On Group Invariant Solutions to the Maxwell Dirac Equations

by

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Submitted in fulfilment of the requirements for the Degree of

Master of Science

University of Tasmania

November 2007

Contents

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Acknowledgements

I thank my supervisor, Dr Peter Jarvis for his years of tutelage and advice, Dr Hilary Booth, (1956-2005) who was there at the start, and for valuable guidance from Professor Bob Delbourgo, University of Tasmania, and Professor Gerd Rudolph, Universität Leipzig. I thank my father, John, for proofreading the manuscript.

Dedication

To Janet;
wife, companion, friend, mother of my children,
who demonstrated her love for me by supporting me in this work,
even though sometimes she wondered where my heart lay.

Abstract

This work constitutes a study on group invariant solutions of the Maxwell Dirac equations for a relativistic electron *spinor* in its own self-consistent electromagnetic field. First, the Maxwell Dirac equations are written in a gauge independent tensor form, in terms of bilinear Dirac currents and a *gauge independent total four-potential.* A requirement of this form is that the length of the current vector be non-zero. In this form they are amenable to the study of solutions invariant under subgroups of the Poincaré group without reference to the Abelian gauge group. In particular, all subgroups of the Poincaré group that generate 4 dimensional orbits by *transitive* action on Minkowski space, and the corresponding invariant vector fields are identified, which will constitute invariant solutions merely if various constants satisfy a set of algebraic equations. For each such subgroup, the possibility of solutions to both the full Maxwell Dirac equations and to a classical approximation to the self-field equations is determined. Of the 19 classes of simply transitive subgroups, only one class yielded a solution. That solution of the Maxwell Dirac equations depends upon a parameter κ and there exists a gauge in which the solution may be written

$$\psi = \sqrt{\frac{\rho}{2}}\left(e^{+\frac{1}{2}\kappa y} + \gamma^3 e^{-\frac{1}{2}\kappa y}\right)\phi$$

where ϕ is a constant spinor satisfying the relations $\overline{\phi}\gamma^0\phi = 1$, $(1 - i\gamma^2)\phi = 0$, $(1 - \gamma^0\gamma^3)\phi = 0$, and where γ^μ generate the Dirac algebra via $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The length of the four-current, which must be positive, is given by

$$\rho = \frac{\kappa^2}{\mu_0 q^2}(mc^2 - \frac{1}{2}\kappa c\hbar)$$

where m is the mass of the electron and q is its charge. The self-consistent electromagnetic four-potential in this gauge is given by

$$A_\mu = -(\mu_0 q\rho/\kappa^2)(\cosh\kappa y, 0, 0, -\sinh\kappa y).$$

The scalar density $\overline{\psi}\psi$ is constant over all space and equals ρ while the pseudoscalar density $\overline{\psi}i\gamma_5\psi$ is zero. The relativistic electron four-current exhibits unbounded laminar stream flow pointing in the z-direction, $J_\mu = (\rho\cosh\kappa y, 0, 0, -\rho\sinh\kappa y)$. The electric field $\mathbf{E} = (0, (c\mu_0 q\rho/\kappa)\sinh\kappa y, 0)$ is y-pointing and the magnetic field, $\mathbf{B} = ((\mu_0 q\rho/\kappa)\cosh\kappa y, 0, 0)$, is x-pointing. The axial current, $K_\mu = (0, \rho, 0, 0)$, is constant and parallel to the magnetic field for $\kappa > 0$, anti-parallel for $\kappa < 0$. This advances the known closed-form solutions of the Maxwell Dirac equations beyond null-current or massless solutions.

List of symbols

Space time coordinates	$x_\mu = (ct, x, y, z)$
Metric tensor	$\eta_{\mu\nu}$ =diag$(1, -1, -1, -1)$
The Einstein summation convention	$J_\mu K^\mu = \Sigma_{\mu=0..3} J_\mu K^\mu$
Co(variant) four-vector	$V^\mu = \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}$
[Contra(gradient)]vector	$V_\mu = \begin{pmatrix} V^0 & -V^1 & -V^2 & -V^3 \end{pmatrix}$
Three-vector	$\mathbf{V}$
Complex conjugation	z^*
Hermitian conjugation	$\psi^\dagger = \psi^{*T}$
Pauli matrices	σ^μ, $\sigma^0..\sigma^3$
	$\sigma_\mu = \eta_{\mu\nu}\sigma^\nu$
Dirac spinor	$\psi = \psi^\alpha \in \mathbb{C}^4,\ \alpha = 1..4$
Dirac conjugate	$\overline{\psi} = \psi^\dagger \gamma^0$
Dirac gamma matrices	$\gamma^\mu, \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$
	$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$
Gamma 5 matrix	$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$
Levi Civita tensor	$\varepsilon^{\mu\nu\rho\sigma} = +1$ if $\{\mu, \nu, \rho, \sigma\}$ is even
	$= -1$ if it is an odd permutation
	$\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon_{\mu\nu\rho\sigma}$
Charge conjugation matrix	C
Charge conjugate spinor	$\psi_c = C\gamma^0\psi^*$
Parity operator	P
Charge conjugation operator	$\mathcal{C}$
Time reversal operator	T
Phase (change) in the spinor	θ
Momentum operator	$p_\mu \equiv i\hbar\partial_\mu$

Speed of light	c
Planck's constant	$h = 2\pi\hbar$
Charge of the electron	e
Permittivity of free space	ε_0
Permeability of free space	$\mu_0 = (c^2\varepsilon_0)^{-1}$
Fine structure constant	$\alpha = e^2/4\pi\varepsilon_0\hbar c$
Charge of particle	q
Mass of particle (electron)	m
Compton wavelength	$\lambda_c = h/mc = 2\pi\hbar/mc$
Dirac scalar density	$\sigma = \overline{\psi}\psi$
Dirac pseudo scalar	$\omega = \overline{\psi}i\gamma^5\psi$
Arg of $\overline{\psi}\psi + i\left(\overline{\psi}i\gamma^5\psi\right)$	β where $\sigma + i\omega = \rho e^{i\beta}$
Scalar length of four-current	ρ
Four-current probability / number density	$J^\nu = \overline{\psi}\gamma^\nu\psi$
Four-current charge density	$\mathcal{J}^\nu = qJ^\nu$
Four-velocity, for the current as for a fluid	$u^\nu = J^\nu/\rho$
Axial current	$K^\nu = \overline{\psi}\gamma^5\gamma^\nu\psi$
Normalised axial current	$k^\nu = K^\nu/\rho$
Rank 2 spin current tensor	$S^{\mu\nu} = \overline{\psi}\sigma^{\mu\nu}\psi$
Gauge dependent currents	$M^\nu + iN^\nu = \overline{\psi}_c\gamma^\nu\psi$
Gauge dependent normalised tetrad legs	$m^\nu = M^\nu/\rho,\;\; n^\nu = N^\nu/\rho$
Tetrad as a matrix	$\mathbf{e}^\mu_{\;\alpha} = (u^\mu, m^\mu, n^\mu, k^\mu)$
Dual rank 2 tensor of a tensor $T^{\mu\nu}$	${}^*T^{\mu\nu} = -\frac{1}{2}\varepsilon^{\mu\nu}_{\;\;\alpha\beta}T^{\alpha\beta}$
Rank 2 spin plane tensor	$H^{\mu\nu} = m^\mu n^\nu - n^\mu m^\nu$
Dual of spin plane tensor	${}^*H^{\mu\nu} = u^\mu k^\nu - k^\mu u^\nu$
Rank 2 electromagnetic field tensor	$F^{\mu\nu}$
Electric field 3-vector	$\mathbf{E}$
Magnetic field 3-vector	$\mathbf{B}$
Electromagnetic 4-potential	A^ν
Gauge independent total potential	$\mathfrak{B}^\nu$
D'Alembertian	$\Box^2 = (\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2})$

