e^{-iq}|0\rangle = 0$ if $p \leq -1$. Thus we finally obtain

$$
h_k\, s_{(n,m)} = \sum_{i=-1}^{n}\sum_{j=0}^{m} s_{(n+m+k-i-j,i,j)}, \tag{5.24}
$$

where the expression on the right involves non-standard S-functions which can be turned into standard S-functions using the modification rules described in Chapter 2. For k smaller than m or n, many terms on the right hand side of (5.24) cancel amongst themselves, so that in this particular case, it is not a very efficient formula. However we shall soon derive a formula which is more efficient when k is small. This method can be extended to the general case, given by the combinatorial expression (5.22), with the result being

$$
h_k\ s_{(n_1,\ldots,n_p)} = \sum_{i_1=1}^{n_1+p}\ \sum_{i_2=1}^{n_2+p-1}\cdots\sum_{i_p=1}^{n_p+1} s_{(k-p+i_1+\cdots+i_p,n_1+1-i_1,\ldots,n_p+1-i_p)}. \tag{5.25}
$$

In the case of the product of a complete symmetric function and a one-hook S-function, we can use the exchange relation

$$
\eta(z)\psi^*(w) = \left(\frac{z}{w}-1\right)\psi^*(w)\eta(z),
$$

to show that

$$
h_n\, s_{(a|b-1)} = s_{(a+n|b-1)} + s_{(n+a-1|b)} + \sum_{i=1}^{a}\left(s_{(n+i-1,a-i|b-1,0)} + s_{(n+i-2,a-i|b,0)}\right). \tag{5.26}
$$

For $n \geq a$ the partitions in the above expression are standard, and the result could also easily have been derived by adding n boxes to the hook diagram $(a|b)$ in the prescribed manner. When $n < a$, the terms on the right start cancelling each other out. Again, we will be able to derive a more efficient expression for this case.

We would now like to start multiplying S-functions on the left by the two-part S-function $s_{(n,m)}$. The generating function for these S-functions takes the form

$$
R(z_1,z_2) = \sum_{n,m\geq 0} s_{(n,m)}\, z_1^n z_2^m = \left(1-\frac{z_2}{z_1}\right)\exp\left(\sum_{k\geq 1}\frac{p_k}{k}(z_1^k+z_2^k)\right).
$$

Again, we can decompose this into free fermionic currents and an annihilation operator. Indeed $R(z_1,z_2)=\psi(z_1)\psi(z_2)\eta(z_1,z_2)$, where

$$\eta(z_1,z_2)=\exp\left(\sum_{n\geq 1}\frac{\partial}{\partial p_n}(z_1^{-n}+z_2^{-n})\right)z_1^{-\alpha_0-1}z_2^{-\alpha_0}e^{-2iq}.$$

Using the exchange relation

$$\eta(z_1,z_2)\psi(w)=\left(\frac{w/z_1}{1-w/z_1}\right)\left(\frac{w/z_2}{1-w/z_2}\right)\ \psi(w)\eta(z_1,z_2),$$

we can follow the previous example and show that

$$s_{(n,m)}\ h_k=\sum_{i=1}^{k+1}\sum_{j=1}^{k+2-i}s_{(n+i-1,m+j-1,k+2-i-j)},\tag{5.27}$$

which is a more efficient expression than (5.24) for small k. In a similar manner, one can derive the result

$$s_{(n,m)}s_{(p,q)}=\sum_{i_1=1}^{p+2}\sum_{i_2=1}^{p+3-i_1}\sum_{i_3=1}^{q+1}\sum_{i_4=1}^{q+2-i_3}s_{(i_1+i_3+n-2,i_2+i_4+m-2,p+2-i_1-i_2,q+2-i_3-i_4)}.\tag{5.28}$$

Generally, we can consider multiplication on the left by the general S-function $s_{(n_1,\ldots,n_p)}$ through use of the generating function

$$R(z_1,\ldots,z_p)=\sum_{n_1,\ldots,n_p\geq 0}s_{(n_1,\ldots,n_p)}\,z_1^{n_1}\cdots z_p^{n_p}=\prod_{i<j}\left(1-\frac{z_j}{z_i}\right)\exp\left(\sum_{k\geq 1}\frac{p_k}{k}(z_1^k+\cdots z_p^k)\right).$$

This allows us to write $R(z_1,\ldots,z_p)=\psi(z_1)\cdots\psi(z_p)\eta(z_1,\ldots,z_p)$ where

$$\eta(z_1,\ldots,z_p)=\exp\left(\sum_{n\geq 1}\frac{\partial}{\partial p_n}(z_1^{-n}+\cdots z_p^{-n})\right)z_1^{-\alpha_0-p+1}\cdots z_p^{-\alpha_0}e^{-piq}.$$

Thus by using the relation

$$\eta(z_1,\ldots,z_p)\psi(w)=\prod_{j=1}^{p}\left(\frac{w/z_j}{1-w/z_j}\right)\psi(w)\eta(z_1,\ldots,z_p),$$

we obtain the result

$$s_{(n_1,\ldots,n_p)}h_k=\sum_{i_1=1}^{k+1}\sum_{i_2=1}^{k+2-i_1}\cdots\sum_{i_p=1}^{k+p-i_1-\cdots-i_{p-1}}s_{(i_1+n_1-1,i_2+n_2-1,\ldots,i_p+n_p-1,k+p-i_1-\cdots-i_p)}\tag{5.29}$$

Let us now turn our attention to multiplication on the left by one-hook S-functions. They have the generating function

$$H(z;w)=\sum_{n,m\geq 0}s_{(n|m-1)}\,z^n(-w)^m=\frac{1}{1-z/w}\ \exp\left(\sum_{k\geq 1}\frac{p_k}{k}(z^k-w^k)\right).\tag{5.30}$$

That is, $H(z;w) = \psi^*(w)\psi(z)\tau(z;w)$ where

$$\tau(z;w) = \left(\frac{w}{z}\right)^{\alpha_0} \exp\left(\sum_{k\geq 1} \frac{\partial}{\partial p_k}(z^{-k} - w^{-k})\right).$$

Upon using

$$\tau(z;w)\psi(y) = \left(\frac{w-y}{z-y}\right)\psi(y)\tau(z;w), \tag{5.31}$$

we see that

$$s_{(n|m-1)}h_p = \sum_{i=0}^{p} s_{(n+1+i,p+1-i,1^{m-2})} + \sum_{i=0}^{p-1} s_{(n+1+i,p-i,1^{m-1})}, \tag{5.32}$$

which is a more efficient version of (5.26) in the case when p is small.

Note that until now, we have not found it necessary to use the anti-commutation relations for free fermions. As a final example of the method described above, and one in which the anti-commutation relations are needed, let us consider the product of two one-hook S-functions. By using the generating function (5.30), along with the exchange relation (5.31) and

$$\tau(z;w)\psi^*(y) = \left(\frac{z-y}{w-y}\right)\psi^*(y)\tau(z;w), \tag{5.33}$$

we see that

$$\begin{aligned}(-1)^{m+q}s_{(n|m-1)}s_{(p|q-1)} = \sum_{i,j\geq 0}\Big(&\psi^*_{-m-i}\psi_{n+j}\psi^*_{-q+i}\psi_{p-j} - \psi^*_{-m-i}\psi_{n+j+1}\psi^*_{-q+i+1}\psi_{p-j} \\ &-\psi^*_{-m-i-1}\psi_{n+j}\psi^*_{-q+i}\psi_{p-j-1} + \psi^*_{-m-i-1}\psi_{n+j+1}\psi^*_{-q+i+1}\psi_{p-j-1}\Big)|0\rangle .\end{aligned}$$

Consider for a moment, the first term in the above expression. For it to be non-zero, we require $0 \leq j \leq q$. We also require that *either* $0 \leq i \leq q-1$ *or* $i = p+q-j$. In this latter case, we must use relation $\{\psi^*_i, \psi_i\} = 1$ to get rid of the annihilation operator ψ^*_{p-j}, yielding the term $\sum_{j=0}^{p}\psi^*_{-m-p-q+j}\psi_{n+j}|0\rangle$. After treating the other terms in a similar fashion, and gathering like terms, we end up with the result

$$\begin{aligned}s_{(n|m-1)}s_{(p|q-1)} \quad = \quad & s_{(n+p,m+q-1)} + s_{(n+p+1,m+q-2)} + \sum_{i=1}^{q-1} s_{(n,p|m+i-1,q-i-1)} \\ + \quad & \sum_{i=1}^{p-1} s_{(n+i+1,p-i-1|m-1,q-1)} + \sum_{j=0}^{p}\sum_{i=0}^{q-2} s_{(n+j+1,p-j|m+i-1,q-i-2)} \\ + \quad & \sum_{j=0}^{p-1}\sum_{i=0}^{q-1} s_{(n+j,p-j-1|m+i,q-i-1)} + 2\sum_{j=0}^{p-1}\sum_{i=0}^{q-2} s_{(n+j+1,p-j-1|m+i,q-i-2)}.\end{aligned} \tag{5.34}$$

Let us remark that instead of considering S-function multiplication, we can consider S-function division (i.e. skewing) and derive similar formulae by consider the generating functions for $D(s_\mu)$. These generating functions will be purely functions of $\frac{\partial}{\partial p_n}$ and hence can be applied directly to the generating function for s_λ, crunched together using the standard rules yielding a formula for the skew function $s_{\lambda/\mu}$ in terms of non-standard S-functions, which can then be converted into standard ones using the modification rules.

5.4 S-function plethysms

In this section, we shall be interested in a particular operation among S functions, that of (outer) *plethysm*. Littlewood's original definition [195] was couched in terms of invariant matrices of elements of the group $GL(n)$, and there is a large amount of literature (see for example [225–230]) devoted to the problem of their computation. There is an alternative definition [33] involving a particular substitution process which we now describe. Given a symmetric function $f(x) = \sum_\alpha c_\alpha x^\alpha$ expressed in terms of monomials $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ define new variables $\{y_1, y_2, \ldots\}$ by

$$\prod_i (1 + y_i z) = \prod_\alpha (1 + x^\alpha z)^{c_\alpha}. \tag{5.35}$$

This allows one to define the plethysm of two symmetric functions f, $g \in \Lambda(x)$ by

$$(f \otimes g)(x) = g(y_1, y_2, \ldots).$$

This operation is right-distributive, but not left-distributive [195]:

$$(f + g) \otimes s_\lambda = \sum_\rho (f \otimes s_{\lambda/\rho})(g \otimes s_\rho). \tag{5.36}$$

An important result in the theory of plethysms is [33]

$$s_\lambda \otimes s_\mu = \sum_\rho a^\rho_{\lambda\mu} s_\rho, \tag{5.37}$$

where $a^\rho_{\lambda\mu}$ are non-negative integers, and the sum is over all partitions of weight $|\rho| = |\lambda| \cdot |\mu|$.

There is a standard method for recursively computing plethysms based on the identity [195]

$$\sum_{n=0}^{|\lambda||\mu|} D_{(n)}(s_\lambda \otimes s_\mu) = \left(\sum_{m=0}^{|\mu|} s_{\mu/(m)} \right) \otimes s_\lambda, \tag{5.38}$$

where D_λ is the adjoint (skew) operator $D_\lambda s_\sigma = s_{\sigma/\lambda}$. Although this method can sometimes be ambiguous, Butler and King [231] used (5.38) to develop an unambiguous method for calculating these plethysms. While this procedure is useful for calculating plethysms of S-functions of small weight, it is not very practical for larger weights. It is our intention here to outline another way of calculating plethysms which allow us to obtain some quite general results.

By taking the logarithm of (5.35) it follows that $p_n(y) = f(x^n)$. Thus to calculate the plethysm $s_\lambda \otimes s_\mu$, one expresses $s_\mu(x)$ as a multinomial in the power sums $p_1(x)$, $p_2(x), \ldots$ and then makes the substitution $p_j(x) \to s_\lambda(x^j)$. Thus a knowledge of how to express $s_\lambda(x^j)$ in terms on S-functions with argument x, along with the Littlewood–Richardson rule for multiplying the S-functions together is sufficient (in theory) to be able to calculate any plethysm.

We will soon see how the B-F correspondence and its relation to S-functions will enable us to derive an algorithm for calculating plethysms of the type $s_\lambda \otimes$

$p_r = s_\lambda(x_1^r, x_2^r, \ldots)$, which, as explained above, allows one to calculate more general plethysms. The use of vertex operators in investigating S-function plethysms is not new: various stability properties of the plethysm operation have been studied [232] in this way. The emphasis in our approach however, is on the explicit calculation of plethysms.

Let us examine, as a warm-up, expansions of the functions $h_n(x^2)$ and $e_n(x^2)$ in terms of functions of the argument x . By examining the generating function for complete symmetric functions, given by (2.4) and (2.7), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n(x^2) z^{2n} &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_{2n}(x) z^{2n}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{1+(-1)^n}{n} p_n(x) z^n\right) \\
&= \left(\sum_{k=0}^{\infty} h_k(x) z^k\right)\left(\sum_{l=0}^{\infty} (-1)^l h_l(x) z^l\right),
\end{aligned}$$

so that we obtain

$$h_n(x^2) = \sum_{k=0}^{2n} (-1)^k h_k(x) h_{2n-k}(x). \tag{5.39}$$

The Littlewood-Richardson rule between two complete symmetric functions provides us with

$$h_k h_{2n-k} = \sum_{j=0}^{k} s_{(2n-j,j)}. \tag{5.40}$$

Substituting (5.40) into (5.39) and rearranging the sum, we obtain the result [225]

$$h_n(x^2) = \sum_{j=0}^{n} (-1)^j \, s_{(2n-j,j)}(x). \tag{5.41}$$

Similarly, considering the generating function for the elementary symmetric functions (2.3) and repeating the above process, we get

$$e_n(x^2) = \sum_{j=0}^{n} (-1)^{j+n} \, s_{(2^j, 1^{2n-2j})}(x). \tag{5.42}$$

So, by examining the generating functions for $h_n(x) \equiv s_{(n)}(x)$ and $e_n(x) \equiv s_{(1^n)}(x)$ we have been able to determine expansions of the functions $s_{(n)}(x^2)$ and $s_{(1^n)}(x^2)$ in terms of S-functions. As shown in Appendix B the functions $s_{(n)}(x^2)$ can also be expanded in terms of elementary Q-functions and replicated S-functions.

How do we go about treating S-functions associated with more general partitions ? As we saw in section 5.3, the creation part of a product of vertex operators of the form (5.7) is a multilinear generating function for S-functions associated with partitions of a particular shape. Hence it might be possible to modify these generating functions (vertex operators) to extract the requisite information. This is indeed the case as we shall now see.

5.4.1 The case $r = 2$

Let us examine the case $r = 2$ in some detail, as it is this case which yields the most explicit results. Define the following vertex operators

$$
\begin{aligned}
\Psi(z) &= \exp\left(2{\sum_n}' \frac{p_n(x)}{n} z^n\right) \exp\left(-{\sum_n}' \frac{\partial}{\partial p_n(x)} z^{-n}\right) e^{iq} z^{2\alpha_0}, \\
\Psi^*(z) &= \exp\left(-2{\sum_n}' \frac{p_n(x)}{n} z^n\right) \exp\left({\sum_n}' \frac{\partial}{\partial p_n(x)} z^{-n}\right) z^{-2\alpha_0} e^{-iq},
\end{aligned} \tag{5.43}
$$

where ${\sum_n}' \equiv \sum_{n\, even}$. Note that the vertex operators (5.43) are just the operators (5.7) with $x \to x^2$ and $z \to z^2$. Thus if we write $\Psi(z) = \sum_{n\in\mathbb{Z}} \Psi_n z^{2n}$, $\Psi^*(z) = \sum_{n\in\mathbb{Z}} \Psi^*_n z^{-2n}$, then

$$
\varrho\left(\Psi^*_{-j_1} \cdots \Psi^*_{-j_r} \Psi_{i_s} \cdots \Psi_{i_1} |\, 0\rangle\right) = (-1)^{j_1+\cdots j_r + l(l-1)/2}\; s_\lambda(x^2),
$$

where λ is given by (5.10). Thus if we could express the vertex operators $\Psi(z)$, $\Psi^*(z)$ in terms of the the operators $\psi(z)$, $\psi^*(z)$ we could use the B-F correspondence to write $s_\lambda(x^2)$ in terms of functions $s_\mu(x)$. To this end, define

$$
\widehat{\psi}(z) = \psi(z)\psi(-z), \qquad \widehat{\psi^*}(z) = \psi^*(z)\psi^*(-z). \tag{5.44}
$$

Using (5.7) we can write these out as vertex operators of the form

$$
\begin{aligned}
\widehat{\psi}(z) &= 2\ \exp\left(2{\sum_n}' \frac{p_n(x)}{n} z^n\right) \exp\left(-2{\sum_n}' \frac{\partial}{\partial p_n(x)} z^{-n}\right) e^{2iq}(-1)^{\alpha_0} z^{2\alpha_0+1}, \\
\widehat{\psi^*}(z) &= 2\ \exp\left(-2{\sum_n}' \frac{p_n(x)}{n} z^n\right) \exp\left(2{\sum_n}' \frac{\partial}{\partial p_n(x)} z^{-n}\right) (-1)^{\alpha_0-1} z^{-2\alpha_0-1} e^{-2iq}.
\end{aligned}
$$

By comparison with (5.43) we see that

$$
\Psi(z) = \widehat{\psi}(z)\xi(z), \qquad \Psi^*(z) = \widehat{\psi^*}(z)\xi^*(z), \tag{5.45}
$$

where

$$
\begin{aligned}
\xi(z) &= \frac{1}{2}\, z\, e^{-iq}(-1)^{\alpha_0-1} \exp\left({\sum_n}' \frac{\partial}{\partial p_n(x)} z^{-n}\right), \\
\xi^*(z) &= (-1)^{\alpha_0+1} z e^{iq} \exp\left(-{\sum_n}' \frac{\partial}{\partial p_n(x)} z^{-n}\right).
\end{aligned} \tag{5.46}
$$

Now note the crucial fact that the operators $\xi(z)$ and $\xi^*(z)$ are functions only of the operators $\partial/\partial p_n(x)$, acting on the space $\Lambda(x)$. Thus if we are given a string of vertex operators $\Psi(z)$ or $\Psi^*(z)$ acting on the vacuum, we can write them out according to (5.45) and then shuffle the operators $\xi(z)$ and $\xi^*(z)$ to the right (picking up various factors along the way) which will then disappear into the vacuum. This will leave us with a generating-type function solely composed of the currents $\psi(z)$, $\psi(-z)$, $\psi^*(z)$ and $\psi^*(-z)$ from which, one can extract S-functions of the argument x. Let us now examine the details of this procedure.

Using (5.44) we can expand $\widehat{\psi}(z)$ as a power series $\widehat{\psi}(z) = \sum_{n\in\mathbb{Z}} \widehat{\psi}_n\, z^{2n-1}$, where

$$\widehat{\psi}_n = \sum_{j\in\mathbb{Z}}(-1)^j \psi_{2n-1-j}\psi_j = 2\sum_{j\leq n-1}(-1)^j \psi_{2n-1-j}\psi_j. \tag{5.47}$$

If we now use the fact that $h_n(x^2) = \Psi_n|0\rangle$, we have

$$\begin{aligned} h_n(x^2) &= \frac{1}{2\pi i}\oint \frac{dz}{z} z^{-2n}\Psi(z)|0\rangle = -\frac{1}{4\pi i}\oint \frac{dz}{z} z^{-2n+1}\widehat{\psi}(z)\, e^{-iq}|0\rangle \\ &= -\frac{1}{2}\widehat{\psi}_n e^{iq}|0\rangle\,, \end{aligned}$$

where the contour in the above integrals encircles the origin. However, noting that $\psi_n\, e^{ikq}|0\rangle = 0$ for $n \leq k$, we have, using (5.47)

$$h_n(x^2) = -\sum_{j=-1}^{n-1}(-1)^j \psi_{2n-1-j}\psi_j e^{-iq}|0\rangle\,.$$

Upon using the fact that $\psi_i\psi_j e^{kiq}|0\rangle = s_{(i-k-1,j-k)}(x)$, we recover (5.41). In a similar manner we can consider $s_{(n-1,m)}(x^2) = \Psi_n\Psi_m|0\rangle$, and get

$$\begin{aligned} \Psi_n\Psi_m|0\rangle &= \frac{1}{(2\pi i)^2}\oint \frac{dz}{z} z_1^{-2n} z_2^{-2m}\widehat{\psi}(z_1)\xi(z_1)\widehat{\psi}(z_2)\xi(z_2)|0\rangle \\ &= \frac{1}{4}\sum_{p=-1}^{m-1}\widehat{\psi}_{n+m-1-p}\widehat{\psi}_p\, e^{-2iq}|0\rangle\,, \end{aligned}$$

where we have used the property $\xi(z)\widehat{\psi}(w) = -w^2(1-w^2/z^2)^{-1}\widehat{\psi}(w)\xi(z)$. Hence

$$s_{(n-1,m)}(x^2) = \sum_{p=0}^{m}\sum_{j=-2}^{n+p-1}\sum_{k=-2}^{m-2-p}(-1)^{j+k}s_{(k-1,2m-2p-3-k,j+1,2n+2p+1-j)}(x). \tag{5.48}$$

As for the results of section 5.3, the partitions that occur in the right hand side of (5.48) may be non-standard, and hence must be modified using the standard rules. By carefully examining the summation occuring in (5.48), and using the modification rules for S-functions, we can write out the cases $m = 1$ and $m = 2$ explicitly, with the results

$$\begin{aligned} s_{(n-1,1)}(x^2) &= \sum_{j=2}^{n-1}(-1)^j s_{(2n-2-j,j,2)}(x) - \sum_{j=1}^{n-1}(-1)^j s_{(2n-2-j,j,1^2)}(x) \\ &+ \sum_{j=2}^{n}(-1)^j s_{(2n-j,j)}(x) - s_{(2n-2,1^2)}(x), \end{aligned} \tag{5.49}$$

$$\begin{aligned} s_{(n-2,2)}(x^2) &= \sum_{j=4}^{n-2}(-1)^j s_{(2n-4-j,j,4)}(x) - \sum_{j=3}^{n-2}(-1)^j s_{(2n-4-j,j,3,1)}(x) - s_{(2n-4,3,1)}(x) \\ &+ \sum_{j=2}^{n-2}(-1)^j s_{(2n-4-j,j,2,2)}(x) + \sum_{j=2}^{n-1}(-1)^j s_{(2n-2-j,j,2)}(x) + s_{(2n-5,3,2)}(x) \\ &- \sum_{j=3}^{n-1}(-1)^j s_{(2n-2-j,j,1^2)}(x)\sum_{j=4}^{n}(-1)^j s_{(2n-j,j)}(x) - s_{(2n-6,3,3)}(x) \\ &+ s_{(2n-5,2^2,1)}(x). \end{aligned} \tag{5.50}$$

In the above expressions, the partitions are in standard form provided $n \geq 2$ and $n \geq 4$ respectively.