2 × 2 complex matrices	$M_2(\mathbb{C})$
The multiplicative group of the complex numbers $\mathbb{C} \sim \{0\}$	$\mathbb{C}^\times$
The group of invertible 2 × 2 complex matrices	$GL_2(\mathbb{C})$
The group of 2 × 2 complex matrices with unit determinant	$SL_2(\mathbb{C})$
The group of matrices preserving the $(1, -1, -1, -1)$ metric and with unit determinant	$SO(1, 3)$
Dirac 4 Spinor as a matrix $\in M_2(\mathbb{C})$	$\mathbf{\Psi}$
Quaternionic conjugate	$\bar{\mathbf{\Psi}}$, $\overline{\sigma}^k = -\sigma^k$
Trace of a matrix, $\mathbf{\Psi}$	$Tr\,(\mathbf{\Psi})$
Group	G
member of a group	g
Subgroup	S
Action of a member of a group on a vector	$g \cdot x^\mu$, $g \cdot V^\mu$
Cartesian space, as of independent variables	X
Space of dependent variables	U
Manifold	M
Orbit of a subgroup S acting on a vector space	$\mathcal{O}_S$
Isotropy group of the action of S at x	S_x
Number of independent variables	p
Dimension of an orbit	s
Rank of a group invariant solution	r
Order of a Lie group, S	l
Order of the isotropy group of S at a point x	d
Semi-direct product	$\ltimes$
Cartesion product	$\times$
Lie algebra associated with a Lie group	$\mathfrak{g}$
- Lie algebra of $\mathbb{C}^\times \otimes SO(1, 3)$	$\mathfrak{c} + \mathfrak{so}(1, 3)$
- Lie algebra of $GL_2(\mathbb{C})$	$\mathfrak{gl}_2(\mathbb{C})$
Generator in a Lie algebra	X_A
Lorentz group	$\mathcal{L}$
Poincare group	$\mathcal{P}$
Rotation generator of $\mathcal{L}$,$\mathcal{P}$	$\mathbf{J}_x..\mathbf{J}_z$
Boost generator of $\mathcal{L}$,$\mathcal{P}$	$\mathbf{K}_x..\mathbf{K}_z$
Translational generator of $\mathcal{P}$	$\mathbf{P}_t..\mathbf{P}_z$
Generators of $\mathcal{L}$ defined by PWZ [52]	$B_1..B_6$
Translational generators of $\mathcal{P}$ defined by PWZ [52]	$X_1..X_4$
Light cone coordinates	$l_+ = t + z$ $l_- = t - z$
Commutative group of 4 d translations	$T(4)$
Translational subgroups of $T(4)$	T_{xyz}, T_{l+}

Chapter 1

Introduction

1.1 Dirac theory

The Dirac equation is the relativistically correct wave equation for spin half particles, such as electrons. It gives more accurate predictions of atomic spectra than those calculated from the Schrödinger equation. This is especially so for atoms with large atomic number where the electron's 'speed' is significant compared with the speed of light, c. It has been pointed out that some of the special chemical and physical properties of elements such as Silver (Ag), Gold (Au) and Platinum (Pt), can be attributed to the relativistic behaviour of the electron [9], in accordance with the Dirac equation. The Dirac equation also correctly predicts that the electron should possess 2 spin states and an antiparticle state. Under minimal coupling with the electromagnetic field, the Dirac equation predicts that the electron will behave as if it has magnetic moment of $g(e/2m)S$, where e, m and S are the charge, mass and spin of the electron respectively, and $g = 2$ is the classical electron *g-value*. From the time of its discovery in 1928 until 1951, the Dirac equation was supported by a vast body of experimental evidence. It seemed that the Dirac equation was the perfect marriage of the modern theories of quantum mechanics and special relativity.

But even though the Dirac equation predicts anti-electrons (positrons), it does not by itself provide a consistent framework for particle creation and annihilation. The negative energy states pose difficulties in the theory and it is usually considered to be only a single-particle theory [36]. Eventually, inspired by W E Lamb's discovery in 1951 that the g-value of the electron was in fact 2.00232[1], the theory of Quantum Electrodynamics (QED) was developed by Feynman, Schwinger and Tomonaga. QED gives excellent predictions regarding real and virtual particle and antiparticle creation and annihilation, and is one of the most successful theories ever. Nowadays, QED is regarded as the full quantum field theory of electrons, positrons, and photons, and the study of the Dirac equation itself, coupled with Maxwell's equations, is done in the context of 'classical' or 'semiclassical' field theory.

[1] For a recent measurement more accurate than 1 part in 10^{12} see [24],[49] and the accessible account in [23].

1.2 Maxwell Dirac equations

Physically we know that not only is the electron's motion *subject* to the electromagnetic field; the electron is also a *source* of the electromagnetic field. This self-field is a source of mathematical difficulty in both semiclassical theory and QED. Without self-field, the electromagnetic potential in which the electron moves can be prescribed as an external field, and the Dirac equation is linear for the spinor ψ. An arbitrary solution can be formed from a sum of eigenstates, and a probability normalisation condition of the form $\int \psi^\dagger \psi d^3x = 1$ can be imposed at will. This ensures that the particle so described has unit charge and probability density. But if the electron current is included as a source term in Maxwell's equations, the coupled equations are *nonlinear.* Problems that include self-field have been traditionally solved by perturbation. For instance, in atomic problems, the electromagnetic potential is dominated by the contribution from the nucleus, to which electron self-field is treated as a perturbation. In such cases, perturbation corrections arise in powers of α/π, where the fine structure constant $\alpha = e^2/4\pi\varepsilon_0 c\hbar \approx 1/137$.

There have been a number of investigations looking at how the semiclassical electron reacts to its self-field in the absence of *any external potential,* for example [63],[28],[12],[13],[17],[16],[19],[43],[54],[11],[56],[55],[25],[45]. If we were not forearmed with the knowledge that the electron is in fact a 'particle', we might expect that the electron cloud would blow itself apart with self-repulsion. However, relativistic and wave mechanical effects might instead lead to some kind of nonlinear self-gathering of the electron into a localised wave or soliton. One motivation of these investigations is to obtain an insight into wave-particle duality. Another motivation has been to reassess the foundations on which QED is constructed, looking at whether an alternative formulation might be possible without perturbation theory. The system of the semiclassical electron reacting only to its own field is described by the Maxwell Dirac (M-D)[2] equations of spinor electrodynamics. Firstly, the Dirac equation in covariant form for an electron moving in the presence of an electromagnetic potential, A_μ:

$$qA_\mu \gamma^\mu \psi + mc^2 \psi = i\hbar c \partial_\mu \gamma^\mu \psi \tag{1.1}$$

where $q = -e$ is the charge of the electron, m is its mass, ψ is a Dirac spinor,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \in \mathbb{C}^4,$$

and γ^μ are 4x4 complex matrices satisfying the commutation relations of a complex Clifford algebra with Minkowski signature:

$$\gamma^\mu \in M_4(\mathbb{C}), \qquad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \ \mu = 0..3 \tag{1.2}$$

A useful representation for the γ^μ is the γ_5-diagonal or chiral representation

[2] sometimes referred to as the D-M equations

[36]:

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where σ^μ are the Pauli matrices,

$$\sigma^0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Secondly, the Maxwell equations in covariant form for an electromagnetic field where the electron four-current is the source:

$$\partial_\nu F^{\nu\mu} = \mu_0 \mathcal{J}^\mu = \mu_0 q \overline{\psi} \gamma^\mu \psi. \tag{1.3}$$

The electromagnetic field tensor is the curl of the four-potential :

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{1.4}$$

The homogeneous Maxwell's equations

$$\partial^\rho F^{\nu\mu} + \partial^\nu F^{\mu\rho} + \partial^\mu F^{\rho\nu} = 0 \tag{1.5}$$

are immediately satisfied as a consequence of (1.4).

A significant amount of mathematical analysis has been devoted to the M-D equations (1.1),(1.3) and (1.4). It is a very complicated system consisting of 18 first-order real partial differential equations in the same number of real dependent variables with the 4 space time coordinates as the independent variables. A significant milestone was the solution of the Cauchy problem; the proof, published in 1997[19], of the existence of solutions with finite probability density, localised in an asymptotic sense. Yet the search for explicit realistic self-field configurations goes on.

Many have been seeking localised solutions, solitary waves, or 'solitons', stationary in time. The various studies have indeed shown that the equations support localised solutions. These investigators have considered it sufficient to require that the integrated probability density is finite and not necessarily equal to 1. It may be hoped that by varying some bare charge or mass parameters, a soliton might be obtained whose gross behviour exhibits the mass and charge that are normally encountered. Wakano[63] examined 'attraction by a repulsive potential', using a dominant spherically symmetric electrostatic potential with approximations. Lisi[43] repeated this work, showing that although magnetic effects were not spherically symmetric they were small in comparison with the electrostatic. Lisi also found a way to require unit normalisation of the electron probability. Radford[54] published a self-consistent field solution, spherically symmetric with respect to gauge independent quantities. Radford showed that the only truly spherically symmetric solutions required the electron wave function to be 'bound' by a monopole, which, in accordance with Dirac's prediction, had strength $1/e$. The solution was obtained numerically, and the electron number density, integrated over all space, was many times unit probability. Radford and Booth[11] also published a cylindrically symmetric self-consistent electrostatic field solution. This solution was also obtained numerically, and exhibited some of the characteristics of ‘attraction by a repulsive potential’ as in Wakano and Lisi’s work. Das and Kay[17] found a family of exact plane wave solutions for $m = 0$. Note that these last three mentioned configurations are invariant under actions of certain subgroups of the Poincaré group, and are in fact *Group Invariant Solutions*. Fushchich and Zhdanov [22][20] have performed a comprehensive study of the theory of group invariant spinor fields as solutions to nonlinear wave equations, and in the case of the M-D equations, published a family of exact light speed plane wave solutions of the M-D equations where $m \neq 0$ but where the length of the current vector was 0. Nevertheless, the range of possible group invariant fields that might potentially satisfy the M-D equations is very large, and these efforts have barely scratched the surface.