For the general case we have

$$\begin{aligned} s_{(n_1-p+1,n_2-p+2,\ldots,n_{p-1}-1,n_p)}(x^2) &= \Psi_{n_1}\Psi_{n_2}\cdots\Psi_{n_p}|0\rangle \\ &= \frac{(-1)^p}{(2\pi i)^p}\sum_{k_1,\ldots,k_p}\oint\frac{dz}{z}z_1^{2(k_1-n_1)}z_2^{2(k_2-n_2+1)}\cdots z_p^{2(k_p-n_p+p-1)}\times \\ &\qquad\qquad \times\prod_{i<j}\left(1-\frac{z_j^2}{z_i^2}\right)\widehat{\psi}_{k_1}\cdots\widehat{\psi}_{k_p}\,e^{-piq}|0\rangle \\ &= (-1)^p\prod_{i<j}(1-R_{ij})^{-1}\,\widehat{\psi}_{n_1}\widehat{\psi}_{n_2-1}\cdots\widehat{\psi}_{n_p-p+1}\,e^{-piq}|0\rangle\,, \end{aligned} \tag{5.51}$$

where R_{ij} acts as a raising operator:

$$R_{ij}\widehat{\psi}_{\lambda_1}\cdots\widehat{\psi}_{\lambda_i}\cdots\widehat{\psi}_{\lambda_j}\cdots\widehat{\psi}_{\lambda_p} = \widehat{\psi}_{\lambda_1}\cdots\widehat{\psi}_{\lambda_i+1}\cdots\widehat{\psi}_{\lambda_j-1}\cdots\widehat{\psi}_{\lambda_p}.$$

One can now use (5.47) to rewrite the terms occuring in (5.51) in terms of free fermions, which can then be turned into S-functions of argument x. Unfortunately, it is not worthwhile writing down explicit expressions like (5.48) for partitions of length greater than two.

There is however, a neat formula expressing one-hook S-functions with argument x^2 in terms of one and two-hook S-functions of argument x. Recall that $\Psi^*_{-j}\Psi_i|0\rangle = (-1)^j s_{(i|j-1)}(x^2)$. By using (5.45) and the relation $\xi^*(z)\widehat{\psi}(w) = (z^{-2}-w^{-2})\widehat{\psi}(w)\xi^*(z)$, we have $(-1)^j s_{(i|j-1)}(x^2) = \frac{1}{4}(\widehat{\psi^*}_{-j}\widehat{\psi}_i - \widehat{\psi^*}_{-j+1}\widehat{\psi}_{i+1})|0\rangle$. Expressing these back in terms of free fermions, we obtain after some algebra

$$\begin{aligned} (-1)^j s_{(i|j-1)}(x^2) &= \sum_{p=1}^{j}\sum_{q=0}^{i-1}(-1)^{p+q+1}s_{(q,2i-1-q|2j-p,p-1)}(x) + s_{(2i|2j-1)}(x) \\ &+ \sum_{p=1}^{j-1}\sum_{q=0}^{i}(-1)^{p+q}s_{(q,2i+1-q|2j-2-p,p-1)}(x) - s_{(2i+1|2j-2)}(x). \end{aligned} \tag{5.52}$$

Note that all the partitions occuring in the above equation are standard, and no modification is necessary. The extension of this method to p-hook S-functions is now obvious although again, it is not worthwhile to write down any explicit cases.

5.4.2 The general case

Let us now extend the above construction to enable us to calculate $s_\lambda(x^r)$ for generic (integral) values of r. Let $\sum_n{}'' \equiv \sum_{n\equiv 0\ (\mathrm{mod}\ r)}$, and define the vertex operator

$$\Psi(z) = \exp\left(r\sum_n{}''\frac{p_n(x)}{n}\,z^n\right)\exp\left(-\sum_n{}''\frac{\partial}{\partial p_n(x)}\,z^{-n}\right)\;e^{iq}z^{r\alpha_0}.$$

with a similar expression for $\Psi^*(z)$ which we will not be using. If we express this in terms of its modes by $\Psi(z) = \sum_{n\in\mathbb{Z}} \Psi_n z^{rn}$, then the B-F correspondence tells us that

$$\varrho\left(\Psi_{i_1}\cdots\Psi_{i_s}|0\rangle\right) = s_\lambda(x^r), \tag{5.53}$$

where $\lambda = (i_1 - s + 1, i_2 - s + 2, \ldots, i_s)$. Note that in this expression, we have reordered the modes to absorb the factor of $(-1)^{i_1+\cdots i_s}$ occuring in (5.9). This is possible because the modes Ψ_n and Ψ_m^* still obey the anti-commutation relations for free fermions.

Let ω be a primitive r'th root of unity. That is, $\omega^r = 1$ and

$$1 + \omega^k + \omega^{2k} + \cdots + \omega^{(r-1)k} = \begin{cases} r, & k \equiv 0 \pmod r \\ 0, & \text{else} \end{cases}.$$

Now define

$$\begin{aligned} \widehat{\psi}(z) &= \psi(z)\psi(\omega z)\cdots\psi(\omega^{r-1}z) \\ &= \prod_{j=1}^{r-1}(1-\omega^j)^{n-j} \ \exp\left(r{\sum_n}'' \frac{p_n(x)}{n}\, z^n\right)\exp\left(-r{\sum_n}''\frac{\partial}{\partial p_n(x)}\, z^{-n}\right)\times \\ &\quad \times e^{riq}\ z^{r\alpha_0 + r(r-1)/2}\ \omega^{r(r-1)\alpha_0/2 + r(r-1)(r-2)/6}, \end{aligned}$$

which we have written in a normal-ordered form, shifting all of the differential operators to the right. Then $\Psi(z) = \widehat{\psi}(z)\xi(z)$ where

$$\begin{aligned} \xi(z) &= \prod_{j=1}^{r-1}(1-\omega^j)^{j-n}\ z^{r(r-1)/2}\ \omega^{r(r-1)(2r-1)/6}\ e^{-(r-1)iq}\ \omega^{-r(r-1)\alpha_0/2}\times \\ &\quad \times\exp\left((r-1){\sum_n}''\frac{\partial}{\partial p_n(x)}\, z^{-n}\right). \end{aligned}$$

If we expand the current $\widehat{\psi}(z)$ into its modes

$$\widehat{\psi}(z) = \sum_{p\in\mathbb{Z}}\widehat{\psi}_p\ z^{rp - r(r-1)/2},$$

then

$$\widehat{\psi}_p = \sum_{\substack{i_1,i_2,\ldots,i_r \\ i_1+\cdots+i_r = rp - r(r-1)/2}} \omega^{i_2 + 2i_3 + \cdots + (r-1)i_r}\psi_{i_1}\cdots\psi_{i_r}. \tag{5.54}$$

Using the results of Appendix C, we can rewrite this in the form

$$\widehat{\psi}_p = r\sum_{\substack{i_1<i_2<\ldots<i_r \\ i_1+\cdots+i_r = rp - r(r-1)/2}}\left(\sum_{\sigma\in S_{r-1}}(\mathrm{sgn}\sigma)\ \omega^{i_{\sigma(2)}+2i_{\sigma(3)}+\cdots+(r-1)i_{\sigma(r)}}\right)\psi_{i_1}\cdots\psi_{i_r}, \tag{5.55}$$

where S_{r-1} denotes the symmetric group on the elements $\{2,3,\ldots,r\}$. From here on, we can simply mimic the $r = 2$ case. That is, given a string of vertex operators $\Psi(z)$ generating an S-function $s_\lambda(x^r)$ according to (5.53), we decompose it into the operators $\widehat{\psi}(z)$ and $\xi(z)$ and then move the annihilation-type operators $\xi(z)$ to the right using the relation

$$\xi(u)\widehat{\psi}(v) = \left(\frac{v^r}{1-\frac{v^r}{u^r}}\right)^{r-1} \widehat{\psi}(v)\xi(u).$$

This leaves us with a generating function composed just of free fermions which can be decomposed into S-functions of argument x.

Example

Let us demonstrate the above with an example of how to handle $h_n(x^r)$, which is associated with the state $\Psi_n|0\rangle$. Now,

$$\Psi_n|0\rangle = \oint \frac{dz}{z} z^{-rn}\widehat{\psi}(z)\xi(z)|0\rangle = \frac{\omega^{r(r-1)(2r-1)/6}}{\prod_{j=1}^{r-1}(1-\omega^j)^{n-j}}\widehat{\psi}_n\, e^{-(r-1)iq}|0\rangle\,, \tag{5.56}$$

So that if we now use the fact that

$$\psi_{i_1}\cdots\psi_{i_r}\, e^{ikq}|0\rangle = s_{(i_1-(r-1)-k,i_2-(r-2)-k,\ldots,i_r-k)}(x),$$

we are able to express $h_n(x^r)$ in terms of S-functions with argument x. It is still, however, too difficult to write down an explicit result valid for arbitrary r. In the case $r=3$, with $\omega^3=1$, using (5.55) and (5.56) we have

$$h_n(x^3) = \frac{1}{\omega^2-\omega} \sum_{\substack{-2\le i_1<i_2<i_3\\ i_1+i_2+i_3=3n-3}} \left(\omega^{i_2+2i_3}-\omega^{i_3+2i_2}\right) s_{(i_3,i_2+1,i_1+2)}(x), \tag{5.57}$$

so that, for example

$$\begin{aligned} h_3(x^3) \;&=\; s_{(9)}(x)-s_{(81)}(x)+s_{(711)}(x)+s_{(63)}(x)-s_{(621)}(x)-s_{(54)}(x)+s_{(522)}(x)\\ &\quad +s_{(441)}(x)-s_{(432)}(x)+s_{(333)}(x), \end{aligned}$$

which may be checked explicitly by noting that, in terms of power sums, both sides are equal to $(p_3^3+3p_6p_3+2p_9)/6$. It is interesting to note that not all of the terms on the right hand side of (5.57) are non-trivial (e.g. the ones corresponding to $(i_1,i_2,i_3) = (-2,1,7)$ or $(-1,2,5)$).

5.4.3 Examples

Using the above results we are in a position to calculate some plethysms by brute force (from the definition). For example, if we want to calculate $s_{(n)}\otimes s_{(2)}$, we write $s_{(2)} = \frac{1}{2}(p_2+p_1^2)$, so that

$$s_{(n)}\otimes s_{(2)} = \frac{1}{2}\left(h_n(x^2)+h_n^2(x)\right).$$

Upon using (5.40) and (5.41), we recover the well-known result

$$s_{(n)}\otimes s_{(2)} = \sum_{j=0}^{[n/2]} s_{(2n-2j,2j)}, \tag{5.58}$$

where $[x]$ denotes the integer part of x. Similarly, using $s_{(1^2)} = \frac{1}{2}(p_1^2 - p_2)$ we see that

$$s_{(n)} \otimes s_{(1^2)} = \sum_{j=1}^{[(n+1)/2]} s_{(2n-2j+1,2j-1)}. \tag{5.59}$$

One could go on to calculate $s_{(n)} \otimes s_{(3)}$ from the definition

$$s_{(n)} \otimes s_{(3)} = \frac{1}{3}h_n(x^3) + \frac{1}{2}h_n(x^2)h_n(x) + \frac{1}{6}(h_n(x))^3,$$

by combining the equations (5.40), (5.41) and (5.57) along with the formula

$$s_{(2n-j,j)}(x)\, h_n = \sum_{p=0}^{n} \sum_{q=\max\,(0,n-j-p)}^{\min\,(n-p,2n-2j)} s_{(2n-j+p,j+q,n-p-q)}(x),$$

which follows from the Littlewood-Richardson rule. In a similar manner, one can obtain explicit expressions for the plethysms $s_{(n)} \otimes s_{(21)}$ and $s_{(n)} \otimes s_{(1^3)}$ by using $s_{(21)} = (p_1^3 - p_3)/3$ and $s_{(1^3)} = p_1^3/6 - p_2 p_1/2 + p_3/3$.

We can also write down explicit expressions for the plethysms $s_{(n-1,1)} \otimes s_{(2)}$ and $s_{(n-1,1)} \otimes s_{(1^2)}$ through the use of (5.49) and the (outer) product

$$\begin{aligned} s^2_{(n-1,1)} &= \sum_{j=2}^{n} s_{(2n-j,j)} + 2\sum_{j=2}^{n-1} s_{(2n-1-j,j,1)} + \sum_{j=2}^{n-1} s_{(2n-2-j,j,2)} \\ &+ \sum_{j=1}^{n-1} s_{(2n-2-j,j,1^2)} + s_{(2n-2,1^2)}. \end{aligned} \tag{5.60}$$

Thus we have

$$\begin{aligned} s_{(n-1,1)} \otimes s_{(2)} &= \sum_{j=0}^{[\frac{n-2}{2}]} s_{(2n-2-2j,2j+2)} + \sum_{j=1}^{n-2} s_{2n-2-j,j+1,1)} \\ &+ \sum_{j=1}^{[\frac{n-1}{2}]} s_{(2n-2-2j,2j,2)} + \sum_{j=0}^{[\frac{n-2}{2}]} s_{(2n-3-2j,2j+1,1^2)}, \end{aligned} \tag{5.61}$$

$$\begin{aligned} s_{(n-1,1)} \otimes s_{(1^2)} &= \sum_{j=0}^{[\frac{n-3}{2}]} s_{(2n-3-2j,2j+3)} + \sum_{j=1}^{n-2} s_{(2n-2-j,j+1,1)} + s_{(2n-2,1^2)} \\ &+ \sum_{j=1}^{[\frac{n-2}{2}]} s_{(2n-3-2j,2j+1,2)} + \sum_{j=1}^{[\frac{n-1}{2}]} s_{(2n-2-2j,2j,1^2)}. \end{aligned} \tag{5.62}$$

In a similar manner we can calculate the plethysms $s_{(n,2)} \otimes s_{(2)}$ and $s_{(n,2)} \otimes s_{(1^2)}$. From the Littlewood-Richardson rule we have

$$
\begin{aligned}
s_{(n,2)}^2 \;=\;& \sum_{j=0}^{n-2} s_{(2n-j,j+4)} + 2\sum_{j=0}^{n-3} s_{(2n-1-j,j+4,1)} + 3\sum_{j=0}^{n-4} s_{(2n-2-j,j+4,2)} \\
+\;& 2\sum_{j=0}^{n-4} s_{(2n-3-j,j+4,3)} + \sum_{j=0}^{n-4} s_{(2n-4-j,j+4,4)} + \sum_{j=0}^{n-2} s_{(2n-1-j,j+3,1^2)} \\
+\;& \sum_{j=0}^{n-4} s_{(2n-2-j,j+2,2^2)} + 2\sum_{j=0}^{n-3} s_{(2n-2-j,j+3,2,1)} + \sum_{j=0}^{n-3} s_{(2n-3-j,j+3,3,1)} \\
+\;& s_{(2n,3,1)} + s_{(2n,2^2)} + 2s_{(2n-1,3,2)} + s_{(n+1,n+1,2)} + s_{(2n-2,3^2)} + s_{(2n-1,2^2,1)}.
\end{aligned}
$$

Combining this with the expression for $s_{(n,2)}(x^2)$ given by (5.50), we end up with

$$
\begin{aligned}
s_{(n,2)} \otimes s_{(2)} =& \sum_{j=0}^{[(n-2)/2]} s_{(2n-2j,2j+4)} + \sum_{j=0}^{n-3} s_{(2n-1-j,j+4,1)} + s_{(2n,2^2)} \\
+\;\; 2& \sum_{j=0}^{[(n-4)/2)]} s_{(2n-2-2j,2j+4,2)} + \sum_{j=-1}^{[(n-5)/2)]} s_{(2n-3-2j,2j+5,2)} + \frac{(1-(-1)^n)}{2}\, s_{(n+1,n+1,2)} \\
+\;& \sum_{j=0}^{n-4} s_{(2n-3-j,j+4,3)} + \sum_{j=0}^{[(n-4)/2]} s_{(2n-4-2j,2j+4,4)} + \sum_{j=0}^{[(n-2)/2]} s_{(2n-1-2j,2j+3,1^2)} \\
+\;& \sum_{j=0}^{[(n-2)/2]} s_{(2n-2-2j,2j+2,2^2)} + \sum_{j=0}^{n-3} s_{(2n-2-j,j+3,2,1)} + \sum_{j=0}^{[(n-3)/2]} s_{(2n-3-2j,2j+3,3,1)},
\end{aligned}
$$

$$
\begin{aligned}
s_{(n,2)} \otimes s_{(1^2)} =& \sum_{j=0}^{[(n-3)/2]} s_{(2n-1-2j,2j+5)} + \sum_{j=-1}^{n-3} s_{(2n-1-j,j+4,1)} + \frac{(1+(-1)^n)}{2}\, s_{(n+1,n+1,2)} \\
+\;& \sum_{j=0}^{[(n-4)/2]} s_{(2n-2-2j,2j+4,2)} + 2\sum_{j=0}^{[(n-5)/2]} s_{(2n-3-2j,2j+5,2)} + \sum_{j=-1}^{n-4} s_{(2n-3-j,j+4,3)} \\
+\;& \sum_{j=0}^{[(n-5)/2]} s_{(2n-5-2j,2j+5,4)} + \sum_{j=0}^{[(n-3)/2]} s_{(2n-2-2j,2j+4,1^2)} + \sum_{j=0}^{[(n-3)/2]} s_{(2n-3-2j,2j+3,2^2)} \\
+\;& \sum_{j=-1}^{n-3} s_{(2n-2-j,j+3,2,1)} + \sum_{j=0}^{[(n-4)/2]} s_{(2n-4-2j,2j+4,3,1)}.
\end{aligned}
$$

As a final example, using equations (5.34) and (5.52) we have the plethysms

$$
\begin{aligned}
s_{(n|m-1)} \otimes s_{(2)} \;=\;& \sum_{j=0}^{n-1}\sum_{i=0}^{m-2} s_{(2n-j,j|2m-2-i,i)} + \sum_{p=1}^{m}\sum_{q=0}^{n-1} \varepsilon_{p,q,m} s_{(2n-1-q,q|2m-p,p-1)} \\
&+ \sum_{p=1}^{m-1}\sum_{q=0}^{n} (1-\varepsilon_{p,q,m}) s_{(2n+1-q,q|2m-2-p,p-1)} + s_{(2n|2m-1)}, \\
s_{(n|m-1)} \otimes s_{(1^2)} \;=\;& \sum_{j=0}^{n-1}\sum_{i=0}^{m-2} s_{(2n-j,j|2m-2-i,i)} + \sum_{p=1}^{m}\sum_{q=0}^{n-1} (1-\varepsilon_{p,q,m}) s_{(2n-1-q,q|2m-p,p-1)} \\
&+ \sum_{p=1}^{m-1}\sum_{q=0}^{n} \varepsilon_{p,q,m} s_{(2n+1-q,q|2m-2-p,p-1)} + s_{(2n+1|2m-2)},
\end{aligned}
$$

where $\varepsilon_{p,q,m} = 1$ (resp. 0) if $p+q+m$ is even (resp. odd).

5.5 Hall-Littlewood functions

The final type of boson-fermion correspondence we shall look at is the B-F correspondence between certain deformed Heisenberg generators and the generalized fermions of Jing [221], in which the states in the fermionic Fock space can be associated to Hall-Littlewood symmetric functions. The deformed Heisenberg operators $\tilde{\alpha}_n$ considered by Jing obey the commutation relations

$$[\tilde{\alpha}_n, \tilde{\alpha}_m] = \frac{n}{1-t^{|n|}}\,\delta_{n+m,0}, \qquad n \neq 0, \tag{5.63}$$

along with operators $\tilde{\alpha}_0$ and q which satisfy (5.6). These operators, for $n \neq 0$, can again be realized on the ring $\Lambda(x)[t]$ of symmetric functions over $\mathbb{Q}(t)$, through the association

$$\tilde{\alpha}_{-n} = (1-t^n)p_n(x), \qquad \tilde{\alpha}_n = n\frac{\partial}{\partial p_n(x)}. \tag{5.64}$$

Instead of the vertex operators used in [221], we use a slightly different version, with the inclusion of the operators $\tilde{\alpha}_0$ and q, so that they reproduce the free fermion currents (5.7) when $t=0$,

$$\begin{aligned}
\varphi(z) &= \exp\left(\sum_{n=1}^{\infty}\frac{1-t^n}{n}p_n(x)\,z^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{\partial}{\partial p_n(x)}\,z^{-n}\right)\,e^{iq}z^{\tilde{\alpha}_0},\\
\varphi^*(z) &= \exp\left(-\sum_{n=1}^{\infty}\frac{1-t^n}{n}p_n(x)\,z^n\right)\exp\left(\sum_{n=1}^{\infty}\frac{\partial}{\partial p_n(x)}\,z^{-n}\right)\,z^{-\tilde{\alpha}_0}e^{-iq}.
\end{aligned} \tag{5.65}$$

Let φ_n, φ_n^* denote the modes of these currents, so that $\varphi(z) = \sum_n \varphi_n z^n$ and $\varphi^*(z) = \sum_n \varphi_n^* z^{-n}$. By forming the generating function $Z(p,q) = \varphi(p)\varphi^*(q)$ as was done in [113] and taking the appropriate limit, it can be seen that for $n \neq 0$, the generators

$$\tilde{H}_n = \frac{1}{(1-t)(1-t^{|n|})}\sum_{i\in\mathbb{Z}} :\varphi_i\varphi_{i+n}^*: \tag{5.66}$$

are identical to those given in (5.64) and hence satisfy the commutation relations (5.63). From the normal-ordering relations

$$\varphi(z)\varphi^*(w) = \frac{w}{z}\frac{z-tw}{z-w}:\varphi(z)\varphi^*(w):, \qquad \varphi^*(w)\varphi(z) = \frac{w-tz}{w-z}:\varphi^*(w)\varphi(z):,$$
$$\varphi(z)\varphi(w) = z\,\frac{z-w}{z-tw}:\varphi(z)\varphi(w):, \qquad \varphi^*(z)\varphi^*(w) = \frac{1}{w}\frac{z-w}{z-tw}:\varphi^*(z)\varphi^*(w):.$$

and following the techniques in [221], one can derive the anti-commutation relations

$$\begin{aligned}
\{\varphi_n, \varphi_m\} &= t\varphi_{n+1}\varphi_{m-1} + t\varphi_{m+1}\varphi_{n-1},\\
\{\varphi_n^*, \varphi_m^*\} &= t\varphi_{n-1}^*\varphi_{m+1}^* + t\varphi_{m-1}^*\varphi_{n+1}^*,\\
\{\varphi_n, \varphi_m^*\} &= t\varphi_{n-1}\varphi_{m-1}^* + t\varphi_{m+1}^*\varphi_{n+1} + (1-t)^2\delta_{n,m}.
\end{aligned} \tag{5.67}$$

which yield the free fermion relations in the limit $t \to 0$. Equations (5.65) and (5.66) represent Jing's generalized B-F correspondence between the deformed Heisenberg generators (5.63) and the deformed fermions (5.67).