This thesis investigates the systematic application of the theory of group invariant solutions to the semiclassical Maxwell Dirac equations. This aspect, rather than the mathematical analysis, or localisation, or the asymptotic polarity, is the interest of the current work. The equations are retained as wave and field equations, without recourse to Lagrangian formulations. The thesis is restricted to the semiclassical theory and makes no use of QED. As in the above-mentioned prior M-D studies, all wave functions are assumed to be commuting rather than anticommuting. Consequently, the results have limited applicability.

1.3 Group invariant solutions

In order to understand what group invariant solutions are and why they might be of interest, it is necessary to explain the *transformation group of a set of differential equations*, the very notion through which Lie groups were discovered by Lie himself. A set of invertible infinitesimal transformations of solutions of a given set of differential equations that produce other solutions of the same equations is called a *transformation group of the set of differential equations.* In the case of *ordinary* differential equations (ODEs), the important result is that if the maximal transformation group of a set of ODEs is *solvable*, then it follows that the equations can be solved by successive quadratures (integrations). But while Lie group theory cannot offer a similar panacea for *partial* differential equations (PDEs), it does offer a method for finding some solutions, often exact explicit solutions, where possibly none was known. These are the group invariant solutions, and they are found by studying the system when it has been reduced by the action of subgroups of the maximal transformation group of the equations[42][51][50].

If the original equation has p independent variables, and if the action of S is *projectable* and generates an s dimensional orbit in X, then the equation for the solution which is group invariant under S is a differential equation in $p - s$ independent variables. If $s = p - 1$ then the group invariant solution, if it exists, can be found as the solution of an ODE. If $s = p$, then the group invariant solution, if it exists, is found by solving an algebraic equation for a set of constants. For PDEs, these are often the only explicit solutions that are known.

The aim of the present work is to identify group invariant solutions of the M-D equations in a systematic way. Ibragimov gives the Maxwell Dirac equations in his second volume[35], quoting their maximal invariance group (for $m \neq 0$) as a direct product of the Poincaré group and the Abelian gauge group, a result that should surprise no mathematical physicist.

It should be clear by now that having obtained a group invariant solution, or in fact *any* solution, then the existence of the maximal group of transformations of the set of equations allows us to extend that solution to a general form using transformations expressed in terms of the parameters of the entire group. In the case where the symmetry group is the Poincaré group, this is nothing more than the accepted physical principle that we can translate, rotate or boost any solution of the Maxwell Dirac equations.

A complete list of conjugacy equivalence classes of $s-$parameter subgroups of the group of the equation is called an *optimal* system of $s-$parameter subgroups. A complete list of $s-$parameter group invariant solutions is called an *optimal* system of solutions. The conjugacy equivalence classes of the Poincaré group have been calculated by Patera Winternitz and Zassenhaus[52], (henceforth referred to as PWZ), with full knowledge of their potential application for group invariant solutions.

Before closing this section, we note that there are other productive applications of Lie group theory to PDEs. The identification of conserved quantities in the M-D system is one example[26]. The separation of variables for the Dirac equation in the context of Lie group theory is another[38]. Furthermore, there are extensions of the method of group invariant solutions to more general symmetries[1] and partial invariance[51]. None of these extensions will be considered in the current work.

1.4 A gauge independent formulation of the Maxwell Dirac equations

The investigation commences in chapter 2 with the synthesis of a gauge independent formulation of the M-D equations. In the context of the theory of group invariant solutions, this means that the complete set of equations will have invariance simply under the Poincaré group, $\mathcal{P}$, rather than $\mathcal{P}$ extended by the Abelian gauge group. This will make it easier to study group invariant solutions, and will provide simpler access to a potentially larger set of solutions than would otherwise be possible under this method. We use the specific approach of Mickelsson[47], based on treating the Dirac spinor as being homomorphic to an element of $\mathbb{C}^{\times} \times SO(1,3)$, in which the bilinear Dirac currents are related to the components of a normalised tetrad. This is possible providing that the length of the current vector is non-zero. A significant feature of the Dirac equation rewritten in this form is that it is possible to invert the equation to express the four-potential as a function of the bilinear current densities and their derivatives. The coupled equations are still gauge dependent in this form and would normally require some form of gauge fixing. However, as pointed out by Mickelsson, Rudolph and Kijowksi[39], it is noted that the four-potential can be grouped with the only other gauge dependent term, to define a 'gauge independent total potential', $\mathfrak{B}_\mu$. The four-curl can be taken of $\mathfrak{B}_\mu$ to calculate the normal electomagentic field tensor by adding two gauge independent terms additional to the regular curl. The entire gauge independent formulation of the spinor electrodynamics closely follows the work of Takabayasi[58].

It is shown that the set of 18 first-order equations is now in terms of 20 gauge independent tensor quantities, which satisfy 3 additional algebraic relations. The imbalance in the number of equations and unknowns is resolved by pointing out one redundant equation in the set of 18. 10 of the quantities are the components of the electromagnetic field tensor and gauge independent total potential. The remaining 10 quantities are Dirac bilinear currents: The normalised four-current, u_μ, the normalised axial current k_μ, the scalar density σ and the pseudo scalar density ω. We term this set of equations the *gauge independent Maxwell Dirac equations.* Thus we have chosen to use these algebraically equivalent equations to the Maxwell Dirac equations, in terms of variables which are unaffected by $U(1)$ gauge transformations, and therefore invariant under the action of that group, and have completed a significant step towards finding group invariant solutions.

Chapter 2 continues with a study of the complete set of coupled gauge independent tensor Maxwell Dirac equations, including a count of variables, independent algebraic and differential equations, and constraints on solutions. We review the discrete symmetries of the system, including the well known shortcomings of charge conjugation as applied to commuting wave functions. We argue that the M-D equations are only valid for states that are predominantly electrons and consider it prudent to rule out positron-like solutions on physical grounds except with an arbitrary change in the sign of q through the entire set of equations.

In chapter 3, we show that as $\hbar \longrightarrow 0$, the gauge independent Dirac system simplifies to an equation expressing an exact energy-momentum balance between the (gauge independent) potential and the mass $\times$ velocity, and show that any solution to this equation is a solution for the equations of a relativistic charged fluid or dust with no internal pressures. This is compatible with the Maxwell Vlasov equations for a dilute plasma of single charge carrying species and sharp fluid velocity profile. This equation is coupled with the Maxwell equation to form a self-field classical approximation system,

$$\Box^2 u^\mu - \partial^\mu \partial_\nu u^\nu = \left(\frac{mc^2}{\mu_0 q^2}\right) \rho u^\mu.$$

We will show that when $\frac{\partial u^\mu}{\partial t} = 0$, and $\rho =$constant, solutions of these equations satisfy Helmholtz's equation componentwise. However, the coupled system is still nonlinear by virtue of the normalisation constraint on the relativistic fluid velocity, $u^\mu u_\mu = 1$. We write down a solution of this self-field classical approximation by inspection. This solution demonstrates how the self-repulsion can be balanced at a classical level in an unbounded sliding laminar current flow configuration where an associated perpendicular magnetic field provides a force that exactly balances the electrostatic repulsion. This simplified classical problem will guide and inspire our search for solutions to the full M-D system.

1.5 Group invariant solutions of the gauge independent equations

After a quick review of the theory and definitions of group invariant solutions in chapter 4, we then turn to the symmetry groups of the classical self-field and gauge independent Maxwell Dirac systems in chapter 5. Because the development to this point has eliminated gauge dependent quantities, the symmetry group of the gauge independent Maxwell Dirac equations is simply the Poincaré group (the Lorentz group extended by translations), with no extension to arbitrary gauge transformation functions required. Using the method of group invariant solutions to PDEs, our initial aim is to use those subgroups of the Poincaré group that generate 3 dimensional orbits when acting on $\mathbb{R}^4$, since these will reduce the number of independent variables from 4 to 1, leaving a set of ODEs for solution. However, we discover that even with this simplification, the number of equations and variables is far too large to deal with.

Instead, we turn to subgroups of the Poincaré group with *transitive* action— those that generate 4 dimensional orbits when acting on $\mathbb{R}^4$-since these will reduce the number of independent variables from 4 to 0, leaving a set of algebraic equations for solution. In fact, we notice that the solution already given for the classical self-field equations is invariant under the action of the transitive Poincaré subgroup with Lie generators $\{\mathbf{P}_t, \mathbf{P}_x, \mathbf{P}_z, \mathbf{P}_y + \kappa\mathbf{K}_z\}$. This subgroup generates 4 dimensional orbits when acting on $\mathbb{R}^4$. There are 45 families of conjugacy classes of transitive subgroups, 15 of whose action on vectors is uniform through all space giving fields with no variation and only one trivial zero solution. Of the remaining 30 cases, it is sufficient to test solutions for 19 that are subgroups of the other 11. Vectors invariant under 4 dimensional orbits are known functions of (t, x, y, z) based upon the 4 components of a fixed constant reference four-vector. For the M-D equations, by employing the aforementioned inversion to eliminate the vector potential, we can write the entire set in terms of just 2 scalars and 2 reference four-vectors, a total of 10 constants. When these forms are substituted back into M-D equations, a solution will exist if and only if the 10 constants satisfy 10 algebraic relations.