Analogous to the $t = 0$ case where each state in the fermionic Fock space was associated with an S-function of a particular shape, the states in the Fock space generated by the generalized fermions can be associated to Hall-Littlewood functions. Indeed, if the bra and ket vacua are defined as in (5.2), then we have the equivalence [221]

$$\varrho(\varphi_{\lambda_1}\varphi_{\lambda_2}\cdots\varphi_{\lambda_{p-1}}\varphi_{\lambda_p}|0\rangle) = Q_\lambda(x;t), \tag{5.68}$$

where $\lambda = (\lambda_1 - p + 1, \lambda_2 - p + 2, \ldots, \lambda_p)$. By using this equivalence, we shall be able to derive some identities between S-functions and Hall-Littlewood functions.

5.5.1 Inverse Kostka-Foulkes matrices

It turns out that we can use the techniques of the previous section, where we decomposed a vertex operator in terms of products of free fermionic currents and an annihilation-type operator, to find a simple way of calculating the inverse Kostka-Foulkes matrix elements $K^{-1}_{\lambda\mu}(t)$, where

$$P_\lambda(x;t) = \sum_\mu K^{-1}_{\lambda\mu}(t)\; s_\mu(x).$$

Write $\varphi(z) = \widehat{\varphi}(z)\,\vartheta(z)$ where $\widehat{\varphi}(z) = \psi(z)\psi^*(tz)$ and

$$\vartheta(z) = (1-t)t^{\alpha_0 - 1}e^{iq}z^{\alpha_0}\exp\left(-\sum_{n=1}^{\infty} t^{-n}\frac{\partial}{\partial p_n(x)}z^{-n}\right).$$

Again, we have the crucial fact that $\vartheta(z)$ is a function only of differential operators. Now observe that we can write $\widehat{\varphi}(z) = \sum_{n\in\mathbb{Z}}\widehat{\varphi}_n z^n$, where

$$\widehat{\varphi}_n = \sum_{j\in\mathbb{Z}} t^j\,\psi_{n-j}\psi^*_{-j}. \tag{5.69}$$

Thus

$$\begin{aligned}
\varphi_n|0\rangle &= \frac{(1-t)}{2\pi i}\oint\frac{dz}{z}z^{-n}\widehat{\varphi}(z)\vartheta(z)|0\rangle = \widehat{\varphi}_n\,e^{iq}|0\rangle \\
&= (1-t)\sum_{j=0}^{n-1} t^j\,\psi_{n-j}\psi^*_{-j}\,e^{iq}|0\rangle\,,
\end{aligned}$$

and we recover once again the result

$$P_{(n)}(x;t) = \sum_{k=0}^{n-1}(-t)^k\,s_{(n-1-k|k)}(x).$$

Similarly, we can use the fact that

$$\vartheta(z)\widehat{\varphi}(w) = \frac{tz-w}{z-w}\;\widehat{\varphi}(w)\vartheta(z),$$

to obtain

$$\varphi_n\varphi_m|0\rangle = t^2(1-t)^2\left(\widehat{\varphi}_{n-1}\widehat{\varphi}_m + (1-t^{-1})\sum_{j=0}^{m-1}\widehat{\varphi}_{n+j}\widehat{\varphi}_{m-j-1}\right)e^{2iq}|0\rangle\,.$$

If we now use (5.69) to convert everything on the right hand side of the above equation back into free fermions, and use the fact that

$$\widehat{\varphi}_0\;e^{2iq}|0\rangle = \frac{t^{-1}}{1-t}\;e^{2iq}|0\rangle\,,$$

provided $|t|<1$, we find that

$$\begin{aligned}
Q_{(n-1,m)}(x;t) &= (1-t)^2\left\{\sum_{q=-1}^{n+m-3}(-t)^q\;s_{(n+m-3-q|q+1)}(x)\right.\\
&+\left(\sum_{j=-1}^{-1}+(1-t^{-1})\sum_{j=0}^{m-2}\right)\left(\sum_{q=-1}^{m-j-3}(-1)^{n+j+q}t^{2q+n+j+2}\;s_{(m-j-3-q|n+j+1+q)}(x)\right.\\
&-\sum_{q=-1}^{m-j-3}(-1)^q t^{2q+j+3-m}\;s_{(n+m-3-q|q+1)}(x)\\
&\left.\left.-\sum_{p=-1}^{n+j-2}\sum_{q=-1}^{m-j-3}(-t)^{p+q+2}s_{(m-j-3-q,n+j-2-p|p+1,q+1)}(x)\right)\right\}. \qquad (5.70)
\end{aligned}$$

Thus we are able to express two-part Hall-Littlewood functions in terms of one and two-hook S-functions. One can explicitly write out the cases $m=1,2$ with the result that, for $n>2$

$$\begin{aligned}
P_{(n-1,1)}(x;t) &= \sum_{k=1}^{n-2}(-t)^{k-1}s_{(n-1-k|k)}(x)+(-t)^{n-2}(1+t)s_{(0|n-1)}(x)\\
&+\sum_{k=1}^{n-3}(-t)^k s_{(n-3-k,0|k,0)}(x),
\end{aligned}$$

while for $k>4$ we have

$$\begin{aligned}
P_{(k-2,2)}(x;t) &= \sum_{q=1}^{k-4}(-t)^q s_{(k-2-q|q+1)}(x)+(1-t)\sum_{p=0}^{k-4}(-t)^p s_{(k-3-p,0|p+1,0)}(x)\\
&+\sum_{p=0}^{k-4}(-t)^{p+1}\left(s_{(k-4-p,1|p+1,0)}(x)+s_{(k-3-p,1|p,0)}(x)\right)-(-t)^{k-1}(1+t)s_{(0|k-1)}(x)\\
&+(-t)^{k-3}(1+t)s_{(1|k-2)}(x).
\end{aligned}$$

When $k=4$, we see that

$$P_{(22)}(x;t) = \frac{1}{(1-t)(1-t^2)}Q_{(22)}(x;t) = s_{(2^2)}(x) - t\,s_{(21^2)}(x) + t^3\,s_{(1^4)}(x).$$

which gives the correct result, which can be checked by computing the inverse of the Kostka-Foulkes matrix for $|\lambda| = 4$ appearing in Macdonald [33]. For the general case we have

$$\varphi_{n_1} \cdots \varphi_{n_p}|0\rangle = (1-t)^p\, t^{p(p-1)} \prod_{i<j} \frac{1-t^{-1}R_{ij}}{1-R_{ij}}\, \widehat{\varphi}_{n_1-p+1} \cdots \widehat{\varphi}_{n_p}\, e^{ipq}|0\rangle\,,$$

and so application of (5.69) will allow one to express $P_\lambda(x;t)$ where λ is a p part partition in terms of one, two, up to p-hook S-functions, although it is not feasible to write down general expressions for more complicated partitions than the ones above. As in the S-function case, there is an exception however – one-hook partitions. We must first though, ascertain how one-hook Hall-Littlewood functions are represented by fermionic states.

As a first guess, we might expect that the state $\varphi^*_{-j_1} \cdots \varphi^*_{-j_r} \varphi_{i_s} \cdots \varphi_{i_1}|0\rangle$ can be associated with an r-hook Hall-Littlewood function. Let us examine the simplest state $\varphi^*_{-j}\varphi_i|0\rangle$. From the vertex operators (5.65) we have

$$\varrho(\varphi^*_{-j}\varphi_i|0\rangle) = f_{i,j}(x;t) \equiv q_j(0/x;t)q_i(x;t) + (1-t)\sum_{k=1}^{i} q_{j+k}(0/x;t)q_{i-k}(x;t), \quad (5.71)$$

where the supersymmetric functions $q_n(x/y;t)$ are defined by their generating function (3.32). The function $f_{i,j}(x;t)$ has the one-hook S-function $(-1)^j s_{(i|j-1)}(x)$ as its limit when $t \to 0$, but does it represent a one-hook Hall-Littlewood function ? In the special case $t = -1$, by using $q_n(0/x;-1) = (-1)^n q_n(x;-1)$, it follows that $f_{i,j}(x;-1) = Q_{(i,j)}(x)$, a two-part Q-function. For the case of generic t, consider the case of functions associated to partitions of weight 3 (which are all one-hook partitions). From the definition (5.71), we have

$$\begin{aligned}
f_{0,3}(x;t) &= -\frac{1}{6}(1-t)^3 p_{(111)} + \frac{1}{2}(1-t)(1-t^2)p_{(21)} - \frac{1}{3}(1-t^3)p_{(3)},\\
f_{1,2}(x;t) &= \frac{1}{6}(1-t)^3(t+2)p_{(111)} - \frac{t}{2}(1-t)(1-t^2)p_{(21)} - \frac{1}{3}(1-t)(1-t^3)p_{(3)},\\
f_{2,1}(x;t) &= -\frac{1}{6}(1-t)^3(2t+1)p_{(111)} - \frac{1}{2}(1-t)(1-t^2)p_{(21)} - \frac{1}{3}(1-t)(1-t^3)p_{(3)}.
\end{aligned}$$

However, under the Hall-Littlewood inner product (2.37) we see that

$$\langle\, f_{0,3}, f_{1,2}\,\rangle = \langle\, f_{0,3}, f_{2,1}\,\rangle = 0, \qquad \langle\, f_{1,2}, f_{2,1}\,\rangle = t^2(1-t)^2,$$

so these functions are not orthogonal, as Hall-Littlewood functions should be.

Now, there is actually a formula which expresses one-hook Macdonald functions $Q_{(n,1^k)}(x;q;t)$ in terms of the elementary functions $g_n(x;q,t)$ and $e_n(x)$ [192]. By taking the limit $q \to 0$ in that formula, we deduce that

$$Q_{(n,1^k)}(x;t) = \gamma_k(t)\, \sum_{j=0}^{k} (-1)^j\, q_{n+j}(x;t)\, e_{k-j}(x), \qquad \gamma_k(t) = (1-t)(1-t^2)\cdots(1-t^k). \quad (5.72)$$

Using this, along with the vertex operators $\varphi(z)$ and $\psi^*(z)$ we see that

$$\begin{aligned}\varrho(\varphi_m\psi^*_{-n}|0\rangle) &= (-1)^{n-1}q_{m+1}e_{n-1}+(-1)^{n-2}q_{m+2}e_{n-2}+\cdots+q_{m+n} \\ &= (-1)^{n-1}\gamma_{n-1}^{-1}(t)\, Q_{(m+1,1^{n-1})}(x;t). \end{aligned} \tag{5.73}$$

Thus it is the states $\varphi_m\psi^*_{-n}|0\rangle$ that are connected with one-hook Hall-Littlewood functions, **not** the states $\varphi^*_{-j}\varphi_i|0\rangle$. The path is now clear as to how one can turn these back into S-functions. First, note the exchange relation

$$\vartheta(z)\psi^*(w) = \frac{w}{(tz-w)}\,\psi^*(w)\vartheta(z).$$

Thus using the integral technique, we see that

$$\varphi_m\psi^*_{-n}|0\rangle = (1-t)\sum_{j=1}^{n} t^{-j}\widehat{\varphi}_{m+j}\psi^*_{j-n}e^{iq}|0\rangle\,.$$

Upon using (5.69) to express everything in terms of free fermions, and then passing back to S-functions, we arrive at the result

$$\begin{aligned}P_{(m+1,1^{n-1})}(x;t) &= \sum_{j=1}^{n}\sum_{k=0}^{m+j-1}(-t)^{k-j}\; s_{(m+j-k,0|k-1,n-j-1)}(x) \\ &\quad +(-t)^m\frac{(1-t^n)}{(1-t)}s_{(0|n+m-1)}(x), \end{aligned} \tag{5.74}$$

which expresses one-hook Hall-Littlewood functions in terms of one and two-hook S-functions.

One can check, by converting $\varphi_1\varphi_0\psi^*_{-2}\psi^*_{-1}|0\rangle$ into S-functions using the above technology, that this state does correctly represent the two-hook function $Q_{(1,0|1,0)}(x;t)$. Thus we are led to conjecture that for r-hook Hall-Littlewood functions,

$$\varrho(\varphi_{i_1}\cdots\varphi_{i_r}\psi^*_{-j_1}\cdots\psi^*_{-j_r}|0\rangle) = Q_{(i_1,\ldots,i_r|j_1-1,\ldots,j_r-1)}(x;t). \tag{5.75}$$

As far as we know however, there is no analogue of (5.72) for hook partitions with Frobenius rank greater than one, so proving the above conjecture would seem to be quite difficult.

5.6 Hall-Littlewood plethysms

In this section we will extend the definition of plethysm to encompass Hall-Littlewood functions. We define the plethysm between two functions $f,\,g\in\Lambda[t]$ to mean express g as a multinomial in power sums and then make the substitution $p_j(x)\to f(x^j;t^j)$. That is, if $g(x;t)=\sum_\mu a_\mu(t)p_\mu(x)$, then

$$f\otimes g = \sum_\mu a_\mu(t)f(x^{\mu_1};t^{\mu_1})f(x^{\mu_2};t^{\mu_2})\cdots f(x^{\mu_{\ell(\mu)}};t^{\mu_{\ell(\mu)}}). \tag{5.76}$$

Note that this definition includes "ordinary" (outer) plethysm between functions $f \in \Lambda \subset \Lambda[t]$. Let us give an example. If we consider the generating function for $\sum_n q_n(x^2;t^2)z^{2n}$ along with the product formula

$$q_{n-k}q_k = Q_{(n-k,k)} + (1-t)\sum_{j=0}^{k-1} Q_{(n-j,j)},$$

we find that

$$q_n(x^2;t^2) = (1+t)\sum_{j=0}^{n-1}(-1)^j Q_{(2n-j,j)}(x;t) + (-1)^n Q_{(n,n)}(x;t). \qquad (5.77)$$

Hence we have

$$q_n \otimes s_{(2)} = \begin{cases} Q_{(2n)} - tQ_{(2n-1,1)} + Q_{(2n-2,2)} - \cdots - tQ_{(n+1,n-1)} + Q_{(n,n)} & \text{if } n \text{ is even} \\ Q_{(2n)} - tQ_{(2n-1,1)} + Q_{(2n-2,2)} - \cdots + Q_{(n+1,n-1)} & \text{if } n \text{ is odd} \end{cases}.$$

and

$$q_n \otimes q_2 = \begin{cases} a(t)Q_{(2n)} + b(t)Q_{(2n-1,1)} + a(t)Q_{(2n-2,2)} \\ \quad + \cdots + b(t)Q_{(n+1,n-1)} + (1-t)Q_{(n,n)} & \text{if } n \text{ is even} \\ Q_{(2n)} + b(t)Q_{(2n-1,1)} + a(t)Q_{(2n-2,2)} \\ \quad + \cdots + b(t)Q_{(n+1,n-1)} + t(t-1)Q_{(n,n)} & \text{if } n \text{ is odd} \end{cases}.$$

where $a(t) = (1-t)(1+t^2)$ and $b(t) = 2t(t-1)$. Note that in these examples, all of the coefficients in expansion of the plethysm are elements of $\mathbb{Z}[t]$. Suppose we had defined the plethysm operation $\tilde{\otimes}$ with $p_j(x) \to f(x^j;t)$ instead of (5.76). Then it appears that the coefficients in the plethysm

$$Q_\lambda \tilde{\otimes} Q_\mu = \sum_\nu \tilde{a}^\nu_{\lambda\mu}(t)Q_\nu$$

are no longer polynomials in t with integer coefficients. For example

$$\begin{aligned} q_2(x^2;t) &= Q_{(4)}(x;t) - Q_{(31)}(x;t) + \frac{1+t^2}{1-t^2}Q_{(22)}(x;t) - \frac{t}{1-t}Q_{(21^2)}(x;t) \\ &\quad + \frac{t}{1-t^4}Q_{(1^4)}(x;t), \end{aligned}$$

so that

$$\begin{aligned} q_2\tilde{\otimes}q_2 &= \tfrac{1}{2}(1-t)(t^2-t+2)Q_{(4)} + \tfrac{1}{2}t(1-t)(t-3)Q_{(31)} + (1-t+t^2)Q_{(2^2)} \\ &- \tfrac{1}{2}t(t+1)Q_{(21^2)} + \frac{t}{2(1+t^2)}Q_{(1^4)}. \end{aligned}$$

We will now show that the procedure developed in section 5.4 can be extended so as to calculate $Q_\lambda(x^r;t^r)$, which together the multiplication rule for Hall-Littlewood functions, will enable us to calculate (in principle) plethysms of the form

$$Q_\lambda \otimes Q_\mu = \sum_\nu a^\nu_{\lambda\mu}(t)Q_\nu.$$

In particular, it will provide strong evidence for the coefficients $a^{\nu}_{\lambda\mu}(t) \in \mathbb{Z}[t]$.

Let us begin, once again, with the case $r = 2$. Let $\Upsilon(z) = \sum_n \Upsilon_n z^{2n}$ be the vertex operator $\varphi(z)$ in (5.65) with $x \to x^2$ and $z \to z^2$. Write $\Upsilon(z) = \check{\varphi}(z)\check{\xi}(z)$ where $\check{\varphi}(z) = \varphi(z)\varphi(-z)$ and

$$\check{\xi}(z) = \frac{1+t}{2}\, z\, e^{-iq}(-1)^{\alpha_0 - 1} \exp\left({\sum_n}' \frac{\partial}{\partial p_n(x)}\, z^{-n} \right).$$

We also write $\check{\varphi}(z) = \sum_n \check{\varphi}_n z^{2n-1}$ where

$$\begin{aligned} \check{\varphi}_n &= \sum_{j\in\mathbb{Z}} (-1)^j \varphi_{2n-1-j}\varphi_j \\ &= (-1)^n \left(\frac{-2}{1+t}\varphi_n\varphi_{n-1} + 2\sum_{j=1}^{\infty} \varphi_{n+j}\varphi_{n-1-j} \right), \end{aligned} \tag{5.78}$$

where we have used the anti-commutation relations (5.67) recursively to order the modes correctly. Thus the equation

$$\Upsilon_n|0\rangle = -\frac{1+t}{2}\, \check{\varphi}_n\, e^{-iq}|0\rangle\,,$$

upon application of (5.78), correctly reproduces (5.77). Similarly, using the exchange relation

$$\check{\xi}(z)\check{\varphi}(w) = -w^2 \left(\frac{z^2 - t^2w^2}{z^2 - w^2} \right) \check{\varphi}(w)\check{\xi}(z)$$

enables us to derive the relation

$$\Upsilon_n\Upsilon_m|0\rangle = \frac{(1+t)^2}{4} \left(\check{\varphi}_n\check{\varphi}_{m-1} + (1-t^2)\sum_{j=1}^{m} \check{\varphi}_{n+j}\check{\varphi}_{m-1-j} \right) e^{-2iq}|0\rangle\,.$$

Use of (5.78) will enable us to decompose this into generalized fermions, and hence will allow us to express $Q_{(n-1,m)}(x^2;t^2)$ in terms of functions $Q_\lambda(x;t)$, where $\ell(\lambda) \leq 4$. The resulting partitions will however, be non-standard and hence will need to be modified using the modification rule (2.36). If we consider the simplest case when $m = 1$, we find after some algebra,

$$\begin{aligned} Q_{(n-1,1)}(x^2;t^2) &= (1+t)\sum_{j=1}^{n-2}(-1)^{j+1}Q_{(2n-2-j,j,1^2)}(x;t) + (-1)^n Q_{(n-1,n-1,1^2)}(x;t) \\ &+ \ (1+t)^2\sum_{j=2}^{n-2}(-1)^j Q_{(2n-4-j,j,j,2)}(x;t) + (-1)^{n+1}(1+t)Q_{(n-1,n-1,2)}(x;t) \\ &+ \ (1+t)(1-t^2)\sum_{j=0}^{n-1}(-1)^j Q_{(2n-j,j)}(x;t) + (-1)^n(1-t^2)Q_{(n,n)}(x;t) \\ &+ \ (t+1)(t^2-1)\left[Q_{(2n)}(x;t) - Q_{(2n-1,1)}(x;t)\right] - (1+t)Q_{(2n-2,1^2)}(x;t) \\ &+ \ t(1+t)^2\left[Q_{(2n-2,2)}(x;t) - Q_{(2n-3,2,1)}(x;t)\right]. \end{aligned}$$

Note that we reproduce (5.49) when $t = 0$, as we should.

Like the S-function case, we can extend the above considerations to the case of generic r, and the details are almost exactly the same. Thus we have, for example when $\omega^3 = 1$,

$$Q_{(n)}(x^3;t^3) = \frac{1+2t^2-t(2+t^2)\omega+t(t-1)\omega^2}{3(\omega-\omega^2)} \sum_{i_1+i_2+i_3=3n-3} \omega^{i_2+2i_3} Q_{(i_1,i_2+1,i_3+2)}(x;t)$$

This is not quite as compact as the S-function case given by (5.57), but it can in principle be treated using the modification rules (2.36).

Finally, let us remark that it might be possible to extend the above definition of plethysm to Macdonald functions by defining for any two symmetric functions $f(x;q,t)$, $g(x;q,t) \in \Lambda_F$, where $F = \mathbb{Q}(q,t)$, the plethysm $f \otimes g$ to mean: expand g in terms of power sums, and then make the substitution $p_j(x) \to f(x^j;q^j,t^j)$. As an example, let us calculate the plethysm $Q_{(n)} \otimes p_2 = Q_{(n)}(x^2;q^2,t^2)$. From the generating function for single-row Macdonald functions, we have

$$\begin{aligned} \sum_{n=0}^{\infty} Q_{(n)}(x^2;q^2,t^2) z^{2n} &= \exp\left(\sum_{n=1}^{\infty} \frac{(1+(-1)^n)}{n} \left(\frac{1-t^n}{1-q^n} \right) p_n(x)\, z^n \right) \\ &= \left(\sum_{k=0}^{\infty} Q_{(k)}(x;q,t) z^k \right) \left(\sum_{l=0}^{\infty} Q_{(l)}(x;q,t)(-z)^l \right), \end{aligned}$$

hence

$$Q_{(n)}(x^2;q^2,t^2) = \sum_{k=0}^{2n} (-1)^k Q_{(k)}(x;q,t) Q_{(2n-k)}(x;q,t).$$

By apply the Pieri formula (2.69) to this expression and then resumming it, using the method used to derive (2.72) and (2.82), with the aid of the summation formula

$${}_2\phi_1(t, q^{-2n}; q^{1-2n}/t; q, -q/t) = \frac{(q;q^2)_n\ (t^2;q^2)_n}{(t;q)_{2n}}, \tag{5.79}$$

which follows directly from the Bailey-Daum summation formula [201, Eq. 1.8.1], we see that

$$Q_{(n)}(x^2;q^2,t^2) = \sum_{k=0}^{n} (-1)^{n-j} \frac{(q;q^2)_n\ (t^2;q^2)_n}{(t;q)_{2n}}\ Q_{(n+k,n-k)}(x;q,t). \tag{5.80}$$

Chapter 6

Symmetric Functions and Quantum Affine Algebras

In this chapter, we demonstrate the utility of symmetric function techniques in providing concrete realizations of the quantum affine algebra $U_q(\widehat{sl(2)})$ and hence the means to calculate traces of products of currents of this algebra. In the case of level one realizations of $U_q(\widehat{sl(2)})$, these traces have been used in the calculation of correlation functions of the spin $\frac{1}{2}$ XXZ quantum spin chain [188], and we hope that the traces calculated here in the level k case might be of use in the calculation of correlation functions of the spin $\frac{k}{2}$ XXZ chain [190].

The key to this construction is to note that the Fock spaces used in the free field realization of $U_q(\widehat{sl(2)})$ at level k can alternatively be described as rings of symmetric functions spanned by particular sets of Kerov's generalized symmetric functions [192]. Not only that, it turns out that matrix elements of currents of the algebra in this basis can also be described in terms of these same functions. This allows a technique of King's [193] to be used to calculate traces of products of these currents.

6.1 Generalized symmetric functions

In order to perform certain vertex operator trace calculations, we shall find it necessary to introduce some symmetric functions which are a specialization of the symmetric functions introduced by Kerov [192]. To describe Kerov's symmetric functions, let $v = (v_1, v_2, \ldots)$ be a sequence of real numbers and define an inner product on the power sum symmetric functions by

$$\langle\, p_\lambda, p_\mu \,\rangle = \delta_{\lambda\mu}\, z_\lambda\, v_\lambda, \tag{6.1}$$

where $v_\lambda = v_n^{m_n} \cdots v_1^{m_1}$ for a partition of the form $\lambda = (n^{m_n} \cdots 1^{m_1})$. Define functions $P_\lambda(x; v)$ which span the ring Λ_F, where $F = \mathbb{Q}(v_1, v_2, \ldots)$ by requiring that they are pairwise orthogonal, and that the transition matrix between these functions and the monomial symmetric functions $m_\mu(x)$, where $|\mu| = |\lambda|$, is unitriangular (upper triangular with one's on the diagonal). That is,

$$P_\lambda(x; v) \;=\; m_\lambda(x) + \sum_{\mu<\lambda} u_{\lambda\mu}(v)\, m_\mu(x), \tag{6.2}$$

$$\langle\, P_\lambda(x;v), P_\mu(x;v)\,\rangle_v \;=\; 0, \qquad \text{for } \lambda \neq \mu, \tag{6.3}$$

where the ordering in the above sum is the standard lexicographic ordering. By using the Gram-Schmidt orthogonalization procedure, one can derive a unique orthogonal basis for Λ_F. Hall-Littlewood symmetric functions correspond to the case when $v_n = (1-t^n)^{-1}$, and Maconald's functions to the case $v_n = (1-q^n)/(1-t^n)$. For our purposes, we shall be interested in the case

$$v_n = \alpha\,\left(\frac{q^{\kappa n} - q^{-\kappa n}}{q^{2n} - q^{-2n}}\right), \tag{6.4}$$

where we assume $\alpha \in \mathbb{R}$ and $\kappa \in \mathbb{Z}$. We shall denote the symmetric functions so obtained $P_\lambda(x;q,\kappa,\alpha)$. From the definition (6.4) of the number v_n, we see that the functions defined by (6.2), (6.3) are either a slight generalization of Jack symmetric functions or of Macdonald's symmetric functions. Indeed, when $q \to 1$, $P_\lambda(x;q,\kappa,\alpha) \to P_\lambda^{(\kappa\alpha/2)}(x)$ in the notation of Chapter 2. In fact, when $\kappa = 2$, these functions are Jack symmetric functions for *all* values of q. Similarly when $\alpha = 1$, $P_\lambda(x;q,\kappa,\alpha)$ are identical to the Macdonald's function $P_\lambda(q^{2\kappa}, q^4)$, again in the notation of Chapter 2. As mentioned in Chapter 2, a very interesting property of Macdonald's functions is that, if they are defined by (6.2), (6.3) with the inner product (2.51), then the definition is independent of the total order appearing in the sum in (6.2) *if* the total order is compatible with the dominance order defined by (2.53). It is not known, however, whether the functions $P_\lambda(x;q,\kappa,\alpha)$ share this property. For what follows, we shall always be using the total lexicographic ordering.

6.1.1 Cauchy identities and skew functions

Once we have constructed the functions $P_\lambda(x;q,\kappa,\alpha)$, we can define their dual functions $Q_\lambda(x;q,\kappa,\alpha) \equiv b_\lambda(q,\kappa,\alpha)\,P_\lambda(x;q,\kappa,\alpha)$, where $b_\lambda(q,\kappa,\alpha) = ||P_\lambda(x;q,\kappa,\alpha)||^{-2}$, such that

$$\langle\, P_\lambda(x;q,\kappa,\alpha), Q_\mu(x;q,\kappa,\alpha)\,\rangle = \delta_{\lambda\mu}.$$

The exact form of $b_\lambda(q,\kappa,\alpha)$ is unknown, but as we shall see, it is not necessary for our purposes. We can see however, from the relationship with Macdonald's functions [39] when $\alpha = 1$, that

$$b_\lambda(q,\kappa,1) = q^{(\kappa-2)|\lambda|}\prod_{x\in\lambda}\frac{1-q^{2\kappa a(x)+4l(x)+4}}{1-q^{2\kappa a(x)+4l(x)+2\kappa}},$$

where $a(x)$ (respectively $l(x)$) is the arm-length (resp. leg-length) of the node $x \in \lambda$. Let us now develop a Cauchy formula for the functions $P_\lambda(x;q,\kappa,\alpha)$. Firstly, we have the result

$$\begin{aligned}\sum_\lambda \frac{1}{z_\lambda(q,\kappa,\alpha)} p_\lambda(x)\,p_\lambda(y) &= \prod_{i,j}\frac{(x_iy_jq^{\kappa+2};q^{2\kappa})_\infty^{1/\alpha}}{(x_iy_jq^{\kappa-2};q^{2\kappa})_\infty^{1/\alpha}} \\ &= \exp\left(\frac{1}{\alpha}\sum_{n>0}\frac{q^{\kappa n}-q^{-\kappa n}}{q^{2n}-q^{-2n}}\,p_n(x)p_n(y)\right), \end{aligned} \tag{6.5}$$

where we have denoted $z_\lambda(q,\kappa,\alpha) = z_\lambda v_\lambda$ for the particular choice (6.4) of v_λ, and

$$(x;q)^a_\infty = \prod_{j=0}^{\infty}(1-xq^j)^a.$$

Equation (6.5) is proved by a standard calculation (see Macdonald [39] for example). From this we obtain the following Cauchy identity

$$\sum_\lambda P_\lambda(x;q,\kappa,\alpha)\; Q_\lambda(y;q,\kappa,\alpha) = \prod_{i,j} \frac{(x_i y_j q^{\kappa+2};q^{2\kappa})^{1/\alpha}_\infty}{(x_i y_j q^{\kappa-2};q^{2\kappa})^{1/\alpha}_\infty}. \tag{6.6}$$

The functions $P_\lambda(x;q,\kappa,\alpha)$ form a basis for the ring Λ_F, so there exist structure constants $f^\lambda_{\mu\nu} \equiv f^\lambda_{\mu\nu}(q,\kappa,\alpha)$ (actually rational *functions* of the indeterminates q and α) such that

$$P_\mu(x;q,\kappa,\alpha)P_\nu(x;q,\kappa,\alpha) = \sum_\lambda f^\lambda_{\mu\nu} P_\lambda(x;q,\kappa,\alpha),$$

or equivalently

$$Q_\mu(x;q,\kappa,\alpha)Q_\nu(x;q,\kappa,\alpha) = \sum_\lambda \bar{f}^\lambda_{\mu\nu} Q_\lambda(x;q,\kappa,\alpha),$$

where

$$\bar{f}^\lambda_{\mu\nu} = \frac{b_\mu(q,\kappa,\alpha)b_\nu(q,\kappa,\alpha)}{b_\lambda(q,\kappa,\alpha)} f^\lambda_{\mu\nu}.$$

Using these coefficients we can define the skew functions

$$\begin{aligned} P_{\lambda/\mu}(x;q,\kappa,\alpha) &= \sum_\nu \bar{f}^\lambda_{\mu\nu} P_\nu(x;q,\kappa,\alpha), \\ Q_{\lambda/\mu}(x;q,\kappa,\alpha) &= \sum_\nu f^\lambda_{\mu\nu}(q) Q_\nu(x;q,\kappa,\alpha). \end{aligned}$$

It then follows from (6.6) that we can define *compound* functions in the indeterminates x and y via

$$\begin{aligned} P_\lambda(x,y;q,\kappa,\alpha) &= \sum_\sigma P_{\lambda/\sigma}(x;q,\kappa,\alpha) P_\sigma(y;q,\kappa,\alpha), \\ Q_\lambda(x,y;q,\kappa,\alpha) &= \sum_\sigma Q_{\lambda/\sigma}(x;q,\kappa,\alpha) Q_\sigma(y;q,\kappa,\alpha). \end{aligned}$$

6.1.2 Replicated functions

For the purposes of the next section, we shall find it useful to introduce the symmetric functions defined by (6.2), but with replicated variables, as was done in Chapter 3. We want to be able to define the function $P_\lambda(x^{(\tau)};q,\kappa,\alpha)$ so that when $\tau = m$, an integer, we have

$$P_\lambda(x^{(m)};q,\kappa,\alpha) = P_\lambda(\overbrace{x_1,\ldots,x_1}^{m},\overbrace{x_2,\ldots,x_2}^{m},\ldots;q,\kappa,\alpha).$$

Before we proceed with the definition, we must introduce the transition matrix $Y^\mu_\lambda \equiv Y^\mu_\lambda(q,\kappa,\alpha)$ between the power sums and the functions P_λ,

$$p_\lambda(x) = \sum_\mu Y^\mu_\lambda \; P_\mu(x;q,\kappa,\alpha).$$

The functions Y^μ_λ have been studied by Srinivasan [233] in the case $\alpha = 1$ (the Macdonald case). From the Cauchy identities (6.5) and (6.6), it follows that we have orthogonality relations of the form

$$\sum_\rho \frac{1}{z_\rho(q,\kappa,\alpha)} Y^\lambda_\rho Y^\mu_\rho = b_\lambda(q,\kappa,\alpha)\delta_{\lambda\mu}, \tag{6.7}$$

$$\sum_\lambda \frac{1}{b_\lambda(q,\kappa,\alpha)} Y^\lambda_\rho Y^\lambda_\sigma = z_\rho(q,\kappa,\alpha)\delta_{\rho\sigma}. \tag{6.8}$$

Let us now show that $Y^{(n)}_\mu = 1$ for all partitions $\mu \vdash n$. Using the first orthogonality relation (6.7), we see that

$$Q_\lambda(x;q,\kappa,\alpha) = \sum_\mu \frac{1}{z_\mu(q,\kappa,\alpha)} Y^\lambda_\mu \; p_\mu(x). \tag{6.9}$$

Now we know that the transition matrix between the functions P_λ and the monomials m_μ is upper uni-triangular. Also we have the fact that $m_\mu(x_1,\ldots,x_n) = 0$ if $\ell(\mu) > n$. Thus we have for *one* variable z,

$$P_\lambda(z;q,\kappa,\alpha) = \begin{cases} z^n & \text{if } \lambda = (n), \\ 0 & \text{otherwise.} \end{cases}$$

and so from the Cauchy identity (6.6) we have

$$\begin{aligned} \sum_{n=0}^\infty Q_{(n)}(x;q,\kappa,\alpha) z^n &= \prod_i \left(\frac{(x_i z q^{\kappa+2}; q^{2\kappa})_\infty}{(x_i z q^{\kappa-2}; q^{2\kappa})_\infty} \right)^{1/\alpha} \\ &= \exp\left(\frac{1}{\alpha} \sum_{n>0} \frac{q^{\kappa n} - q^{-\kappa n}}{q^{2n} - q^{-2n}} \; p_n(x)\; z^n \right). \end{aligned}$$

This implies that $Q_{(n)}(x;q,\kappa,\alpha) = \sum_{\mu\vdash n} z_\mu^{-1}(q,\kappa,\alpha) p_\mu(x)$ and so comparison with (6.9) leads us to conclude that $Y^{(n)}_\mu = 1$.

In exactly the same way as was done in Chapter 3, we can use these orthogonality relations to derive a formula for the symmetric function $P_\lambda(x^{(m)};q,\kappa,\alpha)$ where each of the indeterminates in the argument is repeated m times, in the form of

$$P_\lambda(\overbrace{x_1,\ldots,x_1}^{m}, \overbrace{x_2,\ldots,x_2}^{m}, \ldots; q,\kappa,\alpha) = \sum_\mu B_{\lambda\mu}(m) \; P_\mu(x;q,\kappa,\alpha),$$

where $B_{\lambda\mu}(m)$ is a polynomial in the "variable" m of the form

$$B_{\lambda\mu}(m) = \sum_\sigma \frac{1}{z_\sigma(q,\kappa,\alpha)} Y^\lambda_\sigma Y^\mu_\sigma \; m^{\ell(\sigma)}.$$

Thus we can consistently define our functions $P_\lambda(x^{(\tau)};q,\kappa,\alpha)$ of the replicated variable $x^{(\tau)}$ to be

$$P_\lambda(x^{(\tau)};q,\kappa,\alpha)=\sum_\mu B_{\lambda\mu}(\tau)\ P_\mu(x;q,\kappa,\alpha). \tag{6.10}$$

Again, we can repeat the analysis of Chapter 3 to derive the Cauchy identity

$$\sum_\lambda P_\lambda(x^{(\tau)};q,\kappa,\alpha)Q_\lambda(y^{(\eta)};q,\kappa,\alpha)=\prod_{i,j}\left(\frac{(x_iy_jq^{\kappa+2};q^{2\kappa})_\infty}{(x_iy_jq^{\kappa-2};q^{2\kappa})_\infty}\right)^{\tau\eta/\alpha}, \tag{6.11}$$

which we shall need in the next section.

Example
From the calculations we have carried out using Mathematica, the transition matrices Y and B are very complicated, even for partitions of low weight. We'll only show here results for $|\lambda|=2$. In this case we have

$$Y=\begin{pmatrix} 1 & 1 \\ \dfrac{2(q^4-1)(q^{2\kappa}+1)}{\theta(1,\alpha)} & \dfrac{\alpha(q^4+1)(1-q^{2\kappa})}{\theta(1,\alpha)} \end{pmatrix},$$

where $\theta(x,y)=(x+y)(q^{2\kappa+4}-1)+(x-y)(q^4-q^{2\kappa})$. Also

$$b^{-1}_{(2)}(q,\kappa,\alpha)=\frac{2\alpha^2q^{4-2\kappa}(q^{4\kappa}-1)(q^{2\kappa}-1)}{(q^4-1)\theta(1,\alpha)},$$
$$b^{-1}_{(1^2)}(q,\kappa,\alpha)=\frac{\alpha^2}{2}\left(\frac{q^\kappa-q^{-\kappa}}{q^2-q^{-2}}\right)+\frac{\alpha}{2}\ \frac{q^{2\kappa}-q^{-2\kappa}}{q^4-q^{-4}}.$$

The matrix $B_{\lambda\mu}(\tau)$ appearing in the definition (6.10) of replicated symmetric functions takes the form

$$B=\begin{pmatrix} \dfrac{\tau\ \theta(\tau,\alpha)}{\theta(1,\alpha)} & \dfrac{2\alpha\tau(1-\tau)(q^8-1)(q^{4\kappa}-1)}{(\theta(1,\alpha))^2} \\ \dfrac{\tau(1-\tau)}{2} & \dfrac{\tau\ \theta(1,\alpha\tau)}{\theta(1,\alpha)} \end{pmatrix}.$$

6.2 Vertex operator traces

We shall see in the next section that there are (homogeneous) vertex operator realizations of level k representations of $U_q(\widehat{sl(2)})$. It is our intention to introduce here a very general vertex operator which will describe the currents of this realization, and which we will be able to connect with the symmetric functions $P_\lambda(x;q,\kappa,\alpha)$ introduced in section 6.1. We define these vertex operators as

$$V(z_1^{(\tau_1)},\dots,z_n^{(\tau_n)};w_1^{(\eta_1)},\dots,w_n^{(\eta_n)})=\exp\left(\sum_{m>0}\frac{1}{mv_m}p_m(\tau_1z_1^m+\cdots\tau_nz_n^m)\right)\times$$
$$\times\exp\left(\sum_{m>0}\frac{1}{mv_m}D(p_m)(\eta_1w_1^m+\cdots\eta_nw_n^m)\right), \tag{6.12}$$

where D is the adjoint operator with respect to the inner product (6.1). That is,

$$D(p_m) = m\, v_m\, \frac{\partial}{\partial p_n}.$$

Following the standard calculation [234], the matrix elements of the above vertex operator in a basis of Kerov symmetric functions take the form

$$\langle\, P_\mu(x;v), V\ Q_\nu(x;v)\,\rangle = \sum_\xi P_{\mu/\xi}(z_1^{(\tau_1)},\ldots,z_n^{(\tau_n)};v) Q_{\nu/\xi}(w_1^{(\eta_1)},\ldots,w_n^{(\eta_n)};v). \quad (6.13)$$

Suppose we want to calculate the regularized trace of the vertex operator V over the space Λ_F. That is, we want to calculate

$$S_{p/1} = \sum_{\mu\nu} p^{|\mu|}\ P_{\mu/\nu}(z_1^{(\tau_1)},\ldots,z_n^{(\tau_n)};v)\ Q_{\mu/\nu}(w_1^{(\eta_1)},\ldots,w_n^{(\eta_n)};v). \quad (6.14)$$

Let us follow the method in reference [193, 202]. Define

$$\begin{aligned} S_{p/r} &= \sum_{\mu\nu} p^{|\mu|} r^{|\nu|}\ P_{\mu/\nu}(z_1^{(\tau_1)},\ldots,z_n^{(\tau_n)};v)\ Q_{\mu/\nu}(w_1^{(\eta_1)},\ldots,w_n^{(\eta_n)};v), \\ A_{\lambda\mu} &= \sum_{\xi} p^{|\xi|}\ P_{\xi/\lambda}(z_1^{(\tau_1)},\ldots,z_n^{(\tau_n)};v)\ Q_{\xi/\mu}(w_1^{(\eta_1)},\ldots,w_n^{(\eta_n)};v). \end{aligned}$$

Suppose that the Kerov functions with replicated arguments obey a very general Cauchy identity

$$\sum_\lambda r^{|\lambda|} P_\lambda(x^{(\tau)};v) Q_\lambda(y^{(\eta)};v) = J_r^{\tau\eta}(x,y;v),$$

so that for the functions $P_\lambda(x;v)$ with v_λ defined by (6.4) for example, the expression on the right has the form

$$J_r^{\tau\eta}(x,y;v) = \prod_{i,j} \left(\frac{(x_i y_j q^{\kappa+2} r; q^{2\kappa})_\infty)}{(x_i y_j q^{\kappa-2} r; q^{2\kappa})_\infty)} \right)^{\tau\eta/\alpha}. \quad (6.15)$$