This is a very unusual application of the theory of group invariant solutions, and can only ever yield solutions where scalar quantities are constant. Nevertheless, it has turned out to be very useful here, where the system is so very complex and the number of subgroups of the transformation group is so large.

In chapter 6, M-D solutions are checked under each of the 19 mentioned classes of subgroups. In some cases, it was necessary to check only 2 equations in order to demonstrate that they could only be satisfied by a *null* four-current, outside the realm of validity of the gauge independent equations. Other cases could only be checked after extensive computations in Mathematica. Only one case, $\{\mathbf{P}_t, \mathbf{P}_x, \mathbf{P}_z, \mathbf{P}_y + \kappa\mathbf{K}_z\}$, yielded non-null-current solutions to the full M-D System. Having obtained the solution initially in tensor variables, we use

Crawford's result[14] for the reconstruction of a spinor from bilinear densities. As a check, we show that the original Dirac equation is satisfied by this spinor. Although this solution is *not* localised and is unbounded as $|y| \longrightarrow \infty$, it is to the author's knowledge the first ever published explicit closed form solution (in terms of standard functions) of the Maxwell Dirac equations for non-null-current. It is also shown to be consistent with tunnelling through a laminar barrier separating an electron field into two parts, in which the field on one side of the barrier has non zero velocity relative to the other.

Chapter 7 consists of conclusions, observations, prospects and recommendations for further work. Appendix A gives the entire list of conjugacy classes of subalgebras of the Poincaré group, $\mathcal{P}$, as classified by Patera, Winternitz and Zassenhaus [52], hereafter referred to as PWZ . Appendix B gives the results of the calculations of the invariant fields and algebraic relations arising from the reduced equations.

Chapter 2

Gauge independent Maxwell Dirac system

This work concerns group invariant solutions of the coupled Maxwell-Dirac equations. Whereas Maxwell's equations are in terms of tensors, the Dirac equation is in terms of spinors. The first aim of this work is to write the Dirac equation in terms of a set of variables that more closely matches those used in Maxwell's equations, that is, to replace the spinor component variables with tensor currents. Writing the Dirac equation in terms of tensor currents is not new, and has appeared in the literature before[58], [65], [37], [47], [15], [39] and [41]. The Dirac equation can be 'solved' for the potential in terms of tensor variables (demonstrated elsewhere in terms of spinor coordinates [54], [10]), which in principle allows the electromagnetic field to be eliminated from the coupled Maxwell Dirac equations. Proceeding, it is shown that the system of equations can be cast in a gauge independent form, with gauge independent variables. This has also been demonstrated by [58], [37], [47] and [39].

The chapter commences with a review of the Dirac bilinear current densities, the relations between them, and the way in which the spinor can be reconstructed from them. It is shown that the Dirac field at any point can be re-expressed in terms of a complex number (scalar plus imaginary pseudo scalar), and a tetrad field based on Dirac bilinear currents, with the same number of degrees of freedom as the original Dirac field.

Sections 2.2, 2.3 and 2.4 transcribe the Dirac equation to be in terms of these new variables. The approach is unconventional in 3 regards:

1. In section 2.2, the spinor is treated initially not as $\psi \in \mathbb{C}^4$, but as $\mathbf{\Psi} \in M_2(\mathbb{C})$, which was first published by Gürsey [30],[31].

2. In section 2.3, imposing the condition $\det \mathbf{\Psi} \neq 0$, we restrict ourselves to invertible matrices $GL_2(\mathbb{C})$. As shown by Mickelsson[47], using the double covering Lie group homomorphism $GL_2(\mathbb{C}) \longrightarrow \mathbb{C}^\times \times SO(1,3)$ and its tangent map, the Dirac equation can be rewritten as a tensor equation, mathematically equivalent to the standard formulation providing the length of the current vector is non-zero.

3. In section 2.4, the resulting equation is shown to be an inversion for the

electromagnetic potential A_μ in terms of tensor current densities and their derivatives.

The presentation is given in more detail than Mickelsson's original work, exemplifying connections with other known work. Unlike the spinor form of the Dirac equation, the tensor derivations as given in the literature are often extremely lengthy. While it is beyond the scope of this work to give a full overview of such approaches, the author has attempted to bring together some of the salient points of this complicated theory into a succint presentation, focussing in particular on the role of the tetrad. We hope that this will enable the reader to see the merit in such formulations, such as the ability to visualise the Dirac field.

In section 2.5, the gauge-dependent electromagnetic four-potential, A_μ, is grouped with the other gauge-dependent terms to define a new four-vector - $\mathfrak{B}_\mu$, the 'gauge independent total potential', in terms of currents. We show how it is possible to calculate $F^{\mu\nu}$ from $\mathfrak{B}_\mu$, having eliminated A_μ from the problem. Since these equations have eliminated the notion of the phase, θ, of the wave function, it is necessary to impose an extra integral loop condition on solutions. In section 2.6, we then present the complete set of coupled gauge independent tensor Maxwell Dirac equations, including a count of variables, independent algebraic and differential equations, and constraints on solutions.

Section 2.7 discusses how, in the absence of a probability normalisation condition, the fine structure constant effectively plays no role in the problem, being absorbed into the dimensionless variables of the problem. Section 2.8 gives the action of parity transformations on the tensor quantities. The final section, 2.9, reviews the problem of charge conjugation of the the charge current $\mathcal{J}^\mu = q\overline{\psi}\gamma^\mu\psi$ that emerges when ψ is a commuting wave function.

2.1 Bilinear tensor current densities

2.1.1 The four-current

As is well known in the Dirac system, the vector defined by

$$J_\mu \equiv \overline{\psi}\gamma_\mu\psi$$

where

$$\overline{\psi} \equiv \psi^\dagger\gamma^0$$

has the following properties:

1.

$$\begin{aligned} J_0 &= \overline{\psi}\gamma_0\psi = \psi^\dagger\psi \\ &= |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \end{aligned}$$

which is strictly positive definite.

2. The probability conservation relation

$$\partial_\mu J^\mu = 0, \tag{2.1}$$

can be derived directly from the Dirac equation by taking bilinear inner products with $\overline{\psi}$ from the left, followed by taking the imaginary part.

On the basis of these properties, J_μ can be successfully interpreted as the local four-current number density or probability density of the electron cloud.

2.1.2 Dirac algebra and bilinear currents

Based on the commutation relations (1.2), the entire Dirac algebra, a complex Clifford algebra, $Cl_{4,\mathbb{C}}$ can be constructed, by repeated bracketing. The algebra contains 16 generators, which span $M_4(\mathbb{C})$ as a vector space over $\mathbb{C}$. The table below provides the definitions (refer [36]) for these 16 generators, Γ, plus the Dirac bilinear currents (refer [44]) arising from the application of Γ as a density operator $\overline{\psi}\Gamma\psi$. Where required, factors of i are introduced to ensure every defined generator Γ is Hermitian, and therefore each density is a real quantity.

Generators, Γ	Count	Associated bilinear density	Nature of the densities
1	1	$\sigma \equiv \overline{\psi}\psi$	Scalar
γ^μ	4	$J^\mu \equiv \overline{\psi}\gamma^\mu\psi$	Vector
$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	6	$S^{\mu\nu} \equiv \overline{\psi}\sigma^{\mu\nu}\psi$	Rank 2 anti-symmetric
$\gamma^5\gamma^\mu$	4	$K^\mu \equiv \overline{\psi}\gamma^5\gamma^\mu\psi$	Axial Vector
$i\gamma^5 \equiv -\gamma^0\gamma^1\gamma^2\gamma^3$	1	$\omega \equiv \overline{\psi}i\gamma^5\psi$	Pseudo Scalar

All these bilinear densities are tensors, based on Hermitian operators, and 'observables' in the language of quantum mechanics.

2.1.3 Relations between bilinears

In addition to the Dirac bilinear currents defined above, additional bilinears can be defined using the charge conjugate spinor on the left. In particular we define M_μ and N_μ as the real and imaginary parts of

$$M_\mu + iN_\mu \equiv \overline{\psi_c}\gamma_\mu\psi,$$

where ψ_c is the charge conjugate spinor, $\psi_c = C\gamma^0\psi^*$. Note this is a complex quantity since $\overline{\psi_c}\gamma_\mu$ is not expressible as $\psi^\dagger H$ with H a Hermitian operator, and so neither M_μ nor N_μ can be called an 'observable'. They are, in fact, gauge-dependent tensors, in a similar way that A_μ is a gauge-dependent tensor. For the remainder of this section we will refer to them as one of the collective set of tensor currents.

The bilinears $\sigma, \omega, J^\mu, S^{\mu\nu}, K^\mu, M^\mu, N^\mu$ so defined are clearly not independent, all having been derived from just 4 complex (8 real) components of the spinor ψ. A set of algebraic relations does exist between these quantities [37][61][44]. These are often called the Fierz identities, and are attributed to Kofink. Here is a basic set, from which other identities can be derived:

$$J^\mu J_\mu = \sigma^2 + \omega^2 \tag{2.2}$$

$$J^\mu J_\mu = -K^\mu K_\mu = -M^\mu M_\mu = -N^\mu N_\mu \tag{2.3}$$

$$J^\mu K_\mu = 0 \qquad J^\mu M_\mu = 0 \qquad J^\mu N_\mu = 0$$

$$M^\mu N_\mu = 0 \qquad K^\mu M_\mu = 0 \qquad K^\mu N_\mu = 0 \tag{2.4}$$

$$S_{\nu\mu} + i^* S_{\nu\mu} = \left(\frac{\sigma + i\omega}{\rho^2}\right)\left[(M_\nu N_\mu - N_\nu M_\mu) + i\,(J_\nu K_\mu - K_\nu J_\mu)\right] \tag{2.5}$$

where

$$^*S_{\mu\nu} \equiv -\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}S^{\alpha\beta} = \overline{\psi} i\gamma_5\sigma_{\mu\nu}\psi,$$

and using the Levi Civita tensor

$$\varepsilon^{0123} = 1 = -\varepsilon_{0123}.$$

Since the right hand side of (2.2) is necessarily positive, it follows that J^μ is timelike whereas K^μ, M^μ and N^μ are spacelike. Therefore, the length of the current vector can be defined,

$$\rho \equiv \sqrt{J^\mu J_\mu},$$

and at each point x^μ locally there exists a reference frame in which the current will appear to be at rest with rest probability density ρ: $J'_\mu = (\rho, 0, 0, 0)$. Where the particle has charge q, the *charge current* is taken to be

$$\mathcal{J}_\mu = qJ_\mu.$$

We find it convenient to define β as the argument of the complex number formed from the scalar and pseudoscalar Dirac densities:

$$\rho e^{i\beta} = \sigma + i\omega \in \mathbb{C}.$$

We define a relativistic fluid velocity by dividing the current by its length:

$$u^\mu \equiv \frac{J^\mu}{\rho}\ .$$

Then u^μ is a relativistic unit four-vector:

$$u^\nu u_\nu = 1 = u_0^2 - \mathbf{u}^2, \text{ and therefore } u_0 \geqq 1.$$

2.1.4 The tetrad

From the equality in (absolute) length of J^μ, K^μ, M^μ and N^μ, and the orthogonality relations, (2.4), we note that if we were to normalise all these vectors, we would have the columns of an orthonormal matrix $\mathbf{e} \in SO(1,3)$, called a Minkowski *tetrad*, or *vierbein* (four-leg), and so in addition to u^μ, for $\rho \neq 0$, we define:

$$k^\mu \equiv \frac{K^\mu}{\rho}, \qquad m^\mu \equiv \frac{M^\mu}{\rho}, \qquad n^\mu \equiv \frac{N^\mu}{\rho}.$$

Then by virtue of (2.3) and (2.4),

$$\mathbf{e}^\mu_{\ \alpha} \equiv (u^\mu, m^\mu, n^\mu, k^\mu) \in SO(1,3). \tag{2.6}$$

Every member of $SO(1,3)$ defines a Lorentz transformation and $\mathbf{e}^\mu_{\ \alpha}$ can be manipulated in the same way as a rank 2 tensor, with raising and lowering operations on the indexes. In order to avoid confusion, we will always put the tetrad leg selector index as the rightmost index, and use Greek letters from the beginning of the alphabet, α, β... for it.

All these quantities are well defined by the Dirac spinor and have been represented in diagram 2.1. The diagram shows that the orthogonal bilinear currents before normalisation all have the same absolute length as the modulus of the complex quantity $\rho e^{i\beta} = \sigma + i\omega$. It is easy enough to visualise the normalised tetrad simply by dividing each orthogonal current by the same length, ρ. We cannot represent the 4 orthogonal axes of the Minkowski tetrad in a diagram. But in the centre, we show the 3 spacelike legs as an orthonormal triad frame in which J^μ is 'at rest'. On the right, we show J^μ on a time axis and a single space axis representing the direction of the spatial part $\mathbf{J}$. The diagram shows under a hyperbolic rotation of J^μ to an *at rest* current density, it lines up to equal the other lengths.

We define the 'spin plane' and its dual

$$\begin{aligned} H_{\mu\nu} &\equiv \epsilon_{\nu\mu\alpha\beta} u^\alpha k^\beta \\ &= m_\nu n_\mu - n_\nu m_\mu \\ {}^*H_{\mu\nu} &= u_\nu k_\mu - k_\nu u_\mu \end{aligned} \tag{2.7}$$

after Hestenes [33] and Takabayasi[58].

All of the bilinear currents can be expressed in terms of a base set of fields:

$$\{\rho, \beta, (u^\mu, m^\mu, n^\mu, k^\mu)\} \tag{2.8}$$

Having the same form as a Lorentz transformation, the tetrad can be uniquely specified by a total of 6 independent parameters. Therefore, the set of quantities (2.8) contain 8 independent degrees of freedom, the same number of real quantities as the original spinor, ψ. This set uniquely determines ψ up to a sign.

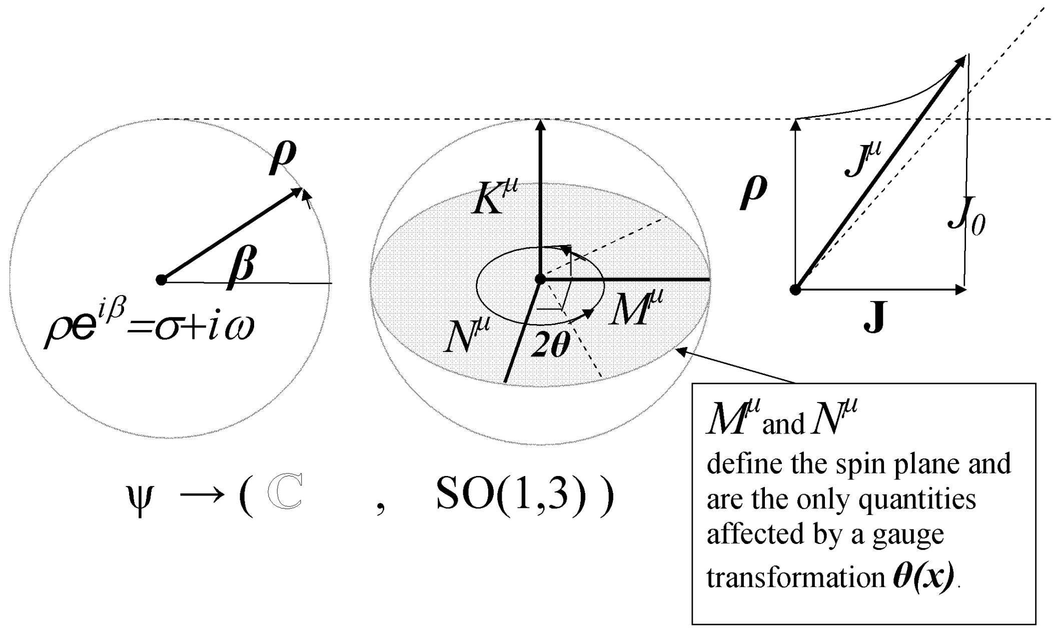


Figure 2.1: Diagrammatic representation of the tensor quantities that equivalently represent the Dirac spinor with the same degrees of freedom. Spacelike K^μ, M^μ, N^μ and timelike J^μ are orthogonal and equal in absolute length to ρ, the modulus of the complex number $\sigma + i\omega$. The 4 vectors can be normalised to form a tetrad $\in SO(1,3)$. In a frame where $\mathbf{J}$ is zero, K^μ, M^μ and N^μ form an orthogonal triad.