We then form the generating function $I = \sum_{\lambda\mu} A_{\lambda\mu} P_\lambda(u) Q_\mu(s)$, obtaining

$$\begin{aligned} I &= \sum_\xi p^{|\xi|}\ P_\xi(z_1^{(\tau_1)},\ldots,z_n^{(\tau_n)},u;v)\ Q_\xi(w_1^{(\eta_1)},\ldots,w_n^{(\eta_n)},s;v) \\ &= \prod_{i,j=1}^n J_p^{\tau_i\eta_j}(z_i,w_j;v) \prod_{k=1}^n J_p^{\tau_k,1}(z_k,s;v) J_p^{1,\eta_k}(u,w_k;v) J_p^{1,1}(u,s;v) \\ &= \prod_{i,j=1}^n J_p^{\tau_i\eta_j}(z_i,w_j;v) \sum_{\sigma,\sigma_1,\sigma_2,\lambda,\mu} p^{|\sigma|+|\sigma_1|+|\sigma_2|} P_{\sigma_1}(z_1^{(\tau_1)},\ldots,z_n^{(\tau_n)};v) \times \\ &\quad \times Q_{\sigma_2}(w_1^{(\eta_1)},\ldots,w_n^{(\eta_n)};v) f^\lambda_{\sigma_2\sigma} P_\lambda(u;v) \bar{f}^\mu_{\sigma_1\sigma} Q_\mu(s;v). \end{aligned}$$

We conclude from this that

$$A_{\lambda\mu} = \prod_{i,j=1}^n J_p^{\tau_i\eta_j}(z_i,w_j;v) \sum_\sigma p^{|\lambda|+|\mu|-|\sigma|} P_{\mu/\sigma}(z_1^{(\tau_1)},\ldots,z_n^{(\tau_n)};v)\ Q_{\lambda/\sigma}(w_1^{(\eta_1)},\ldots,w_n^{(\eta_n)};v).$$

This means that

$$S_{p/r} = \sum_\nu r^{|\nu|}\ A_{\nu\nu} = \prod_{i,j=1}^{n} J_p^{\tau_i\eta_j}(z_i, w_j; v)\ S_{rp^2/p^{-1}},$$

and hence by interating this [193] we arrive at

$$S_{p/1} = \sum_{j=1}^{\infty} \frac{1}{(1-p^j)} \prod_{i,j=1}^{n} J_{p^j}^{\tau_i\eta_j}(z_i, w_j; v). \tag{6.16}$$

Example

In the Hall-Littlewood case, the above trace calculation leads to the particular identities

$$\begin{aligned}\sum_{\mu\nu} t^{|\mu|} P_{\mu/\nu}(x^{(\alpha)}, y^{(\beta)}; t) Q_{\mu/\nu}(w^{(\tau)}, z^{(\eta)}; t) = \prod_{p=1}^{\infty}(1-t^p)^{-1} \prod_{i,j}(1-tx_iw_j)^{-\alpha\tau}\\ \times \prod_{k,l}(1-tx_kz_l)^{-\alpha\eta} \prod_{m,n}(1-ty_mw_n)^{-\beta\tau} \prod_{r,s}(1-ty_rz_s)^{-\beta\eta},\end{aligned}$$

and

$$\begin{aligned}\sum_{\mu\nu} t^{|\mu|} P_{\mu/\nu}(x^{(\alpha)}, y^{(\beta)}; t^{-1}) Q_{\mu/\nu}(w^{(\tau)}, z^{(\eta)}; t^{-1}) = \prod_{p=1}^{\infty}(1-t^p)^{-1} \prod_{i,j}(1-x_iw_j)^{\alpha\tau}\\ \times \prod_{k,l}(1-x_kz_l)^{\alpha\eta} \prod_{m,n}(1-y_mw_n)^{\beta\tau} \prod_{r,s}(1-y_rz_s)^{\beta\eta}.\end{aligned}$$

6.3 Level 1 representations of $U_q(\widehat{sl(2)})$

The aim of this section is to calculate the regularized trace of currents of the free field realization of the level one $U_q(\widehat{sl(2)})$ algebra using symmetric function techniques.

Let us recall the Drinfeld [177] realization of the quantum affine algebra $U_q(\widehat{sl(2)})$. It is generated by the elements $\{E_n^\pm : n \in \mathbb{Z}\}$, $\{H_m : m \in \mathbb{Z}/\{0\}\ \}$, $q^{\pm d}$, $q^{\pm\sqrt{2}H_0}$ and the central elements k, $\gamma^{\pm 1/2}$ which satisfy the following relations

$$\begin{aligned}
[q^{\pm\sqrt{2}H_0}, H_m] = 0, \qquad [H_n, H_m] = \frac{[2n]}{2n}\frac{\gamma^n - \gamma^{-n}}{q-q^{-1}}\,\delta_{n+m,0}, \qquad n \neq 0,\\
q^{\sqrt{2}H_0} E_n^\pm q^{-\sqrt{2}H_0} = q^{\pm 2} E_n^\pm, \qquad [H_n, E_m^\pm] = \pm\sqrt{2}\frac{\gamma^{\mp|n|/2}[2n]}{2n} E_{n+m}^\pm, \qquad n \neq 0,\\
[E_n^+, E_m^-] = \frac{\gamma^{(n-m)/2}\Psi_{n+m} - \gamma^{-(n-m)/2}\Phi_{n+m}}{q-q^{-1}},\\
E_{n+1}^\pm E_m^\pm - q^{\pm 2} E_m^\pm E_{n+1}^\pm = q^{\pm 2} E_n^\pm E_{m+1}^\pm - E_{m+1}^\pm E_n^\pm,\\
q^d E_n^\pm q^{-d} = q^n E_n^\pm, \qquad q^d H_n q^{-d} = q^n H_n,
\end{aligned} \tag{6.17}$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$ is the usual (symmetric) q-number and Ψ_n and Φ_n are the modes of the currents

$$\Psi(z) \equiv \sum_{n\geq 0} \Psi_n z^{-n} = q^{\sqrt{2}H_0} \exp\left(\sqrt{2}(q-q^{-1})\sum_{n>0} H_n z^{-n}\right), \tag{6.18}$$

$$\Phi(z) \equiv \sum_{n\leq 0} \Phi_n z^{-n} = q^{-\sqrt{2}H_0} \exp\left(-\sqrt{2}(q-q^{-1})\sum_{n<0} H_n z^{-n}\right). \tag{6.19}$$

The central element γ takes the value q^k on representations of level k, where k must be a non-negative integer for unitary representations. In the Chevally basis $\{e_i, f_i, t_i\}$ $i = 0, 1$, the relations of this algebra take the familiar form

$$\begin{gathered} t_i t_j = t_j t_i, \qquad [e_i, f_j] = \delta_{ij}\frac{t_i - t_i^{-1}}{q - q^{-1}}, \\ t_i e_i t_i^{-1} = q^2 e_i, \quad t_i f_i t_i^{-1} = q^{-2} e_i, \\ t_i e_j t_i^{-1} = q^{-2} e_j, \quad t_i f_j t_i^{-1} = q^2 f_j, \qquad i \neq j, \end{gathered}$$

where the explicit correspondence between the two sets of generators is given by [190]

$$\begin{gathered} t_0 = \gamma q^{-\sqrt{2}H_0}, \quad e_0 = E_1^- q^{-\sqrt{2}H_0}, \quad f_0 = q^{\sqrt{2}H_0} E_{-1}^+, \\ t_1 = q^{\sqrt{2}H_0}, \qquad e_1 = E_0^+, \qquad f_1 = E_0^-. \end{gathered}$$

Let us recall the Frenkel-Jing construction of level one representations [178]. Suppose we have a deformed Heisenberg algebra with generators $\{\alpha_n : n \in \mathbb{Z} \setminus \{0\}\} \cup \{\alpha_0, Q\}$ satisfying the commutation relations

$$\begin{aligned} [\alpha_n, \alpha_m] &= \frac{[2n][n]}{2n}\,\delta_{n+m,0}, \qquad\qquad n \neq 0, m \neq 0, \\ [Q, \alpha_0] &= i, \end{aligned} \tag{6.20}$$

with all other commutators being zero. Note that as $q \to 1$, we recover the ordinary Heisenberg algebra (3.1). The level 1 free field realization is then [179] given by $H_n = \alpha_n$ along with the currents

$$E^{\pm}(z) \equiv \sum_{n\in\mathbb{Z}} E_n^{\pm} z^{-n-1},$$

where

$$E^{\pm}(z) = \exp\left(\pm\sqrt{2}\sum_{n>0}\frac{q^{\mp n/2}}{[n]}\alpha_{-n}z^n\right)\exp\left(\mp\sqrt{2}\sum_{n>0}\frac{q^{\mp n/2}}{[n]}\alpha_n z^{-n}\right)e^{\pm\sqrt{2}iQ}z^{\pm\sqrt{2}\alpha_0}. \tag{6.21}$$

The currents $\Psi(z)$ and $\Phi(z)$ are just realized by (6.18) and (6.19) with H_n replaced by α_n, and $q^d = q^{-L_0}$ with L_0 given by (6.30). If we define normal ordering $: \ :$ of operators by requiring that annihilation operators α_n, $n > 0$ be moved to the right of creation operators α_n, $n < 0$, then from the definition (6.21) and the commutation relations (6.20) we have

$$
\begin{aligned}
E^{\pm}(z)E^{\pm}(w) &= (z-w)(z-q^{\mp 2}w)\ :E^{\pm}(z)E^{\pm}(w): && (6.22)\\
E^{\pm}(z)E^{\mp}(w) &= \frac{1}{(z-qw)(z-q^{-1}w)}\ :E^{\pm}(z)E^{\mp}(w):\,, && (6.23)
\end{aligned}
$$

where

$$
\begin{aligned}
:E^{\pm}(z)E^{\pm}(w): &= e^{\pm 2\sqrt{2}iQ}(zw)^{\pm\sqrt{2}\alpha_0}\exp\left(\pm\sqrt{2}\sum_{n>0}\frac{q^{\mp n/2}}{[n]}\alpha_{-n}(z^n+w^n)\right)\times\\
&\quad\times\exp\left(\mp\sqrt{2}\sum_{n>0}\frac{q^{\mp n/2}}{[n]}\alpha_n(z^{-n}+w^{-n})\right),\\
:E^{\pm}(z)E^{\mp}(w): &= \left(\frac{z}{w}\right)^{\pm\sqrt{2}\alpha_0}\exp\left(\pm\sqrt{2}\sum_{n>0}\frac{\alpha_{-n}}{[n]}(q^{\mp n/2}z^n-q^{\pm n/2}w^n)\right)\times\\
&\quad\times\exp\left(\mp\sqrt{2}\sum_{n>0}\frac{\alpha_n}{[n]}(q^{\mp n/2}z^{-n}-q^{\pm n/2}w^{-n})\right). \qquad (6.24)
\end{aligned}
$$

The Fock space is spanned by all the monomials in the raising generators of the deformed Heisenberg algebra (6.20)

$$
\alpha_{-n_1}^{k_1}\alpha_{-n_2}^{k_2}\cdots\alpha_{-n_p}^{k_p}|0\rangle\,, \qquad n_1 > n_2 > \cdots > n_p > 0, \qquad (6.25)
$$

with the Fock vacuum $|0\rangle$ being annihilated by all α_n, $n > 0$. We can realize this Fock space in another way, namely on the space of symmetric functions Λ_F, where $F = \mathbb{Q}(q)$. To see this, note that if we set

$$
\alpha_{-n} = \frac{[2n]}{2n}\,p_n, \qquad \alpha_n = [n]\,\frac{\partial}{\partial p_n}, \qquad n > 0, \qquad (6.26)
$$

then the commutation relations (6.20) are fulfilled (the position and momentum operators α_0 and Q effectively decouple from the Fock space, so we can consider them separately). Moreover, we can define an adjoint operator D on the power sums by

$$
D(p_n) = \frac{2n[n]}{[2n]}\,\frac{\partial}{\partial p_n}, \qquad n > 0. \qquad (6.27)
$$

This adjoint operator defines an inner product on the ring Λ_F such that the inner product between two power sums is given by (6.1) where v_λ is given by (6.4) with $\alpha = 2$, $\kappa = 1$. Moreover, this inner product will be consistent with the Fock space inner product on the states (6.25) and the association (6.26). Thus we can consider our Fock space to be spanned by the set of power sums $p_\lambda(x)$, or alternatively, by the functions $Q_\lambda(x;q) \equiv Q_\lambda(x;q,1,2)$.

Consider now the vertex operator

$$
V(z^{(\tau)};w^{(\eta)}) = \exp\left(\tau\sum_{n>0}\frac{[2n]}{2n[n]}p_n(x)z^n\right)\exp\left(\eta\sum_{n>0}\frac{[2n]}{2n[n]}D(p_n(x))w^n\right), \qquad (6.28)
$$

which is a special case of (6.12) in the case where v_λ is given by (6.4) with $\alpha = 2$, $\kappa = 1$. In terms of this operator, we can express the level one currents $E^{\pm}(z)$ as

$$E^{\pm}(z) = V(q^{\mp 1/2} z^{(\pm\beta)}; q^{\mp 1/2} \bar{z}^{(\mp\beta)}) e^{\pm\sqrt{2} iQ} z^{\pm\sqrt{2}\alpha_0}, \qquad \beta = \sqrt{2}, \quad \bar{z} \equiv z^{-1}. \quad (6.29)$$

To calculate the regularized trace of a product of these vertex operators, let

$$L_0 = \bar{L}_0 + \hat{L}_0, \tag{6.30}$$

where

$$\begin{aligned} \bar{L}_0 &= \sum_{n>0} \frac{2n^2}{[2n][n]} \alpha_{-n}\alpha_n = \sum_{n>0} n\, p_n \frac{\partial}{\partial p_n}, \\ \hat{L}_0 &= \tfrac{1}{2}\alpha_0^2, \end{aligned}$$

which has the property that $[L_0, \alpha_{-m}] = m\, \alpha_{-m}$. Equivalently, if $f \in \Lambda_F^n$ is homogeneous of degree n, then $L_0.f = n\, f$. The operator p^{L_0} will be used to regularize the vertex operator traces. Define

$$J_p^{\tau,\eta}(x;y) = \prod_{i,j} (1 - pqx_iy_j)^{-\tau\eta/\alpha} (1 - pq^{-1}x_iy_j)^{-\tau\eta/\alpha}. \tag{6.31}$$

Then from section 6.2, we have that the regularized trace of the product of two vertex operators of the form (6.28) over the space Λ_F is given by

$$\mathrm{tr}\left(p^{\bar{L}_0} V(x^{(\tau)}; y^{(\eta)}) V(z^{(\xi)}; w^{(\zeta)})\right) = \prod_{j=1}^{\infty} \frac{1}{(1-p^j)} J_{p^j}^{\tau,\xi}(x;z)\, J_{p^j}^{\tau,\zeta}(x;w)\, J_{p^j}^{\eta,\xi}(y;z)\, J_{p^j}^{\eta,\zeta}(y;w). \tag{6.32}$$

From this it is clear how to go about calculating the regularized trace of $E^+(z)E^-(w)$. Write

$$\mathrm{tr}\left(p^{L_0} E^+(z) E^-(w)\right) = T \cdot M,$$

where M is the trace over the momentum lattice:

$$M = \mathrm{tr}\left(p^{\hat{L}_0} (\frac{z}{w})^{\sqrt{2}\alpha_0}\right),$$

and T is the trace over the space Λ_F:

$$T = \mathrm{tr}\left(p^{\bar{L}_0}\, V(q^{-1/2} z^{(\sqrt{2})}; q^{-1/2} \bar{z}^{(-\sqrt{2})})\, V(q^{1/2} w^{(-\sqrt{2})}; q^{1/2} \bar{w}^{(\sqrt{2})})\right).$$

For the case of $U_q(\widehat{sl(2)})$, the momentum lattice is just $\mathbb{Z}.\sqrt{2}$ so that

$$M = \sum_{n\in\mathbb{Z}} p^{n^2} \left(\frac{z}{w}\right)^{2n} = \prod_{n=1}^{\infty} (1 - p^{2n})(1 + \tfrac{z}{w} p^{2n-1})(1 + \tfrac{w}{z} p^{2n-1}),$$

where we have used the Jacobi triple product identity (4.10). To calculate the trace T, we first normal order the vertex operators using (6.23) and (6.24) obtaining

$$
\begin{aligned}
T &= \frac{1}{(z-qw)(z-q^{-1}w)}\mathrm{tr}\left(p^{\bar{L}_0}\ V(q^{-1/2}z^{(\sqrt{2})}, q^{1/2}w^{(-\sqrt{2})}; q^{-1/2}\bar{z}^{(-\sqrt{2})}, q^{1/2}\bar{w}^{(\sqrt{2})})\right) \\
&= \prod_{j=1}^{\infty}\frac{1}{(1-p^j)}J_{p^j}^{\sqrt{2},-\sqrt{2}}(q^{-1/2}z; q^{-1/2}\bar{z})J_{p^j}^{\sqrt{2},\sqrt{2}}(q^{-1/2}z; q^{1/2}\bar{w}) \times \\
&\qquad \times J_{p^j}^{-\sqrt{2},-\sqrt{2}}(q^{1/2}w; q^{-1/2}\bar{z})J_{p^j}^{-\sqrt{2},\sqrt{2}}(q^{1/2}w; q^{1/2}\bar{w}) \\
&= \prod_{j=1}^{\infty}\frac{1}{(1-p^j)} \times \frac{(1-p^j)(1-q^2p^j)(1-q^{-2}p^j)}{(1-qp^j\frac{z}{w})(1-q^{-1}p^j\frac{z}{w})(1-qp^j\frac{w}{z})(1-q^{-1}p^j\frac{w}{z})},
\end{aligned}
$$

where we have used (6.31) and (6.32) with $\alpha = 2$. Combining everything together we have

$$
\begin{aligned}
\mathrm{tr}\left(p^{L_0}E^+(z)E^-(w)\right) &= \frac{1}{(z-qw)(z-q^{-1}w)}\prod_{n=1}^{\infty}(1-p^{2n})(1+\tfrac{z}{w}p^{2n-1})(1+\tfrac{w}{z}p^{2n-1}) \times \\
&\times \prod_{j=1}^{\infty}\frac{1}{(1-p^j)}\ \frac{(1-p^j)(1-q^2p^j)(1-q^{-2}p^j)}{(1-qp^j\frac{z}{w})(1-q^{-1}p^j\frac{z}{w})(1-qp^j\frac{w}{z})(1-q^{-1}p^j\frac{w}{z})},
\end{aligned}
$$

and we recover the result of Jing [235], who derived his result by a brute force calculation using various combinatorial identities. We can extend this result to a string of n vertex operators and after combining the contributions from the normal ordering, the trace over the momentum lattice, and the trace over Λ_F, we obtain

$$
\begin{aligned}
&\mathrm{tr}\left(p^{L_0}E^+(z_1)\cdots E^+(z_n)E^-(w_1)\cdots E^-(w_n)\right) = \\
&\prod_{i<j}(z_i-z_j)(z_i-q^{-2}z_j)(w_i-w_j)(w_i-q^2w_j)\prod_{l,m}(z_l-qw_m)^{-1}(z_l-q^{-1}w_m)^{-1} \times \\
&\times \prod_{r=1}^{\infty}(1-p^{2r})\left(1+\frac{z_1\cdots z_n}{w_1\cdots w_n}p^{2r-1}\right)\left(1+\frac{w_1\cdots w_n}{z_1\cdots z_n}p^{2r-1}\right) \times \\
&\times \prod_{j=1}^{\infty}\prod_{a,b}\left(1-p^j\frac{z_a}{z_b}\right)\left(1-p^jq^{-2}\frac{z_a}{z_b}\right)\left(1-p^j\frac{w_a}{w_b}\right)\left(1-p^jq^2\frac{w_a}{w_b}\right) \times \\
&\times \prod_{c,d}\left(1-p^j\frac{z_c}{w_d}\right)^{-1}\left(1-p^j\frac{w_c}{z_d}\right)^{-1},
\end{aligned}
$$

where l, m, i, j, a, b, c, d run from $1,\ldots,n$. It is not clear whether this result could also have been obtained by brute force, as was the case for $n = 1$.

6.4 Level k representations of $U_q(\widehat{sl(2)})$

We shall now extend the results of the previous section to the case of level k, where k is an arbitrary (complex) number, not equal to 0 or -2. There are several different realizations of the level k $U_q(\widehat{sl(2)})$ algebra [183], all requiring three commuting sets of deformed Heisenberg generators; the one we shall use is that given by Matsuo [182]. Let $\{\alpha_n, \bar{\alpha}_n, \beta_n \,:\, n \in \mathbb{Z}\}$ be generators satisfying the commutation relations

$$\begin{aligned} [\alpha_n, \alpha_m] &= \frac{[2n][kn]}{n}\delta_{n+m,0}, \\ [\bar{\alpha}_n, \bar{\alpha}_m] &= -\frac{[2n][kn]}{n}\delta_{n+m,0}, \\ [\beta_n, \beta_m] &= \frac{[2n][(k+2)n]}{n}\delta_{n+m,0}, \end{aligned} \tag{6.33}$$

together with the position and momentum operators

$$[Q_\alpha, \alpha_0] = i, \qquad [Q_{\bar{\alpha}}, \bar{\alpha}_0] = -i, \qquad [Q_\beta, \beta_0] = i,$$

with all other commutation relations being zero. Then the level k free field realization is given by the generators $H_n = \frac{1}{\sqrt{2}}\alpha_n$, along with the currents

$$E^\pm(z) = \frac{\pm 1}{(q-q^{-1})z}\left(A^\pm(z) - B^\pm(z)\right), \tag{6.34}$$

where

$$\begin{aligned} A^\pm(z) = e^{\pm 2(Q_\alpha + Q_{\bar{\alpha}})}\, z^{\pm(\alpha_0+\bar{\alpha}_0)/k} q^{-(\alpha_0 \pm \beta_0)/2} \exp\left(\pm \sum_{m>0} \frac{(\alpha_{-m} + \bar{\alpha}_{-m})}{[km]} q^{\mp km/2} z^m\right) \times \\ \times \exp\left(\sum_{m>0}\left(\mp \frac{(\alpha_m + \bar{\alpha}_m)}{[km]} q^{\mp km/2} - (q-q^{-1}) q^{\pm(k+2)/2} \frac{[m]}{[2m]}\bar{\alpha}_m \right.\right. \\ \left.\left. \mp (q-q^{-1}) q^{\pm k/2}\frac{[m]}{[2m]}\beta_m\right) z^{-m}\right), \end{aligned}$$

and

$$\begin{aligned} B^\pm(z) = e^{\pm 2(Q_\alpha + Q_{\bar{\alpha}})}\, z^{\pm(\alpha_0+\bar{\alpha}_0)/k} q^{(\alpha_0 \pm \beta_0)/2} \exp\left(\sum_{m>0}\left(\pm \frac{(\alpha_{-m} + \bar{\alpha}_{-m})}{[km]} q^{\mp km/2} \right.\right. \\ \left.\left. + (q-q^{-1}) q^{\pm(k+2)/2}\frac{[m]}{[2m]}\bar{\alpha}_{-m} \pm (q-q^{-1}) q^{\pm k/2}\frac{[m]}{[2m]}\beta_{-m}\right) z^m\right) \times \\ \times \exp\left(\mp \sum_{m>0} \frac{(\alpha_m + \bar{\alpha}_m)}{[km]} q^{\mp km/2} z^{-m}\right) \end{aligned},$$

while $q^d = q^{-L_0}$ with L_0 given by (6.45). The Fock space $\mathcal{F}$ of the level k representation of $U_q(\widehat{sl(2)})$ is spanned by the monomials

$$\alpha_{-n_1}^{b_1} \cdots \alpha_{-n_p}^{b_p} \beta_{-m_1}^{c_1} \cdots \beta_{-m_r}^{c_r} \bar{\alpha}_{-l_1}^{d_1} \cdots \bar{\alpha}_{-l_s}^{d_s} |0\rangle,$$

where $n_1 > \cdots > n_p > 0$, $m_1 > \cdots > m_r > 0$ and $l_1 > \cdots > l_s > 0$. Now the Fock space $\mathcal{F}$ is isomorphic to the space $\mathcal{F}_\alpha \otimes \mathcal{F}_\beta \otimes \mathcal{F}_{\bar{\alpha}}$ where $\mathcal{F}_\alpha$, $\mathcal{F}_\beta$ and $\mathcal{F}_{\bar{\alpha}}$ are the Fock spaces associated with the Heisenberg generators $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\bar{\alpha}_n\}$ respectively. Thus, when we come to do the trace over the total Fock space $\mathcal{F}$, we can separate it into traces over each of the individual Fock spaces. We will now show that we can equally well consider each of the above Fock spaces to be spanned by the functions $P_\lambda(x; q, \kappa, \alpha)$ for appropriate choices of the parameters q, κ, and α.