2.1.5 Gauge transformations

Under $U(1)$ gauge transformations, $\psi \longrightarrow \psi e^{i\theta(x)}$, only the four-vector quantities A_μ, (m_μ and n_μ), (M_μ and N_μ) are affected:

$$\begin{aligned} qA_\mu &\longrightarrow qA_\mu + \hbar c \partial_\mu \theta(\mathbf{x}) \\ m_\mu &\longrightarrow \cos(2\theta) m_\mu + \sin(2\theta) n_\mu \\ n_\mu &\longrightarrow \cos(2\theta) n_\mu - \sin(2\theta) m_\mu . \end{aligned}$$

The observable tensor current variables are unaffected

$$\begin{aligned} \rho e^{i\beta} &\longrightarrow \rho e^{i\beta} \\ u_\mu &\longrightarrow u_\mu \\ k_\mu &\longrightarrow k_\mu \\ S_{\mu\nu} &\longrightarrow S_{\mu\nu} . \end{aligned}$$

Later is this chapter, we will rewrite the M-D system in terms of the latter, gauge independent quantities, and eliminate the former, gauge dependent quantities. So rather than writing a formula for reconstrucing ψ from the entire gauge *dependent* set, $\{\rho, \beta, (u^\mu, m^\mu, n^\mu, k^\mu)\}$, (which would be possible up to a *sign*), we will give a formula for reconstructing ψ from the gauge independent quantities $\{\rho, \beta, u^\mu, k^\mu\}$. This will be possible only up to an indeterminate *phase*.

2.1.6 Spinor reconstruction

Crawford[14] has published the following formulae for the reconstruction of a spinor, up to an indeterminate phase, from its *observable* bilinear currents:

$$\psi = \frac{\exp(-i\lambda)}{2\sqrt{\overline{\eta}Z\eta}} Z\eta$$

where λ is an arbitrary phase and η is any spinor such that

$$\sqrt{\overline{\eta}Z\eta} \neq 0 \tag{2.9}$$

at the point in question, and

$$\begin{aligned} Z &= (\sigma + i\omega\gamma_5 + J_\mu\gamma^\mu)(\sigma + i\omega\gamma_5)^{-1}(\sigma + i\omega\gamma_5 + K_\nu\gamma_5\gamma^\nu) \\ &= \rho(\cos\beta + i\sin\beta\gamma_5 + u_\mu\gamma^\mu)(\cos\beta + i\sin\beta\gamma_5)^{-1}(\cos\beta + i\sin\beta\gamma_5 + k_\nu\gamma_5\gamma^\nu) \end{aligned}$$

(For a more cumbersome reconstruction formula that is not manifestly covariant, see Takahashi [60].)

2.2 The Dirac equation over $M_2(\mathbb{C})$

Having introduced the Dirac equation and tetrad in the regular formalism, we now show that the tensor bilinear variables, based fundamentally on the tetrad $\mathbf{e}^\mu_{\ \alpha}$, and the complex number $\rho e^{i\beta}$, arise very elegantly starting from rewriting the Dirac equation acting on members of $M_2(\mathbb{C})$ (the complex quaternions, or biquaternions $\mathbb{B}$). In this form the equation can be solved for the potential A_μ via a trivial matrix inversion, as demonstrated by Mickelsson[47]. Although not referenced by Mickelsson, Gürsey in 1950[30][31] was the first to write the Dirac equation over members of $M_2(\mathbb{C})$. The approach is essentially the same as the van der Waerden bispinor formalism, with the 2 component γ^5 projections, u and $\overline{v}$ arranged side by side in a 2×2 matrix.

2.2.1 Derivation

In the γ^5 diagonal or chiral representation, using the definitions given in chapter 1, the Dirac equation divides neatly into 2 separate 2 x 2 matrix equations:

$$(-qA_0\sigma^0 + qA_k\sigma^k + i\hbar c\partial_0\sigma^0 - i\hbar c\partial_k\sigma^k)\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} + mc^2\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = 0 \tag{2.10}$$

$$(-qA_0\sigma^0 - qA_k\sigma^k + i\hbar c\partial_0\sigma^0 + i\hbar c\partial_k\sigma^k)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} + mc^2\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} = 0 \tag{2.11}$$

The first equation is entirely conjugated,

$$(-qA_0\sigma^0 + qA_k\sigma^{*k} - i\hbar c\partial_0\sigma^0 + i\hbar c\partial_k\sigma^{*k})\begin{pmatrix}\psi_3^*\\ \psi_4^*\end{pmatrix} + mc^2\begin{pmatrix}\psi_1^*\\ \psi_2^*\end{pmatrix} = 0,$$

then premultiplied by $i\sigma^2 = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and using $\sigma^2\sigma^1 = -\sigma^1\sigma^2, \sigma^2\sigma^3 = -\sigma^3\sigma^2, \sigma^2\sigma^2 = \sigma^2\sigma^2$, $\sigma^{1*} = \sigma^1$, $\sigma^{2*} = -\sigma^2$, $\sigma^{3*} = \sigma^3$

$$\left(qA_0\sigma^0 + qA_k\sigma^k + i\hbar c\partial_0\sigma^0 + i\hbar c\partial_k\sigma^k\right)\begin{pmatrix} -\psi_4^* \\ \psi_3^* \end{pmatrix} - mc^2\begin{pmatrix} -\psi_2^* \\ \psi_1^* \end{pmatrix} = 0 \quad (2.12)$$

In the van der Waerden bispinor formalism, the 2 columns are kept separate with the notation:

$$u \equiv u^A \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \overline{v} \equiv v^{\dot{A}} \equiv \begin{pmatrix} -\psi_4^* \\ \psi_3^* \end{pmatrix}$$

and using the lowering convention

$$\overline{u} \equiv u_A = \begin{pmatrix} -\psi_1^* & \psi_2^* \end{pmatrix}, v \equiv v_{\dot{A}} = \begin{pmatrix} \psi_4 & \psi_3 \end{pmatrix}$$

the Dirac equation is rewritten as 2 separate 2 component equations:

$$\begin{aligned} \left(\hbar c\partial_\nu\sigma^{\nu A\dot{A}} + iqA_\nu\sigma^{\nu A\dot{A}}\right)u_A + imc^2\overline{v}^{\dot{A}} &= 0 \\ \left(\hbar c\partial_\nu\sigma^{\nu A\dot{A}} - iqA_\nu\sigma^{\nu A\dot{A}}\right)v_A + imc^2\overline{u}^{\dot{A}} &= 0. \end{aligned}$$

But in the presentation of Gürsey (2.11) and (2.12) can be combined into a single 2 x 2 matrix equation:

$$\begin{aligned} - qA_\nu\sigma^\nu\left(\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, -\begin{pmatrix} -\psi_4^* \\ \psi_3^* \end{pmatrix}\right) + i\hbar c\partial_\nu\sigma^\nu\left(\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} -\psi_4^* \\ \psi_3^* \end{pmatrix}\right) \\ + mc^2\left(\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}, -\begin{pmatrix} -\psi_2^* \\ \psi_1^* \end{pmatrix}\right) = 0 \end{aligned}$$

Where the second column carries a negative sign, the term is rewritten using postmultiplication by σ^3:

$$-qA_\nu\sigma^\nu\begin{pmatrix} \psi_1 & -\psi_4^* \\ \psi_2 & \psi_3^* \end{pmatrix}\sigma^3 + i\hbar c\partial_\nu\sigma^\nu\begin{pmatrix} \psi_1 & -\psi_4^* \\ \psi_2 & \psi_3^* \end{pmatrix} + mc^2\begin{pmatrix} \psi_3 & -\psi_2^* \\ \psi_4 & \psi_1^* \end{pmatrix}\sigma^3 = 0. \quad (2.13)$$

The first 2 two terms are seen to be matrix operations on

$$\mathbf{\Psi} \equiv \begin{pmatrix} \psi_1 & -\psi_4^* \\ \psi_2 & \psi_3^* \end{pmatrix}. \quad (2.14)$$

We can rewrite the final term using the definition of the quaternionic conjugate. The reader should already be familiar with the notion of the concept of the Hermitian conjugate of a matrix $\mathbf{\Psi} \in M_2(\mathbb{C})$, $\mathbf{\Psi}^\dagger \equiv \mathbf{\Psi}^{*T}$. The definition of for the quaternionic conjugate of a matrix $\mathbf{\Psi} \in M_2(\mathbb{C})$,where $\mathbf{\Psi}$ has complex components over a basis of Pauli matrixes, is given thus:

$$\begin{aligned} \text{If } \mathbf{\Psi} &= z_0\sigma^0 + z_k\sigma^k \;, \; z_\mu \in \mathbb{C} \\ \text{then } \bar{\mathbf{\Psi}} &\equiv z_0\sigma^0 - z_k\sigma^k. \end{aligned} \quad (2.15)$$

Decomposing (2.14) in this way, we have $z_0 = \frac{1}{2}(\psi_1 + \psi_3^*)$, $z_1 = \frac{1}{2}(\psi_2 - \psi_4^*)$, $z_2 = \frac{i}{2}(\psi_2 + \psi_4^*)$, $z_3 = \frac{1}{2}(\psi_1 - \psi_3^*)$, giving

$$\bar{\mathbf{\Psi}} = \begin{pmatrix} \psi_3^* & \psi_4^* \\ -\psi_2 & \psi_1 \end{pmatrix}.$$

Where we have a matrix expression, eg MN, we will use the notation $\overline{MN}$ to indicate the quaternionic conjugate of the entire expression. Quaternionic conjugation obeys the same commutation principle as Hermitian conjugation, $\overline{MN} = \overline{N}\ \overline{M}$. If we introduce the quaternionic conjugates of the individual Pauli matrices, $\overline{\sigma}^0 = \sigma^0$, $\overline{\sigma}^k = -\sigma^k$, then the quaternionic conjugate of $z_\mu\sigma^\mu$ is $z_\mu\overline{\sigma}^\mu$.