In order to treat all these cases together, define a set of Heisenberg generators C_n with the commutations relations

$$[C_n, C_m] = \frac{[2n][\kappa n]}{\alpha n}\, \delta_{n+m,0}, \tag{6.35}$$

together with position and momentum operator Q_C and C_0 satisfying

$$[Q_C, C_0] = \alpha\, i.$$

The generators α_n, β_n and $\bar{\alpha}_n$ correspond to the special cases $(\kappa, \alpha) = (k, 1)$, $(k+2, 1)$ and $(k, -1)$ respectively of the generators C_n. If we now set

$$C_n = \frac{[2n]}{\alpha n}\, D(p_n), \qquad D(p_n) \equiv \frac{\alpha n[\kappa n]}{[2n]}\, \frac{\partial}{\partial p_n}, \qquad n > 0, \tag{6.36}$$

$$C_{-n} = \frac{[2n]}{\alpha n}\, p_n, \qquad n > 0, \tag{6.37}$$

then it is seen that, not only are the commutation relations (6.35) satified, but that the above definition of the adjoint operator D correctly reproduces the scalar product (6.1) on the space Λ_F.

Our present goal is to calculate the regularized trace of the operator $E^+(z)E^-(w)$. From (6.34) we see that

$$E^+(z)E^-(w) = \frac{-1}{(q-q^{-1})^2 zw}\, \Big(A^+(z)A^-(w) - A^+(z)B^-(w) \\ - B^+(z)A^-(w) + B^+(z)B^-(w)\Big),$$

so we must calculate the traces of each of the terms in the above equation. First of all, we shall put the above terms in normal ordered form, moving all annihilation operators to the right of the creation operators, and all C_0 operators to the right of the operators Q_C. The result is

$$\begin{aligned}
A^+(z)A^-(w) &= q^{-1}\left(\frac{z-q^{k+2}w}{z-q^k w}\right) : A^+(z)A^-(w) :, \\
A^+(z)B^-(w) &= q^{-1}\frac{(z-q^{k+2}w)(z-q^{-k-2}w)(z-q^{-k-1}w)}{(z-q^k w)(z-q^{-k}w)(z-q^{k-1}w)} : A^+(z)B^-(w) :, \\
B^+(z)A^-(w) &= q : B^+(z)A^-(w) :, \\
B^+(z)B^-(w) &= q\left(\frac{z-q^{-k}w}{z-q^{-k-2}w}\right) : B^+(z)B^-(w) : .
\end{aligned} \tag{6.38}$$

In order to describe the normal ordered expressions occuring in (6.38) in terms of the generalized vertex operators (6.12), we need to introduce some new notation. For the particular choice of v_λ given by (6.4) the vertex operators (6.12) have the form

$$V_C(z^{(\tau)}; w^{(\eta)}) = \exp\left(\tau \sum_{n>0} \frac{p_n(x)}{\alpha n[\kappa n]}\, z^n\right)\, \exp\left(\eta \sum_{n>0} \frac{D(p_n(x))}{\alpha n[\kappa n]}\, w^{-n}\right). \tag{6.39}$$

This operator will account for terms of the form $\exp(\sum_{n>0} \frac{p_n(x)}{[\kappa n]}\ z^n)$, but looking at the explicit expressions for $A^{\pm}(z)$ and $B^{\pm}(z)$, we see that factors of the form $\exp((q-q^{-1})\sum_{n>0} \frac{[n]}{\alpha n} p_n(x)\ z^n)$ will also arise in the normal-ordered expressions. How can we express them in terms of vertex operators of the form (6.39) ? By looking at the expansion

$$(q-q^{-1})\ \frac{[n][\kappa n]}{[2n]}\ z^n = \sum_{s=0}^{\infty} \Big[(q^{4s+\kappa+1}z)^n + (q^{4s+3-\kappa}z)^n - (q^{4s+3+\kappa}z)^n - (q^{4s-\kappa+1}z)^n \Big],$$

we define new variables $\widehat{z}$ through the relation

$$\begin{aligned} p_n(\widehat{z}) &= p_n(\{q^{4s+\kappa+1}z\}_{s=0}^{\infty}) + p_n(\{q^{4s+3-\kappa}z\}_{s=0}^{\infty}) \\ &\quad - p_n(\{q^{4s+3+\kappa}z\}_{s=0}^{\infty}) - p_n(\{q^{4s-\kappa+1}z\}_{s=0}^{\infty}). \end{aligned} \tag{6.40}$$

With the introduction of this notation, we can write the above normal-ordered products as

$$:A^{+}(z)A^{-}(w): = T_1\ M_1, \qquad :A^{+}(z)B^{-}(w): = T_2\ M_2,$$
$$:B^{+}(z)A^{-}(w): = T_3\ M_3, \qquad :B^{+}(z)B^{-}(w): = T_4\ M_4,$$

where

$$M_1 = \left(\frac{z}{w}\right)^{(\alpha_0+\bar{\alpha}_0)/k} q^{-\bar{\alpha}_0}, \qquad M_2 = \left(\frac{z}{w}\right)^{(\alpha_0+\bar{\alpha}_0)/k} q^{-\beta_0},$$
$$M_3 = \left(\frac{z}{w}\right)^{(\alpha_0+\bar{\alpha}_0)/k} q^{\beta_0}, \qquad M_4 = \left(\frac{z}{w}\right)^{(\alpha_0+\bar{\alpha}_0)/k} q^{\bar{\alpha}_0},$$

and

$$\begin{aligned} T_1 &= V_\alpha(q^{-k/2}z, q^{k/2}w^{(-1)}; q^{-k/2}\bar{z}^{(-1)}, q^{k/2}\bar{w})\ V_\beta(0; q^{k/2}\widehat{\bar{z}}^{(-1)}, q^{-k/2}\widehat{\bar{w}}) \\ &\quad V_{\bar{\alpha}}(q^{-k/2}z, q^{k/2}w^{(-1)}; q^{(k+2)/2}\widehat{\bar{z}}^{(-1)}, q^{(-k-2)/2}\widehat{\bar{w}}^{(-1)}, q^{-k/2}\bar{z}^{(-1)}, q^{k/2}\bar{w}), \end{aligned} \tag{6.41}$$

$$\begin{aligned} T_2 &= V_\alpha(q^{-k/2}z, q^{k/2}w^{(-1)}; q^{-k/2}\bar{z}^{(-1)}, q^{k/2}\bar{w})\ V_\beta(q^{-k/2}\widehat{w}^{(-1)}; q^{k/2}\widehat{\bar{z}}^{(-1)}) \\ &\quad V_{\bar{\alpha}}(q^{-k/2}z, q^{k/2}w^{(-1)}, q^{-(k+2)/2}\widehat{w}; q^{-k/2}\bar{z}^{(-1)}, q^{k/2}\bar{w}, q^{(k+2)/2}\widehat{\bar{z}}^{(-1)}), \end{aligned} \tag{6.42}$$

$$\begin{aligned} T_3 &= V_\alpha(q^{-k/2}z, q^{k/2}w^{(-1)}; q^{-k/2}\bar{z}^{(-1)}, q^{k/2}\bar{w})\ V_\beta(q^{k/2}\widehat{z}; q^{-k/2}\widehat{\bar{w}}) \\ &\quad V_{\bar{\alpha}}(q^{-k/2}z, q^{k/2}w^{(-1)}, q^{(k+2)/2}\widehat{z}; q^{-k/2}\bar{z}^{(-1)}, q^{k/2}\bar{w}, q^{-(k+2)/2}\widehat{\bar{w}}^{(-1)}), \end{aligned} \tag{6.43}$$

$$\begin{aligned} T_4 &= V_\alpha(q^{-k/2}z, q^{k/2}w^{(-1)}; q^{-k/2}\bar{z}^{(-1)}, q^{k/2}\bar{w})\ V_\beta(q^{k/2}\widehat{z}, q^{-k/2}\widehat{w}^{(-1)}; 0) \\ &\quad V_{\bar{\alpha}}(q^{-k/2}z, q^{k/2}w^{(-1)}, q^{(k+2)/2}\widehat{z}, q^{(-k-2)/2}\widehat{w}^{(-1)}; q^{-k/2}\bar{z}^{(-1)}, q^{k/2}\bar{w}). \end{aligned} \tag{6.44}$$

Due to the structure of the Fock space $\mathcal{F}$, when we calculate the trace of T_i, we can calculate the traces on the separate spaces $\mathcal{F}_C$.

Let us first consider the momentum traces. Note that due to the presence of the momentum "raising" and "lowering" operators $\exp(\pm 2(Q_\alpha + Q_{\bar{\alpha}}))$ in the definitions (6.34) of the currents $E^{\pm}(z)$, the eigenvalues of α_0 and $\bar{\alpha}_0$ are constrained in such a way that $\alpha_0 + \bar{\alpha}_0$ must be a constant, which we take to be zero. The momentum lattice of the $\bar{\alpha}_0$ and β_0 operators is just the lattice $2\mathbb{Z}$. In order to calculate the trace of the above vertex operators, we will need to regularize them with the operator p^{L_0}, where $L_0 = \bar{L}_0 + \hat{L}_0$ with[1]

$$\begin{aligned}
\bar{L}_0 &= \sum_{n>0} \left(\frac{n^2}{[2n][kn]} \left(\alpha_{-n}\alpha_n - \bar{\alpha}_{-n}\bar{\alpha}_n\right) + \frac{n^2}{[2n][(k+2)n]} \beta_{-n}\beta_n \right), \\
\hat{L}_0 &= \frac{1}{4k}(\alpha_0^2 + \bar{\alpha}_0^2) + \frac{1}{4(k+2)}(\beta_0^2 + 2\beta_0),
\end{aligned} \tag{6.45}$$

which has the property that $[L_0, C_{-n}] = n\, C_{-n}$. Thus for the momentum part of the trace we have from the Jacobi triple product identity (4.10)

$$\begin{aligned}
&\mathrm{tr}\left(p^{\hat{L}_0} M_1\right) = \mathrm{tr}\left(p^{\hat{L}_0} M_4\right) = \left(\sum_{n\in\mathbb{Z}} p^{2n^2/k} q^{2n}\right)\left(\sum_{m\in\mathbb{Z}} p^{(m^2+m)/(k+2)}\right) \\
&= 2\prod_{j=1}^{\infty}(1-p^{4j/k})(1-q^2p^{(4j-2)/k})(1-q^{-2}p^{(4j-2)/k})(1-p^{4j/(k+2)})(1+p^{2j/(k+2)}),
\end{aligned} \tag{6.46}$$

while

$$\begin{aligned}
&\mathrm{tr}\left(p^{\hat{L}_0} M_2\right) = \left(\sum_{n\in\mathbb{Z}} p^{2n^2/k}\right)\left(\sum_{m\in\mathbb{Z}} p^{(m^2+m)/(k+2)} q^{-2m}\right) \\
&= \prod_{j=1}^{\infty}(1-p^{4j/k})(1+p^{(4j-2)/k})^2(1-p^{2j/(k+2)})(1-q^{-2}p^{(2j-2)/(k+2)})(1-q^2p^{2j/(k+2)}),
\end{aligned} \tag{6.47}$$

with the same result for the regularized trace of M_3 but with q replaced by q^{-1}.

The regularized traces of the operators T_i defined above are calculated in Appendix D with the surprisingly simple results

$$\begin{aligned}
\mathrm{tr}\left(p^{\bar{L}_0}\, T_1\right) &= \prod_{j=1}^{\infty} \frac{1}{(1-p^j)(1-q^2p_j)(1-q^{-2}p^j)} \left(\frac{w - q^{-k-2}z}{w - q^{-k}p^j z}\right)\left(\frac{z - q^{k+2}p^j w}{z - q^k p^j w}\right), \\
\mathrm{tr}\left(p^{\bar{L}_0}\, T_2\right) &= \mathrm{tr}\left(p^{\bar{L}_0}\, T_3\right) = \prod_{j=1}^{\infty} \frac{1}{(1-p^j)(1-q^2p_j)(1-q^{-2}p^j)}, \\
\mathrm{tr}\left(p^{\bar{L}_0}\, T_4\right) &= \prod_{j=1}^{\infty} \frac{1}{(1-p^j)(1-q^2p_j)(1-q^{-2}p^j)} \left(\frac{w - q^{k+2}z}{w - q^k p^j z}\right)\left(\frac{z - q^{-k-2}p^j w}{z - q^{-k} p^j w}\right).
\end{aligned} \tag{6.48}$$

Finally, by combining together (6.46), (6.47) and (6.48), we obtain the result for the regularized trace of the operator $E^+(z)E^-(w)$. One can now go on, in theory, to calculate the trace of a string of currents $E^+(z_1)\cdots E^+(z_n)E^-(w_1)\cdots E^-(w_n)$. Due to the fact that each of the currents is the sum of two terms (see (6.34)), the above product is a sum of 2^n terms, each of whose trace can be calculated by the procedure outlined above.

[1]The constraint $\alpha_0 + \bar{\alpha}_0 = 0$ means that we have to define $\hat{L}_0$ in a slightly different way to Matsuo [182] in order to ensure convergence of the momentum traces.

Chapter 7

Conclusion

7.1 Summary

This thesis has dealt with various problems in the theory of symmetric functions and the representation theory of superconformal and quantum affine algebras.

The standard lore on symmetric functions was reviewed in Chapter 2 and a number of new results were obtained. In particular, one of Littlewood's formulae relating inner and outer products of S-functions was reproven via generating function techniques, and various generalizations derived. The inner product of two Hall-Littlewood functions was defined and Littlewood's formula was generalized to this case as well. Finally, the transformations connecting the Macdonald's functions $Q_\lambda(q,t)$, $Q_\mu(q^k,t)$ and $Q_\nu(q,t^k)$ were examined and some general results obtained for functions associated with one-row partitions, using some very non-trivial identities involving basic hypergeometric series.

S-functions with a replicated argument were introduced in Chapter 3 as an aid in determining the nature of the functions dual to compound S-functions under an induced inner product. This was then extended to the Hall-Littlewood case. Schur (Hall-Littlewood) functions with a q-replicated argument were then introduced and their relationship to Macdonald's functions $P_\lambda(q,q^\alpha)$ (respectively $P_\lambda(q,t)$) investigated. Various bases for the ring of symmetric functions Λ_F, $F = \mathbb{Q}(q,t)$ were given and the transition matrices between them listed.

Branching rules for the $N=1$ and $N=2$ superconformal algebras were studied in Chapter 4. Using various infinite product identities, certain winding subalgebra branching rules were calculated in the case where these decompositions were finite and/or multiplicity-free. Tensor product decompositions between certain irreducible representations of the $N=2$ superconformal algebra were also examined using similar techniques, concentrating again on the case where they were finite and/or multiplicity-free.

Chapter 5 saw the boson-fermion correspondence for free and neutral free fermions applied to various symmetric function calculations: the multiplication and skewing of S- and Q-functions by power sums; outer multiplication of S-functions; outer plethysm of S-functions. Jing's generalized boson-fermion correspondence was then applied to the problem of decomposing Hall-Littlewood functions in terms of S-

functions. The operation of outer plethysm was then defined on the ring of Hall-Littlewood functions and the techniques developed earlier generalized to enable their calculation.

Traces of products of currents of the quantum affine algebra $U_q(\widehat{sl(2)})$ were calculated in Chapter 6. We defined a basis of symmetric functions, through which we were able to realize the generators of $U_q(\widehat{sl(2)})$ and calculate traces of products of the currents using symmetric function techniques.

7.2 Outlook

From the work done in this thesis, several problems suggest themselves for further research. The elements of the transition matrices $M_{\lambda\mu}(a,b;k,l)$ between the Macdonald functions $P_\lambda(q^a,t^b)$ and $P_\mu(q^k,t^l)$ appear to be rational functions of polynomials in q and t with integer coefficients. Moreover, the elements of the matrices $M_{\lambda\mu}(1,1;2,1)$ and $M_{\lambda\mu}(1,2;1,1)$ appear to factorize into simple factors of the form $(1-q^n t^m)$. The reason why this is so is not at all apparent, and hence deserves further attention.

The extension of the branching rules considered in Chapter 4 to the $N=3$ and $N=4$ superconformal algebras is also an open problem. In the $N=4$ case, new identities may be needed to tackle the relevant character formulae. The $N=3$ case is not tractable at all at the moment due to the fact that the determinant formulae are still unproven, and the character formulae unknown.

By considering other different specializations of Kerov's symmetric functions, trace calculations in representations of the level k quantum affine algebra $U_q(\widehat{sl(N)})$ should be feasible [184]. The extension of this method to other (quantum) affine algebras and superalgebras should also be practicable, provided the relevant vertex operator realizations [236, 237] of those algebras are on hand. In a similar vein, Kerov's functions should also have applications in describing the principal realization of algebras such as $\widehat{gl(n)}$ [106].

The procedure we have developed for calculating outer plethysms of the form $s_\lambda(x^r) = s_\lambda(x) \otimes p_r(x)$ was particularly suited for deriving explicit expressions for the plethysms $s_\lambda \otimes s_{(2)}$ and $s_\lambda \otimes s_{(1^2)}$. The expressions for more general plethysms are not quite as pretty due to the presence of roots of unity in the expansion $s_\lambda(x^r) = \sum_\mu A_{\lambda\mu} s_\mu(x)$, and so the fact that the plethysm coefficients $a^\lambda_{\mu\nu}$ are non-negative integers is no longer obvious. A problem to look at therefore, is to examine the nature of the coefficients $A_{\lambda\mu}$ and determine whether they are always integers.

Turning to Hall-Littlewood functions, the conjecture (5.75) concerning p-hook Hall-Littlewood functions needs to be (dis)proven. Whether this is achievable by some Hall-Littlewood analogue of the Giambelli formula (2.12) remains to be seen. The extension of the method developed in section 5.3 for multiplying S-functions also should be generalizable to the Hall-Littlewood case, although the resulting non-standard functions will need to be modified according to the fairly nasty rules (2.36). The introduction of outer plethysms of Hall-Littlewood functions also opens up new problems. For example, is there an interpretation of this operation in terms of invariant matrices of matrix representations of some group, thus generalizing Littlewood's original definition of S-function plethysm ? If the answer were yes, then it should be possible to see whether formulae such as

$$s_\lambda \otimes (s_\mu \cdot s_\nu) = (s_\lambda \otimes s_\mu) \cdot (s_\lambda \otimes s_\nu),$$

which cannot easily be deduced from the substitution definition (5.35), also hold in the Hall-Littlewood case.

Finally, let us note that there exist vertex operator realizations ("boson-fermion" correspondences) for S, Q, and Hall-Littlewood functions, but not for Macdonald's functions. The question arises then, what are the constraints on the numbers v_n such that Kerov's symmetric functions have a vertex operator realization ? Looking at it in the other direction, suppose we are given the vertex operator $V(z;z^{-1})$ defined by (6.12) which has a mode expansion $V(z;z^{-1}) = \sum_{n\in\mathbb{Z}} V_n z^n$. Then for a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$, one can define symmetric functions

$$\widehat{P}(x;v) = V_{-\lambda_1} \cdots V_{-\lambda_p} \cdot 1,$$

and ask the question: for what type of numbers v_n, does this definition coincide with the Gram-Schmidt type definition (6.2), (6.3) ?

Appendix A: Product identities

In this appendix we shall prove the identity (4.32) which provides a paradigm for the proof of several other identities used in Chapter 4.