We notice that taking the Hermitian conjugate of the final matrix term in (2.13) yields the quaternionic conjugate of $\mathbf{\Psi}$,

$$\begin{pmatrix} \psi_3 & -\psi_2^* \\ \psi_4 & \psi_1^* \end{pmatrix}^\dagger = \begin{pmatrix} \psi_3^* & \psi_4^* \\ -\psi_2 & \psi_1 \end{pmatrix} = \bar{\mathbf{\Psi}},$$

and so we can write Ψ in the last term using the composition of both conjugates, arriving at Gürsey's Dirac equation:[1]

$$\hbar c \partial_\mu \sigma^\mu \mathbf{\Psi} + qA_\mu\sigma^\mu\mathbf{\Psi} i\sigma^3 - mc^2\bar{\mathbf{\Psi}}^\dagger i\sigma^3 = 0. \tag{2.16}$$

This form has a certain advantages, such as the ease of inversion for A_μ, and the appearance of the tangent space of $GL_2(\mathbb{C})$, which will be used in section 2.3 as part of an elegant transformation into tensor quantities. Another demonstration of the possibilities of this form is given by Hestenes in his book on space time algebra[32][33], quoting Gürsey's equation in the form:

$$\hbar c \Box \mathbf{\Psi} = \left[e\mathbf{A}\mathbf{\Psi} + mc^2\mathbf{\Psi}^*\right] i\sigma^3$$

but setting it in the context of an equation acting on $\mathbf{\Psi} \in Cl_{\mathbb{R}}(3,1)$ (a real Clifford algebra with generators e^μ, and where $\Box = \partial_\mu e^\mu$). In Hestenes' treatment, the * operator has essentially the same outcome as the composite conjugation $\bar{\mathbf{\Psi}}^\dagger$. Mickelsson explicitly defines the operator $\dot{\mathbf{\Psi}}$ to achieve the same outcome, without any reference to Gürsey's work.

2.2.2 Inversion for A_μ in $M_2(\mathbb{C})$

Just as for Hamilton's real quaternions, the quaternionic conjugate can be used to calculate the determinant of $\mathbf{\Psi} = z_\mu\sigma^\mu$

$$\begin{aligned} \det \mathbf{\Psi} &= \bar{\mathbf{\Psi}}\mathbf{\Psi} \\ &= z_0^2 - z_1^2 - z_2^2 - z_3^2, \ \ z_\mu \in \mathbb{C}. \end{aligned}$$

We can easily find for $\mathbf{\Psi}$ given by (2.14),

$$\bar{\mathbf{\Psi}}\mathbf{\Psi} = \rho e^{i\beta} = \sigma + i\omega. \tag{2.17}$$

[1] Instead of using Pauli matrices, Gürsey follows the mathematical convention of using quaternions that square to -1, using the symbol e_3 in place of $i\sigma^3$.

Therefore (providing $\rho \neq 0$):

$$\mathbf{\Psi}^{-1} = \frac{\bar{\mathbf{\Psi}}}{\rho e^{i\beta}}.$$

From now on we will assume that $\det(\mathbf{\Psi}) \neq 0$, equivalently, $\rho \neq 0$, restricting ourselves to

$$\mathbf{\Psi} \in GL_2(\mathbb{C}),$$

the general linear group of invertible 2×2 matrices over $\mathbb{C}$.

Note that since (2.16) is a matrix equation, rather than a spinor equation, we can right multiply it by $\sigma_3^{-1}\mathbf{\Psi}^{-1} = \sigma_3\bar{\mathbf{\Psi}}/\left(\rho e^{i\beta}\right)$, and rearrange the potential term to the LHS to obtain

$$qA_\mu\sigma^\mu = -mc^2\frac{\bar{\mathbf{\Psi}}^\dagger\bar{\mathbf{\Psi}}}{\rho e^{i\beta}} + i\hbar c\frac{\left(\partial_\mu\sigma^\mu\mathbf{\Psi}\right)\sigma_3\bar{\mathbf{\Psi}}}{\rho e^{i\beta}}, \tag{2.18}$$

which provides the inversion for A_ν in terms of spinor components, the components being extracted from $A_\mu\sigma^\mu$ via $A^\nu = Tr(A_\mu\sigma^\mu\sigma^\nu)$. This simple inversion of the Dirac equation for the four-potential in terms of spinor components is equivalent that given by Radford [54] .

2.3 Group homomorphism to tensor variables

We now show that the differential terms in Gürsey's Dirac equation can be elegantly rewritten in terms of derivatives of tensor currents, specifically the tetrad, using the group homomorphism from $GL_2(\mathbb{C})$ to $\mathbb{C}^\times \times SO(1,3)$. This innovation, and the preceding inversion for A_ν in $M_2(\mathbb{C})$, are due to Mickelsson[47].

2.3.1 Bilinear currents in the $M_2(\mathbb{C})$ formalism

Before we write the same equation in terms of tensors, we need to be able to recognise the bilinear currents in the $M_2(\mathbb{C})$ formalism, and understand the group nature of the map to the tetrad. In this formalism, we can easily establish the following identities:

$$\bar{\mathbf{\Psi}}\mathbf{\Psi} = \rho e^{i\beta} \tag{2.19}$$
$$\mathbf{\Psi}^\dagger\mathbf{\Psi} = J_\mu\sigma^\mu \tag{2.20}$$
$$\mathbf{\Psi}^\dagger\sigma_3\mathbf{\Psi} = K_\mu\sigma^\mu \tag{2.21}$$
$$\mathbf{\Psi}^\dagger\sigma_1\mathbf{\Psi} = M_\mu\sigma^\mu \tag{2.22}$$
$$\mathbf{\Psi}^\dagger\sigma_2\mathbf{\Psi} = N_\mu\sigma^\mu \tag{2.23}$$
$$\bar{\mathbf{\Psi}}i\sigma_3\mathbf{\Psi} = \left(S_{\mu\nu} + i^*S_{\mu\nu}\right)\frac{1}{2}\left(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu\right) \tag{2.24}$$

2.3.2 Homomorphism

By separating out the centre,

$$\begin{aligned} GL_2(\mathbb{C}) &\cong \mathbb{C}^\times \times SL_2(\mathbb{C}), \text{ via the map} \\ \Xi(\mathbf{\Psi}) &= \left(\det(\mathbf{\Psi}), \frac{\mathbf{\Psi}}{\sqrt{\det(\mathbf{\Psi})}}\right). \end{aligned}$$

Furthermore, there is a well-known 2 to 1 homomorphism

$$\begin{aligned} \Delta &: SL_2(\mathbb{C}) \longrightarrow SO(1,3), \\ \Delta(\Phi) &= \Lambda^\mu_\nu \\ \text{where } \Lambda^\mu_{\ \alpha} &= \left(Tr\left(\Phi^\dagger \sigma_\alpha \Phi \sigma^\mu\right)\right) \end{aligned}$$

both spaces being representations of the Lorentz group, $\mathcal{L}$. Therefore, there exists a homomorphism Θ being the composition of Ξ and Δ

$$\begin{aligned} \Theta &: GL_2(\mathbb{C}) \longrightarrow \mathbb{C}^\times \times SO(1,3) \\ \Theta(\mathbf{\Psi}) &= \left(\det \mathbf{\Psi}, \frac{Tr \mathbf{\Psi}^\dagger \sigma_\alpha \mathbf{\Psi} \sigma^\mu}{|\det \mathbf{\Psi}|}\right) \\ &= \left(\rho e^{i\beta}, \mathbf{e}^\mu_{\ \alpha}\right) \end{aligned} \tag{2.25}$$

which provides a group homomorphic context for the previously established set of base tensor quantities (2.8). We know no purpose for group multiplication of Dirac spinors *per se*, although we will find in the next section, a use for the tangent map of Θ between Lie algebras

$$\theta : \mathfrak{gl}_2(\mathbb{C}) \longrightarrow \mathfrak{c} + \mathfrak{so}(1,3)$$

We will use the following expression giving any $\bar{\mathbf{\Psi}}^{-1}\partial_\tau \bar{\mathbf{\Psi}} \in \mathfrak{gl}_2(\mathbb{C})$ expanded over the generators of $\mathfrak{gl}_2(\mathbb{C})$ with coefficients expressed in terms of $\left(\rho e^{i\beta}, \mathbf{e}^\mu_{\ \alpha}\right)$

$$\bar{\mathbf{\Psi}}^{-1}\partial_\tau \bar{\mathbf{\Psi}} = \frac{1}{2}\rho^{-1}\left(\partial_\tau \rho\right)\sigma^0 + \frac{1}{2}\partial_\tau \beta i \sigma^0 + \frac{1}{2}\left[\mathbf{e}_{\nu\alpha}\left(\partial_\tau \mathbf{e}^\nu_{\ \beta}\right)\right]\frac{1}{2}\left(\sigma^\alpha \overline{\sigma}^\beta - \sigma^\beta \overline{\sigma}^\alpha\right) \tag{2.26}$$

The first 2 terms are from the Lie algebra for $\mathbb{C}^\times$. The final sum is over the generators $\frac{1}{2}\left(\sigma^\alpha \overline{\sigma}^\beta - \sigma^\beta \overline{\sigma}^\alpha\right)$ of the Lorentz group $SL_2(\mathbb{C})$.