Consider x as being a fixed complex number with $\mid x \mid < 1$ and consider the following function of y

$$f(y) = \prod_{n=1}^{\infty} \frac{(1-x^{3n})^2(1+yx^{3n-3/2})(1+y^{-1}x^{3n-3/2})}{(1+yx^{3n-1/2})(1+y^{-1}x^{3n-1/2})(1+yx^{3n-5/2})(1+y^{-1}x^{3n-5/2})}.$$

As a function of y, this function has simple poles at the points $y = -x^{\pm(3m-1/2)}, m \in \mathbb{Z}_+$ and $y = -x^{\pm(3m-5/2)}, m \in \mathbb{Z}_+$. Let us calculate the residues of these simple poles. First note that for $m > 0$ the formulae

$$\begin{aligned}\prod_{n=1}^{\infty} a_n &= \prod_{n=1}^{\infty} a_{n+m} \prod_{n=1}^{m} a_n \\ &= \frac{\prod_{n=1}^{\infty} a_{n-m}}{\prod_{n=1}^{m} a_{n-m}},\end{aligned}$$

hold provided $a_{n-m} \neq 0$ for $1 \leq n \leq m$. Thus

$$\begin{aligned}\mathrm{Res}(f, -x^{3m-1/2}) &= \lim_{y \to -x^{3m-1/2}} (y + x^{3m-1/2}) f(y) \\ &= -x^{3m-1/2} \prod_{\substack{n=1 \\ n \neq m}} \frac{1}{1-x^{3(n-m)}} \times \\ &\quad \times \prod_{n=1}^{\infty} \frac{(1-x^{3n})^2(1-x^{3(n-m)-1})(1-x^{3(n+m)-2})}{(1-x^{3(n+m)-1})(1-x^{3(n+m)-3})(1-x^{3(n-m)-2})} \\ &= -x^{3m-1/2} \prod_{n=1}^{m} \frac{(1-x^{3n-3m-1})(1-x^{3n-1})}{(1-x^{3n-2})(1-x^{3n-3m-2})} \prod_{n=1}^{m-1} \frac{(1-x^{3n})}{(1-x^{3n-3m})}.\end{aligned}$$

Noting that

$$\prod_{n=1}^{m} a_n = \prod_{n=1}^{m} a_{m+1-n},$$

we see that

$$\begin{aligned}\mathrm{Res}(f, -x^{3m-1/2}) &= -x^{3m-1/2} \prod_{n=1}^{m} \frac{(1-x^{2-3n})(1-x^{3n-1})}{(1-x^{3n-2})(1-x^{1-3n})} \prod_{n=1}^{m-1} \frac{1-x^{3n}}{1-x^{-3n}} \\ &= -x^{3m-1/2} \prod_{n=1}^{m} x^{-(3n-2)+3n-1} \prod_{n=1}^{m-1} -x^{3n} \\ &= (-1)^m x^{(3m^2+5m-1)/2}.\end{aligned}$$

An easy way of calculating $Res(f, -x^{-3m+1/2})$ is to use the symmetry $f(y) = f(y^{-1})$. For a function f with this property, denote by R the residue of f at a.

$$R = \lim_{y \to a} (y-a) f(y).$$

Then

$$\begin{aligned}
\lim_{y\to a^{-1}}(y-a^{-1})f(y) &= \lim_{y\to a^{-1}} -\frac{y}{a}(y^{-1}-a)f(y) \\
&= \lim_{z\to a}\frac{-1}{az}(z-a)f(z) \qquad z\equiv y^{-1} \\
&= \frac{-1}{a^2}R.
\end{aligned}$$

Hence

$$\mathrm{Res}(f,-x^{-3m+1/2}) = (-1)^{m-1}x^{(3m^2-7m+1)/2}.$$

Using similar techniques, we find that

$$\begin{aligned}
\mathrm{Res}(f,-x^{3m-5/2}) &= (-1)^{m}x^{(3m^2+m-3)/2}, \\
\mathrm{Res}(f,-x^{-3m+5/2}) &= (-1)^{m-1}x^{(3m^2-11m+7)/2}.
\end{aligned}$$

Now the function $g(y)$ defined by

$$\begin{aligned}
g(y) = &\sum_{n=1}^{\infty}(-1)^{n-1}x^{(3n^2-7n+1)/2}\frac{1}{y+x^{-3n+1/2}} - \sum_{n=1}^{\infty}(-1)^{n-1}x^{(3n^2+5n-1)/2}\frac{1}{y+x^{3n-1/2}} \\
&+\sum_{n=1}^{\infty}(-1)^{n-1}x^{(3n^2-11n+7)/2}\frac{1}{y+x^{-3n+5/2}} - \sum_{n=1}^{\infty}(-1)^{n-1}x^{(3n^2+n-3)/2}\frac{1}{y+x^{3n-5/2}} \\
&= \sum_{n\in\mathbb{Z}}x^{6n^2+5n+1}\left(\frac{x^{-6n-5/2}}{y+x^{-6n-5/2}} - \frac{x^{6n+5/2}}{y+x^{6n+5/2}}\right) \\
&\quad+\sum_{n\in\mathbb{Z}}x^{6n^2+n}\left(\frac{x^{-6n-1/2}}{y+x^{-6n-1/2}} - \frac{x^{6n+1/2}}{y+x^{6n+1/2}}\right),
\end{aligned}$$

obviously has the same poles as the function $f(y)$ and the same residue at those poles. Hence the function $F(y) = f(y) - g(y)$ is entire i.e. it has no essential singularities in the complex plane (except possibly at infinity). But $\lim_{y\to 0} F(y) = 0 = \lim_{y\to\infty} F(y)$ Hence, by Liouville's theorem, $F(y) = const.$ i.e. $F(y) = F(0) = 0$ (here we have defined $F(0) = 0$ due to the fact that F has a removable singularity at $y = 0$). Thus $f(y) = g(y)$ and the identity (4.32) is proved. The technique outlined above can be used to prove the identities (4.33) and (4.42) occuring on Chapter 4.

Appendix B: Determinant formulae for $h_n(x^2)$

There is an interesting relation between $h_n(x^2)$, the elementary Q-functions $q_n(x)$ and the functions $h_n(x^{(2)}) \equiv h_n(x,x)$, which we would like to point out. Here

$$h_n(x^{(\alpha)}) = \sum_{\lambda \vdash n} \left(\begin{array}{c} \alpha \\ \lambda' \end{array} \right) s_\lambda(x)$$

denotes a replicated S-function(see Chapter 3). For the particular case $\alpha = 2$, this can be simply written as

$$h_n(x^{(2)}) = \sum_{2m+p=n} (p+1) s_{(m+p,m)}(x).$$

Now,

$$\sum_{n=0}^{\infty} h_n(x^{(2)}) z^n = \exp\left(\left(\sum_{n\, even} + \sum_{n\, odd}\right) \frac{2}{n} p_n(x) z^n\right) = \left(\sum_{k=0}^{\infty} h_k(x^2) z^{2k}\right)\left(\sum_{l=0}^{\infty} q_l(x) z^l\right).$$

Hence

$$h_{2n}(x^{(2)}) = \sum_{j=0}^{n} h_{n-j}(x^2)\, q_{2j}(x), \qquad h_{2n+1}(x^{(2)}) = \sum_{j=0}^{n} h_{n-j}(x^2)\, q_{2j+1}(x).$$

Thus, by using Cramer's rule, we can write this as

$$q_{2n}(x) \;=\; (-1)^n \left| \begin{array}{cccccc} 1 & 1 & 0 & \cdots & 0 & 0 \\ h_2(x^{(2)}) & h_1(x^2) & 1 & 0 & \cdots & 0 \\ h_4(x^{(2)}) & h_2(x^2) & h_1(x^2) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ h_{2n-2}(x^{(2)}) & h_{n-1}(x^2) & h_{n-2}(x^2) & \cdots & h_1(x^2) & 1 \\ h_{2n}(x^{(2)}) & h_n(x^2) & h_{n-1}(x^2) & \cdots & h_2(x^2) & h_1(x^2) \end{array} \right|,$$

$$q_{2n+1}(x) \;=\; (-1)^n \left| \begin{array}{cccccc} h_1(x^{(2)}) & 1 & 0 & \cdots & 0 & 0 \\ h_3(x^{(2)}) & h_1(x^2) & 1 & 0 & \cdots & 0 \\ h_5(x^{(2)}) & h_2(x^2) & h_1(x^2) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ h_{2n-1}(x^{(2)}) & h_{n-1}(x^2) & h_{n-2}(x^2) & \cdots & h_1(x^2) & 1 \\ h_{2n+1}(x^{(2)}) & h_n(x^2) & h_{n-1}(x^2) & \cdots & h_2(x^2) & h_1(x^2) \end{array} \right|,$$

$$h_n(x^2) \;=\; (-1)^n \left| \begin{array}{cccccc} 1 & 1 & 0 & \cdots & 0 & 0 \\ h_2(x^{(2)}) & q_2(x) & 1 & 0 & \cdots & 0 \\ h_4(x^{(2)}) & q_4(x) & q_2(x) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ h_{2n-2}(x^{(2)}) & q_{2n-2}(x) & q_{2n-4}(x) & \cdots & q_2(x) & 1 \\ h_{2n}(x^{(2)}) & q_{2n}(x) & q_{2n-2}(x) & \cdots & q_4(x) & q_2(x) \end{array} \right|,$$

$$h_n(x^2) \;=\; \frac{(-1)^n}{[q_1(x)]^{n+1}} \begin{vmatrix} h_1(x^{(2)}) & q_1(x) & 0 & \cdots & 0 & 0 \\ h_3(x^{(2)}) & q_3(x) & q_1(x) & 0 & \cdots & 0 \\ h_5(x^{(2)}) & q_5(x) & q_3(x) & q_1(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ h_{2n-1}(x^{(2)}) & q_{2n-1}(x) & q_{2n-3}(x) & \cdots & q_3(x) & q_1(x) \\ h_{2n+1}(x^{(2)}) & q_{2n+1}(x) & q_{2n-1}(x) & \cdots & q_5(x) & q_3(x) \end{vmatrix}.$$

Appendix C: Fermionic sum rearrangement

In this appendix, we shall prove equation (5.55) concerning the rearrangement of a sum involving free fermions:

$$\sum_{\substack{i_1,i_2,\ldots,i_r\\ i_1+\cdots+i_r=rp-r(r-1)/2}} \omega^{i_2+2i_3+\cdots+(r-1)i_r}\psi_{i_1}\cdots\psi_{i_r}$$
$$= r\sum_{\substack{i_1<i_2<\ldots<i_r\\ i_1+\cdots+i_r=rp-r(r-1)/2}} \left(\sum_{\sigma\in S_{r-1}} (\mathrm{sgn}\sigma)\ \omega^{i_{\sigma(2)}+2i_{\sigma(3)}+\cdots+(r-1)i_{\sigma(r)}}\right)\psi_{i_1}\cdots\psi_{i_r}.$$

where $\omega^r=1$.

Proof
Certainly

$$\text{l.h.s.} = \sum_{\pi\in S_r}\ \sum_{\substack{i_{\pi(1)}<\cdots<i_{\pi(r)}\\ i_1+\cdots+i_r=rp-r(r-1)/2}} \omega^{i_2+2i_3+\cdots+(r-1)i_r}\psi_{i_1}\cdots\psi_{i_r}. \tag{C.1}$$

Now we know that $S_r=\cup_{p=1}^r S_r^p$, where

$$S_r^p=\{\pi\in S_r:\pi(1)=p\}$$

is the set of all permutations which map 1 to the number p. We shall show that if $\pi\in S_r^p$, then the inner sum in (C.1), call it F_p say, is *independent* of p. Let us change variables and let

$$(i_{\pi(1)},i_{\pi(2)},\ldots,i_{\pi(r)})=(j_1,j_2,\ldots,j_r).$$

Thus

$$F_p=\sum_{\substack{i_{\pi(1)}<\cdots<i_{\pi(r)}\\ \pi(1)=p\\ i_1+\cdots+i_r=rp-r(r-1)/2}} \omega^{i_2+2i_3+\cdots+(r-1)i_r}\psi_{i_1}\cdots\psi_{i_r}$$
$$=\sum_{\substack{j_1<\cdots<j_r\\ j_1+\cdots j_r=rp-r(r-1)/2}} \omega^{j_{\pi^{-1}(2)}+\cdots+(p-1)j_{\pi^{-1}(p)}+\cdots+j_{\pi^{-1}(r)}}\psi_{j_{\pi^{-1}(1)}}\cdots\psi_{j_{\pi^{-1}(r)}}.$$

Now, $j_{\pi^{-1}(p)}=j_1$ and $(p-1)j_1=(p-1)(rp-r(r-1)/2-j_1-\cdots-j_r)$, therefore

$$F_p=\sum_{\substack{j_1<\cdots<j_r\\ j_1+\cdots j_r=rp-r(r-1)/2}} (-1)^{(p-1)(r-1)}(-1)^{\pi^{-1}}\omega^{j_{\pi^{-1}(p+1)}+2j_{\pi^{-1}(p+2)}+\cdots+(r-1)j_{\pi^{-1}(p-1)}}\psi_{j_1}\cdots\psi_{j_r},$$

where $(-1)^{\pi^{-1}}$ is the sign of the permutation π^{-1}. Define $\sigma\in S_r$, such that $\sigma(1)=1$, by writing

$$\pi^{-1}=\begin{pmatrix} 1 & \cdots & p-1 & p & p+1 & \cdots & r\\ \sigma(r+2-p) & \cdots & \sigma(r) & 1 & \sigma(2) & \cdots & \sigma(r+1-p)\end{pmatrix}.$$

That is, $\pi^{-1} = \sigma\theta^{r+1-p}$ where $\theta = (1\,2\,\ldots\,r)$ is the unique r-cycle in S_r. Since

$$(-1)^{\theta} = (-1)^{r-1} \Rightarrow (-1)^{\pi^{-1}} = (-1)^{\sigma-(p-1)(r-1)},$$

we get

$$F_p = \sum_{\substack{j_1<\cdots<j_r \\ j_1+\cdots j_r = rp - r(r-1)/2}} (-1)^{\sigma} \omega^{j_{\sigma(2)}+2j_{\sigma(3)}+\cdots+(r-1)j_{\sigma(r)}} \psi_{j_1} \cdots \psi_{j_r},$$

so that F_p is independent of p. Hence each set S_r^p contributes equally to the sum on the r.h.s. of (C.1), and since S_r^1 is isomorphic to the permutation group S_{r-1} on the numbers $\{2, 3, \ldots, r\}$, we finally obtain the result.

Appendix D: Trace calculation

Let us demonstrate how one goes about calculating the traces (6.48) in Chapter 6. For example, let us compute the regularized trace of the operator T_1 given by (6.41). Call the three factors C_1, C_2 and C_3. Using (6.15) (with $\kappa = k$ and $\alpha = 1$) and (6.16), we have

$$\begin{aligned}
C_1 &= \operatorname{tr}\left(p^{\bar{L}_0}\, V_\alpha(q^{-k/2}z, q^{k/2}w^{(-1)}; q^{-k/2}\bar{z}^{(-1)}, q^{k/2}\bar{w})\right) \\
&= \prod_{j=1}^{\infty} \frac{1}{(1-p^j)} \left(\frac{1-q^{-2}p^j}{1-q^2p^j}\right) \prod_{s=0}^{\infty} \left(\frac{1-q^{2k-2+2ks}p^j}{1-q^{2k+2+2ks}p^j}\right)^2 \times \\
&\qquad \times \left(\frac{z-q^{k+2+2ks}p^j w}{z-q^{k-2+2ks}p^j w}\right) \left(\frac{z-q^{k+2+2ks}p^j w}{z-q^{k-2+2ks}p^j w}\right).
\end{aligned}$$

The second factor is just

$$C_2 = \operatorname{tr}\left(p^{\bar{L}_0}\, V_\beta(0; q^{k/2}\widehat{\bar{z}}^{(-1)}, q^{-k/2}\widehat{\bar{w}})\right) = \prod_{j=1}^{\infty} \frac{1}{(1-p^j)},$$

due to the fact that the vertex operator was of the form $1+\mathcal{O}(D(p_n))$, and hence only "counted" the number of states. The final factor C_3 is (apparently) more complicated due to the presence of factors of the form $J^{\pm 1,\pm 1}_{p^j}(z,\widehat{w})$, where $J^{\tau\eta}_p(x,y)$ is given by (6.15). However, from the definition of the "variables" $\widehat{w}$ (6.40), we have, for example, the amazingly simple result

$$\begin{aligned}
J^{1,1}_{p^j}(z,\widehat{w}) &= \prod_{r=0}^{\infty} J^{1,1}_{p^j}(z, q^{4r+\kappa+1}w) J^{1,-1}_{p^j}(z, q^{4r-\kappa+1}w) J^{1,1}_{p^j}(z, q^{4r-\kappa+3}w) \times \\
&\qquad \times J^{1,-1}_{p^j}(z, q^{4r+\kappa+3}w) \\
&= \frac{1-q^{-1}zwp^j}{1-qzwp^j}.
\end{aligned} \tag{D.1}$$

Hence, after some simplification, we have

$$\begin{aligned}
C_3 &= \operatorname{tr}\left(p^{\bar{L}_0}\, V_{\bar{\alpha}}(q^{-k/2}z, q^{k/2}w^{(-1)}; q^{(k+2)/2}\widehat{\bar{z}}^{(-1)}, q^{(-k-2)/2}\widehat{\bar{w}}^{(-1)}, q^{-k/2}\bar{z}^{(-1)}, q^{k/2}\bar{w})\right) \\
&= \prod_{j=1}^{\infty} \frac{(1-p^j)}{(1-q^{-2}p^j)^2} \left(\frac{w-q^{-k-2}p^j z}{w-q^{-k}p^j z}\right) \left(\frac{z-q^{k+2}p^j w}{z-q^k p^j w}\right) \times \\
&\quad \times \prod_{s=0}^{\infty} \left(\frac{(w-q^{k-2+2ks}p^j z}{w-q^{k+2+2ks}p^j z}\right) \left(\frac{(z-q^{k-2+2ks}p^j w}{z-q^{k+2+2ks}p^j w}\right) \left(\frac{(1-q^{2k+2+2ks}p^j}{1-q^{2k-2+2ks}p^j}\right).
\end{aligned}$$

Combining all three factors together we obtain the result

$$\operatorname{tr}\left(p^{\bar{L}_0}\, T_1\right) = \prod_{j=1}^{\infty} \frac{1}{(1-p^j)(1-q^2p_j)(1-q^{-2}p^j)} \left(\frac{w-q^{-k-2}z}{w-q^{-k}p^j z}\right) \left(\frac{z-q^{k+2}p^j w}{z-q^k p^j w}\right).$$

The calculation of T_2 is very similar, the only difference being that one has factors of the form $J^{1,1}_{p^j}(\widehat{z},\widehat{w})$, which, after using (D.1) take the form

$$J^{1,1}_{p^j}(\widehat{z},\widehat{w}) = \frac{1-q^{\kappa}p^j zw}{1-q^{-\kappa}p^j zw}\prod_{s=0}^{\infty}\left(\frac{1-q^{4s+4+\kappa}p^j zw}{1-q^{4s+4-\kappa}p^j zw}\right)^2\left(\frac{1-q^{4s+2-\kappa}p^j zw}{1-q^{4s+2+\kappa}p^j zw}\right)^2.$$

Letting D_1, D_2 and D_3 be the three factors of T_2, we see that $D_1 \equiv C_1$,

$$D_2 = \prod_{j=1}^{\infty}\frac{1}{1-p^j}\left(\frac{z-q^{k+2}p^j w}{z-q^{-k-2}p^j w}\right)\prod_{s=0}^{\infty}\left(\frac{z-q^{4s+k+6}p^j w}{z-q^{4s+2-k}p^j w}\right)^2\left(\frac{z-q^{4s-k}p^j w}{z-q^{4s+4+k}p^j w}\right)^2,$$

and finally

$$\begin{aligned} D_3 = \prod_{j=1}^{\infty}\frac{(1-p^j)}{(1-q^2p^j)(1-q^{-2}p^j)}&\frac{(z-q^{k+2}p^j w)(z-q^{-k-2}p^j w)}{(z-q^{-k}p^j w)^2} \\ &\prod_{s=0}^{\infty}\left(\frac{1-q^{2ks+2}p^j}{1-q^{2ks-2}p^j}\right)\left(\frac{1-q^{2ks+2k+2}p^j}{1-q^{2ks+2k-2}p^j}\right)\left(\frac{w-q^{2ks+k-2}p^j z}{w-q^{2ks+k+2}p^j z}\right)\times \\ &\times\left(\frac{z-q^{2ks+k-2}p^j w}{z-q^{2ks+k+2}p^j w}\right)\left(\frac{z-q^{4s+k+4}p^j w}{z-q^{4s-k+4}p^j w}\right)^2\left(\frac{z-q^{4s-k+2}p^j w}{z-q^{4s+k+2}p^j w}\right)^2. \end{aligned}$$

Combining all the factors together we get the simple result

$$\mathrm{tr}\left(p^{\bar{L}_0}\,T_2\right) = \mathrm{tr}\left(p^{\bar{L}_0}\,T_3\right) = \prod_{j=1}^{\infty}\frac{1}{(1-p^j)(1-q^2p_j)(1-q^{-2}p^j)}.$$

The calculations for T_3 and T_4 are very similar.

Appendix E: Tables of the polynomials $\Gamma^{\lambda}_{\mu\nu}(t)$

$\mu\ \nu$ \ λ	(2)	(1^2)
$(2)(2)$	1	0
$(2)(1^2)$	0	1
$(1^2)(1^2)$	$1-t^2$	$2t$

$\mu\ \nu$ \ λ	(3)	(21)	(1^3)
$(3)(3)$	1	0	0
$(3)(21)$	0	1	0
$(3)(1^3)$	0	0	1
$(21)(21)$	$1-t$	$1+t-t^2$	1
$(21)(1^3)$	0	$(1+t)(1-t^3)$	$t\ (2+t)$
$(1^3)(1^3)$	$(t^2-1)(t^3-1)$	$t(1+t)(2+t)(1-t^3)$	$t^2(1+4\,t+t^2)$

$\mu\ \nu$ \ λ	(4)	(31)
$(4)(4)$	1	0
$(4)(31)$	0	1
$(4)(2^2)$	0	0
$(4)(21^2)$	0	0
$(4)(1^4)$	0	0
$(31)(31)$	$1-t$	1
$(31)(2^2)$	0	$1-t^2$
$(31)(21^2)$	0	$1-t^2$
$(31)(1^4)$	0	0
$(2^2)(2^2)$	$1-t^2$	$t(1-t^2)$
$(2^2)(21^2)$	0	$(1+t)(1-t^2)$
$(2^2)(1^4)$	0	0
$(21^2)(21^2)$	$(1-t)(1-t^2)$	$(1+t)(1-t^2)(1+t-t^2)$
$(21^2)(1^4)$	0	$(t+1)(t^3-1)(t^4-1)$
$(1^4)(1^4)$	$(1-t^2)(1-t^3)(1-t^4)$	$t(1+t)(2+t+t^2)(t^3-1)(t^4-1)$

$\mu\ \nu \quad \lambda$	(2^2)	(21^2)
$(4)(4)$	0	0
$(4)(31)$	0	0
$(4)(2^2)$	1	0
$(4)(21^2)$	0	1
$(4)(1^4)$	0	0
$(31)(31)$	$1-t$	1
$(31)(2^2)$	$t(1-t)$	$1+t$
$(31)(21^2)$	$1-t^2$	$(1+t)(1+t-t^2)$
$(31)(1^4)$	0	$(1-t^4)(1+t+t^2)$
$(2^2)(2^2$	$1+t^2$	$t(1-t^2)$
$(2^2)(21^2)$	$t(t-1)(t^2-1)$	$(1+t)(1+t+t^2-2t^3)$
$(2^2)(1^4)$	$(t^3-1)(t^4-1)$	$2t(1+t+t^2)(1-t^4)$
$(21^2)(21^2)$	$(t^2-1)(-1-t-t^2+2t^3)$	$1+4t+3t^2+t^3-4t^4-3t^5$
$(21^2)(1^4)$	$2t(t^3-1)(t^4-1)$	$t(2+3t)(1+t+t^2)(1-t^4)$
$(1^4)(1^4)$	$t^2(3+t+2t^2)$ $(t^3-1)(t^4-1)$	$-t^2(1+t)(1+t^2)(t^3-1)$ $(1+5t+3t^2+3t^3)$

$\mu\ \nu \quad \lambda$	(1^4)
$(4)(4)$	0
$(4)(31)$	0
$(4)(2^2)$	0
$(4)(21^2)$	0
$(4)(1^4)$	1
$(31)(31)$	0
$(31)(2^2)$	0
$(31)(21^2)$	1
$(31)(1^4)$	$t(2+t+t^2)$
$(2^2)(2^2$	1
$(2^2)(21^2)$	$2t$
$(2^2)(1^4)$	$t^2(3+t+2t^2)$
$(21^2)(21^2)$	$t(3+2t)$
$(21^2)(1^4)$	$t^2(1+5t+3t^2+3t^3)$
$(1^4)(1^4)$	$t^4(3+4t+10t^2+3t^3+3t^4)$

Appendix F: Transformations between bases of Macdonald functions

In this appendix we list the transition matrices between the various standard bases for the ring Λ_F of symmetric functions over the field $F = \mathbb{Q}(q,t)$. The matrices $M_{\lambda\mu}$ displayed in the tables relate the function u_λ in the left-most column to v_μ in the top-most row by $u_\lambda = \sum_\mu M_{\lambda\mu} v_\mu$. It is to be understood that the labels ρ and σ (if they appear) are summed over.

	$R(q,t)$	$J(q,t)$	$\Sigma(q)$
R	1	$\frac{c_\mu(q,t)c_{\mu'}(q,t)}{b_\lambda(t)z_\rho(t)}X^\lambda_\rho(t)X^\mu_\rho(q,t)$	$\frac{1}{b_\lambda(t)z_\rho(t)}X^\lambda_\rho(t)\chi^\mu_\rho$
J	$\frac{1}{\zeta_\rho(q,t)}X^\lambda_\rho(q,t)X^\mu_\rho(t)$	1	$\frac{1}{\zeta_\rho(q,t)}X^\lambda_\rho(q,t)\chi^\mu_\rho$
Σ	$\frac{1}{z_\rho}\chi^\lambda_\rho X^\mu_\rho(t)$	$\frac{1}{c_\mu(q,t)c_{\mu'}(q,t)z_\rho}\chi^\lambda_\rho X^\mu_\rho(q,t)$	1
T	$\frac{1}{z_\rho(t)}\chi^\lambda_\rho X^\mu_\rho(t)$	$\frac{1}{c_\mu(q,t)c_{\mu'}(q,t)z_\rho(t)}\chi^\lambda_\rho X^\mu_\rho(q,t)$	$\frac{1}{z_\rho(t)}\chi^\lambda_\rho\chi^\mu_\rho$
S	$\frac{1}{\zeta_\rho(q,t)}\chi^\lambda_\rho X^\mu_\rho(t)$	$\frac{1}{c_\mu(q,t)c_{\mu'}(q,t)\zeta_\sigma(q,t)}\chi^\lambda_\sigma X^\mu_\sigma(q,t)$	$\frac{1}{\zeta_\sigma(q,t)}\chi^\lambda_\sigma\chi^\mu_\sigma$
Q	$\frac{1}{\zeta_\sigma(q,t)}X^\lambda_\sigma(t)X^\mu_\sigma(t)$	$\frac{1}{c_\mu(q,t)c_{\mu'}(q,t)\zeta_\sigma(q,t)}X^\lambda_\sigma(t)X^\mu_\sigma(q,t)$	$\frac{1}{\zeta_\sigma(q,t)}X^\lambda_\sigma(t)\chi^\mu_\sigma$
p	$\xi_\lambda(q)X^\mu_\lambda(t)$	$\frac{\xi_\lambda(q)}{c_\mu(q,t)c_{\mu'}(q,t)}X^\mu_\lambda(q,t)$	$\xi_\lambda(q)\chi^\mu_\lambda$
s	$\frac{1}{z_\rho(q)}\chi^\lambda_\rho X^\mu_\rho(t)$	$\frac{1}{z_\rho(q)c_\mu(q,t)c_{\mu'}(q,t)}\chi^\lambda_\rho X^\mu_\rho(q,t)$	$\frac{1}{z_\rho(q)}\chi^\lambda_\rho\chi^\mu_\rho$
g	$\frac{1}{z_\rho(t)}K_{\sigma\lambda}\chi^\sigma_\rho X^\mu_\rho(t)$	$\frac{K_{\sigma\lambda}}{z_\rho(t)c_\mu(q,t)c_{\mu'}(q,t)}\chi^\sigma_\rho X^\mu_\rho(q,t)$	$\frac{K_{\sigma\lambda}}{z_\rho(t)}\chi^\sigma_\rho\chi^\mu_\rho$
m	$\frac{K^{-1}_{\lambda\sigma}}{z_\rho(q)}\chi^\sigma_\rho X^\mu_\rho(t)$	$\frac{K^{-1}_{\lambda\sigma}}{z_\rho(q)c_\mu(q,t)c_{\mu'}(q,t)}\chi^\sigma_\rho X^\mu_\rho(q,t)$	$\frac{K^{-1}_{\lambda\sigma}}{z_\rho(q)}\chi^\sigma_\rho\chi^\mu_\rho$

	$T(q,t)$	$S(t)$	$Q(t)$	p
R	$\frac{1}{b_\lambda(t)z_\sigma}X^\lambda_\sigma(t)\chi^\mu_\sigma$	$\frac{X^\lambda_\sigma(t)\chi^\mu_\sigma}{b_\lambda(t)z_\sigma\xi_\sigma(q)}$	$\frac{1}{z_\sigma(q,t)}X^\lambda_\sigma(t)X^\mu_\sigma(t)$	$\frac{X^\lambda_\mu(t)}{b_\lambda(t)z_\mu(q,t)}$
J	$\frac{1}{z_\sigma(q)}X^\lambda_\sigma(q,t)\chi^\mu_\sigma$	$\frac{1}{z_\sigma}X^\lambda_\sigma(q,t)\chi^\mu_\sigma$	$\frac{1}{z_\sigma(t)b_\mu(t)}X^\lambda_\sigma(q,t)X^\mu_\sigma(t)$	$\frac{1}{z_\mu(t)}X^\lambda_\mu(q,t)$
Σ	$\frac{1}{z_\sigma\xi_\sigma(t)}\chi^\lambda_\sigma\chi^\mu_\sigma$	$\frac{1}{z_\sigma\xi_\sigma(q)\xi_\sigma(t)}\chi^\lambda_\sigma\chi^\mu_\sigma$	$\frac{1}{z_\sigma\xi_\sigma(q)b_\mu(t)}\chi^\lambda_\sigma X^\mu_\sigma(t)$	$\frac{1}{z_\mu\xi_\mu(q)}\chi^\lambda_\mu$
T	1	$\frac{1}{z_\sigma\xi_\sigma(q)}\chi^\lambda_\sigma\chi^\mu_\sigma$	$\frac{1}{z_\sigma(q,t)b_\mu(t)}\chi^\lambda_\sigma X^\mu_\sigma(t)$	$\frac{1}{z_\mu(q,t)}\chi^\lambda_\mu$
S	$\frac{1}{z_\sigma(q)}\chi^\lambda_\sigma\chi^\mu_\sigma$	1	$\frac{1}{z_\sigma(t)b_\mu(t)}\chi^\lambda_\sigma X^\mu_\sigma(t)$	$\frac{1}{z_\mu(t)}\chi^\lambda_\mu$
Q	$\frac{1}{z_\sigma(q)}X^\lambda_\sigma(t)\chi^\mu_\sigma$	$\frac{1}{z_\sigma}X^\lambda_\sigma(t)\chi^\mu_\sigma$	1	$\frac{1}{z_\mu(t)}X^\lambda_\mu(t)$
p	$\frac{\xi_\lambda(q)}{\xi_\lambda(t)}\chi^\mu_\lambda$	$\frac{1}{\xi_\lambda(t)}\chi^\mu_\lambda$	$\frac{1}{b_\mu(t)}X^\mu_\lambda(t)$	1
s	$\frac{1}{z_\rho(q,t)}\chi^\lambda_\rho\chi^\mu_\rho$	$\frac{1}{z_\rho\xi_\rho(t)}\chi^\lambda_\rho\chi^\mu_\rho$	$\frac{1}{z_\rho b_\mu(t)}\chi^\lambda_\rho X^\mu_\rho(t)$	$\frac{1}{z_\mu}\chi^\lambda_\mu$
g	$K_{\mu\lambda}$	$\frac{K_{\sigma\lambda}}{z_\rho\xi_\rho(q)}\chi^\sigma_\rho\chi^\mu_\rho$	$\frac{K_{\sigma\lambda}}{b_\mu(t)z_\rho(q,t)}\chi^\sigma_\rho X^\mu_\rho(t)$	$\frac{K_{\sigma\lambda}}{z_\mu(q,t)}\chi^\sigma_\mu$
m	$\frac{K^{-1}_{\lambda\sigma}}{z_\rho(t,q)}\chi^\sigma_\rho\chi^\mu_\rho$	$\frac{K^{-1}_{\lambda\sigma}}{z_\rho\xi_\rho(t)}\chi^\sigma_\rho\chi^\mu_\rho$	$\frac{K^{-1}_{\lambda\sigma}}{z_\rho b_\mu(t)}\chi^\sigma_\rho X^\mu_\rho(t)$	$\frac{K^{-1}_{\lambda\sigma}}{z_\mu}\chi^\sigma_\mu$

	s	$g(q,t)$	m
R	$\frac{1}{b_\lambda(t)z_\sigma(q,t)}X^\lambda_\sigma(t)\chi^\mu_\sigma$	$\frac{K^{-1}_{\mu\sigma}}{z_\rho b_\lambda(t)}X^\lambda_\rho(t)\chi^\sigma_\rho$	$\frac{K_{\sigma\mu}}{b_\lambda(t)z_\lambda(q,t)}X^\lambda_\rho(t)\chi^\sigma_\rho$
J	$\frac{1}{z_\sigma(t)}X^\lambda_\sigma(q,t)\chi^\mu_\sigma$	$\frac{K^{-1}_{\mu\sigma}}{z_\rho(q)}X^\lambda_\rho(q,t)\chi^\sigma_\rho$	$\frac{K_{\sigma\mu}}{z_\rho(t)}X^\lambda_\rho(q,t)\chi^\sigma_\rho$
Σ	$\frac{1}{z_\sigma\xi_\sigma(q)}\chi^\lambda_\sigma\chi^\mu_\rho$	$\frac{K_{\sigma\lambda}}{z_\rho(t)}\chi^\sigma_\rho\chi^\mu_\rho$	$\frac{K_{\sigma\mu}}{z_\rho\xi_\rho(q)}\chi^\sigma_\rho\chi^\lambda_\rho$
T	$\frac{1}{z_\rho(q,t)}\chi^\lambda_\rho\chi^\mu_\rho$	$K^{-1}_{\mu\lambda}$	$\frac{K_{\sigma\mu}}{z_\rho(q,t)}\chi^\lambda_\rho\chi^\sigma_\rho$
S	$\frac{1}{z_\sigma(t)}\chi^\lambda_\sigma\chi^\mu_\sigma$	$\frac{K^{-1}_{\mu\sigma}}{z_\rho(q)}\chi^\lambda_\rho\chi^\sigma_\rho$	$\frac{K_{\sigma\mu}}{z_\rho(t)}\chi^\lambda_\rho\chi^\sigma_\rho$
Q	$\frac{1}{z_\sigma(t)}X^\lambda_\sigma(t)\chi^\mu_\sigma$	$\frac{K^{-1}_{\mu\sigma}}{z_\rho(q)}X^\lambda_\rho(t)\chi^\sigma_\rho$	$\frac{K_{\sigma\mu}}{z_\rho(t)}X^\lambda_\rho(t)\chi^\sigma_\rho$
p	χ^μ_λ	$\frac{\xi_\lambda(q)}{\xi_\lambda(t)}\chi^\sigma_\lambda K^{-1}_{\mu\sigma}$	$\chi^\sigma_\lambda K_{\sigma\mu}$
s	1	$\frac{K^{-1}_{\mu\sigma}}{z_\rho(t,q)}\chi^\lambda_\rho\chi^\sigma_\rho$	$\frac{K_{\sigma\mu}}{z_\rho}\chi^\lambda_\rho\chi^\sigma_\rho$
g	$\frac{K_{\sigma\lambda}}{z_\rho(q,t)}\chi^\sigma_\rho\chi^\mu_\rho$	1	$\frac{K_{\sigma\lambda}K_{\tau\mu}}{z_\rho(q,t)}\chi^\sigma_\rho\chi^\tau_\rho$
m	$\frac{K^{-1}_{\lambda\sigma}}{z_\rho}\chi^\sigma_\rho\chi^\mu_\rho$	$\frac{K^{-1}_{\lambda\sigma}K^{-1}_{\mu\tau}}{z_\rho(t,q)}\chi^\sigma_\rho\chi^\tau_\rho$	1

Appendix G: Summary of tensor product decompositions

We list in this appendix the various tensor product decompositions of the $N = 2$ superconformal algebras calculated in Chapter 4. First we have the T algebra decompositions

$$\begin{aligned}
V_T(\tfrac{1}{3}, \tfrac{1}{16}) \otimes V_T(\tfrac{1}{3}, \tfrac{1}{16}) &= 2V_T(\tfrac{2}{3}, \tfrac{1}{8}) \\
V_T(\tfrac{1}{3}, \tfrac{1}{16}) \otimes V_T(\tfrac{1}{2}, \tfrac{1}{8}) &= 2V_T(\tfrac{5}{6}, \tfrac{3}{16}) \\
V_T(\tfrac{1}{3}, \tfrac{1}{16}) \otimes V_T(\tfrac{1}{2}, \tfrac{1}{16}) &= V_T(\tfrac{5}{6}, \tfrac{1}{8}) \oplus V_T(\tfrac{5}{6}, \tfrac{5}{8}) \\
V_T(\tfrac{1}{3}, \tfrac{1}{16}) \otimes V_T(\tfrac{3}{5}, \tfrac{3}{16}) &= 2V_T(\tfrac{14}{15}, \tfrac{1}{4}) \oplus V_T(\tfrac{14}{15}, \tfrac{7}{4}) \\
V_T(\tfrac{1}{3}, \tfrac{1}{16}) \otimes V_T(\tfrac{3}{5}, \tfrac{7}{80}) &= 2V_T(\tfrac{14}{15}, \tfrac{3}{20}) \oplus V_T(\tfrac{14}{15}, \tfrac{13}{20})
\end{aligned}$$

For the A algebra

$$\begin{aligned}
V_A(\tfrac{1}{3}, 0, 0) \otimes V_A(\tfrac{1}{3}, 0, 0) &= V_A(\tfrac{2}{3}, 0, 0) \oplus V_A(\tfrac{2}{3}, 1, 0) \\
V_A(\tfrac{1}{3}, 0, 0) \otimes V_A(\tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{3}) &= V_A(\tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{3}) \\
V_A(\tfrac{1}{3}, 0, 0) \otimes V_A(\tfrac{1}{3}, \tfrac{1}{6}, -\tfrac{1}{3}) &= V_A(\tfrac{2}{3}, \tfrac{1}{6}, -\tfrac{1}{3}) \\
V_A(\tfrac{1}{3}, \tfrac{1}{6}, -\tfrac{1}{3}) \otimes V_A(\tfrac{1}{3}, \tfrac{1}{6}, -\tfrac{1}{3}) &= V_A(\tfrac{2}{3}, \tfrac{1}{3}, -\tfrac{2}{3}) \oplus V_A(\tfrac{2}{3}, \tfrac{5}{6}, \tfrac{1}{3}) \\
V_A(\tfrac{1}{3}, \tfrac{1}{6}, -\tfrac{1}{3}) \otimes V_A(\tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{3}) &= V_A(\tfrac{2}{3}, \tfrac{1}{3}, 0) \\
V_A(\tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{3}) \otimes V_A(\tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{3}) &= V_A(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}) \oplus V_A(\tfrac{2}{3}, \tfrac{5}{6}, -\tfrac{1}{3}) \\
V_A(\tfrac{1}{3}, 0, 0) \otimes V_A(\tfrac{1}{2}, 0, 0) &= V_A(\tfrac{5}{6}, 0, 0) \oplus V_A(\tfrac{5}{6}, 1, 0) \\
V_A(\tfrac{1}{3}, 0, 0) \otimes V_A(\tfrac{1}{2}, \tfrac{1}{2}, 0) &= V_A(\tfrac{5}{6}, \tfrac{1}{2}, 0) \oplus V_A(\tfrac{5}{6}, \tfrac{5}{2}, 0) \\
V_A(\tfrac{1}{3}, 0, 0) \otimes V_A(\tfrac{1}{2}, \tfrac{1}{4}, -\tfrac{1}{2}) &= V_A(\tfrac{5}{6}, \tfrac{1}{4}, -\tfrac{1}{2}) \oplus V_A(\tfrac{5}{6}, \tfrac{7}{4}, \tfrac{1}{2}) \\
V_A(\tfrac{1}{3}, \tfrac{1}{6}, -\tfrac{1}{3}) \otimes V_A(\tfrac{1}{2}, \tfrac{1}{2}, 0) &= V_A(\tfrac{5}{6}, \tfrac{2}{3}, -\tfrac{1}{3}) \oplus V_A(\tfrac{5}{6}, \tfrac{7}{6}, \tfrac{2}{3}) \\
V_A(\tfrac{1}{3}, \tfrac{1}{6}, -\tfrac{1}{3}) \otimes V_A(\tfrac{1}{2}, 0, 0) &= V_A(\tfrac{5}{6}, \tfrac{1}{6}, -\tfrac{1}{3}) \oplus V_A(\tfrac{5}{6}, 2\tfrac{1}{6}, -\tfrac{1}{3}) \\
V_A(\tfrac{1}{3}, \tfrac{1}{6}, -\tfrac{1}{3}) \otimes V_A(\tfrac{1}{2}, \tfrac{1}{4}, -\tfrac{1}{2}) &= V_A(\tfrac{5}{6}, \tfrac{5}{12}, -\tfrac{5}{6}) \oplus V_A(\tfrac{5}{6}, \tfrac{11}{12}, \tfrac{1}{6})
\end{aligned}$$

while for the P^+ algebra we have

$$\begin{aligned}
V_{P^+}(\tfrac{1}{3}, \tfrac{1}{24}, \tfrac{1}{3}) \otimes V_{P^+}(\tfrac{1}{3}, \tfrac{3}{8}, 0) &= V_{P^+}(\tfrac{2}{3}, \tfrac{5}{12}, -\tfrac{1}{6}) \\
V_{P^+}(\tfrac{1}{3}, \tfrac{1}{24}, \tfrac{1}{3}) \otimes V_{P^+}(\tfrac{1}{3}, \tfrac{1}{24}, \tfrac{1}{3}) &= V_{P^+}(\tfrac{2}{3}, \tfrac{1}{12}, \tfrac{1}{6}) \oplus V_{P^+}(\tfrac{2}{3}, 1\tfrac{1}{12}, \tfrac{1}{6}) \\
V_{P^+}(\tfrac{1}{3}, \tfrac{1}{24}, \tfrac{1}{3}) \otimes V_{P^+}(\tfrac{1}{3}, \tfrac{1}{24}, \tfrac{2}{3}) &= V_{P^+}(\tfrac{2}{3}, \tfrac{1}{12}, \tfrac{1}{2}) \\
V_{P^+}(\tfrac{1}{3}, \tfrac{3}{8}, 0) \otimes V_{P^+}(\tfrac{1}{3}, \tfrac{3}{8}, 0) &= V_{P^+}(\tfrac{2}{3}, \tfrac{3}{4}, -\tfrac{1}{2}) \oplus V_{P^+}(\tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{2}) \\
V_{P^+}(\tfrac{1}{3}, \tfrac{3}{8}, 0) \otimes V_{P^+}(\tfrac{1}{3}, \tfrac{1}{24}, \tfrac{2}{3}) &= V_{P^+}(\tfrac{2}{3}, \tfrac{5}{12}, \tfrac{1}{6}) \\
V_{P^+}(\tfrac{1}{3}, \tfrac{1}{24}, \tfrac{2}{3}) \otimes V_{P^+}(\tfrac{1}{3}, \tfrac{1}{24}, \tfrac{2}{3}) &= V_{P^+}(\tfrac{2}{3}, \tfrac{1}{12}, \tfrac{5}{6}) \oplus V_{P^+}(\tfrac{2}{3}, 1\tfrac{1}{12}, -\tfrac{1}{6})
\end{aligned}$$

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