2.4 Tensor Inversion for A_μ

2.4.1 A tensor version of the Dirac equation

Take traces of equation (2.18)

$$qA^\tau = \frac{-mc^2}{\rho e^{i\beta}} Tr\left(\sigma^\tau \bar{\mathbf{\Psi}}^\dagger \bar{\mathbf{\Psi}}\right) + \frac{\hbar c}{\rho e^{i\beta}} Tr\left(\sigma^\tau \sigma^\nu \left(\partial_\nu \bar{\mathbf{\Psi}}\right) i\sigma^3 \mathbf{\Psi}\right). \tag{2.27}$$

By (2.20), $Tr\left(\sigma^\tau \bar{\mathbf{\Psi}}^\dagger \bar{\mathbf{\Psi}}\right) = J^\tau$ To calculate the second trace, use the identity $\sigma^\tau \sigma^\nu = \frac{1}{2}\left[\sigma^\tau \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\tau\right] + \eta^{\tau\nu}$ to expand:

$$\begin{aligned} Tr\left(\sigma^\tau \sigma^\nu \left(\partial_\nu \bar{\mathbf{\Psi}}\right) i\sigma^3 \mathbf{\Psi}\right) &= Tr\left(\frac{1}{2}\left[\sigma^\tau \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\tau\right]\left(\partial_\nu \bar{\mathbf{\Psi}}\right) i\sigma_3 \mathbf{\Psi}\right) + Tr\left(\left(\partial^\tau \bar{\mathbf{\Psi}}\right) i\sigma_3 \mathbf{\Psi}\right) \\ &\equiv X^\tau + Y^\tau , \end{aligned} \tag{2.28}$$

using the symbols X^τ, Y^τ to label the first and second terms respectively. Taking the first term, the trace is unchanged by quaternionic conjugation:

$$\begin{aligned} X^\tau &= Tr\left(\frac{1}{2}\left[\sigma^\tau\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\tau\right]\left(\partial_\nu\bar{\mathbf{\Psi}}\right)i\sigma_3\mathbf{\Psi}\right) \qquad (2.29)\\ &= Tr(\bar{\mathbf{\Psi}}\overline{\sigma^3}i\partial_\nu\mathbf{\Psi}\overline{\frac{1}{2}\left[\sigma^\tau\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\tau\right]}). \end{aligned}$$

Now $\overline{\sigma^3} = -\sigma^3$ and $\overline{(\sigma^\tau\overline{\sigma}^\nu)} = -\sigma^\tau\overline{\sigma}^\nu$, $\tau \neq \nu$. We also use the cyclicity of the trace to commute the $\frac{1}{2}\left[\sigma^\tau\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\tau\right]$ term to the front.

$$X^\tau = Tr(\frac{1}{2}\left[\sigma^\tau\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\tau\right]\bar{\mathbf{\Psi}}i\sigma^3\partial_\nu\mathbf{\Psi}) \qquad (2.30)$$

Adding (2.29) and (2.30), and halving,

$$\begin{aligned} X^\tau &= \frac{1}{2}Tr\left\{\frac{1}{2}\left[\sigma^\tau\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\tau\right]\left(\left(\partial_\nu\bar{\mathbf{\Psi}}\right)i\sigma^3\mathbf{\Psi} + \bar{\mathbf{\Psi}}i\sigma^3\partial_\nu\mathbf{\Psi}\right)\right\} \qquad (2.31)\\ &= \frac{1}{2}Tr\left\{\frac{1}{2}\left[\sigma^\tau\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\tau\right]\partial_\nu\left(\bar{\mathbf{\Psi}}i\sigma^3\mathbf{\Psi}\right)\right\}. \qquad (2.32) \end{aligned}$$

Using (2.24),

$$\begin{aligned} X^\tau &= \frac{1}{2}Tr\left\{\frac{1}{2}\left[\sigma^\tau\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\tau\right]\partial_\nu\left(\left(S_{\mu\omega} + i^*S_{\mu\omega}\right)\frac{1}{2}\left(\sigma^\mu\overline{\sigma}^\omega - \sigma^\omega\overline{\sigma}^\mu\right)\right)\right\}\\ &= \frac{1}{2}\left[\left(\eta^{\tau\mu}\eta^{\nu\omega} - \eta^{\tau\omega}\eta^{\nu\mu} + i\varepsilon^{\tau\omega\nu\mu}\right)\right]\partial_\nu\left(S_{\mu\omega} + i^*S_{\mu\omega}\right)\\ &= \frac{1}{2}\partial_\nu\left(S^{\nu\tau} + i^*S^{\nu\tau}\right), \qquad (2.33) \end{aligned}$$

since $S_{\mu\omega} + i^*S_{\mu\omega}$ is antisymmetric and self-dual.

The process of resolving the final term, Y^τ of (2.28) in the conventional manner using Fierz identities is very lengthy [41][37]. But it is done quite simply using the tangent map (2.26). By cyclicity of the trace, we move the Ψ term to the front then use the expression for the tangent map and read off the $i\sigma^3$ term. The $(\alpha = 0, \beta = 3)$ (and *vice versa*) term in the sum (2.26) is selected with an imaginary coefficient, while the $(\alpha = 1, \beta = 2)$ (and *vice versa*) term is selected with a real coefficent. The expression finally simplifies by the asymmetry of the Lie generators:

$$\begin{aligned} Y^\tau &= Tr\left(\mathbf{\Psi}\left(\partial^\tau\bar{\mathbf{\Psi}}\right)i\sigma^3\right)\\ &= \rho e^{i\beta}\,Tr\left(\bar{\mathbf{\Psi}}^{-1}\left(\partial^\tau\bar{\mathbf{\Psi}}\right)i\sigma^3\right)\\ &= \rho e^{i\beta}i\left[\frac{1}{4}\mathbf{e}_{\nu 0}\partial^\tau\mathbf{e}^\nu_{\ 3} - \frac{1}{4}\mathbf{e}_{\nu 3}\partial^\tau\mathbf{e}^\nu_{\ 0} + \frac{i}{4}\mathbf{e}_{\nu 1}\partial^\tau\mathbf{e}^\nu_{\ 2} - \frac{i}{4}\mathbf{e}_{\nu 2}\partial^\tau\mathbf{e}^\nu_{\ 1}\right]\\ &= \rho e^{i\beta}i\left[\frac{1}{4}u_\nu\partial^\tau k^\nu - \frac{1}{4}k_\nu\partial^\tau u^\nu + \frac{i}{4}m_\nu\partial^\tau n^\nu - \frac{i}{4}n_\nu\partial^\tau m^\nu\right]\\ &= \rho e^{i\beta}\left[\frac{i}{2}u_\nu\partial^\tau k^\nu - \frac{1}{2}m_\nu\partial^\tau n^\nu\right] \qquad (2.34) \end{aligned}$$

Combining (2.33) and (2.34) into (2.27),

$$qA^{\tau} = -mc^2 \frac{J^{\tau}}{\rho e^{i\beta}} + \frac{\hbar c \partial_{\nu} \left(S^{\tau\nu} + i^* S^{\tau\nu} \right)}{2\rho e^{i\beta}} - \frac{\hbar c}{2} m_{\nu} \partial^{\tau} n^{\nu} + \frac{i\hbar c}{2} u_{\nu} \partial^{\tau} k^{\nu}. \tag{2.35}$$

This is a single complex vector equation, which will be separated into 2 sets of real four-vector equations in a subsequent section. It provides a complete tensor formulation of the Dirac theory valid when $\rho \neq 0$. The presence of the imaginary i, the pseudoscalar and pseudovector in the equation is consistent with the description of fermions. The real part is an inversion for A_{μ} entirely in terms of bilinear Dirac 'currents', and their derivatives. It is a complex combination of Mickelsson's equations 2.8 and 2.9 on page 224 of reference [47]. The imaginary part is a four-vector consistency condition on currents and their derivatives, derived by Zhelnorovich, and also shown by Kaempffer to arise from the equivalence of 2 different inversions for A_{μ}.

2.4.2 Conditions for and alternative inversions for A_{μ}

Equation (2.35), and its predecessor (2.18) in spinor components, shows that providing the length of the four-current, ρ, is non-zero, the motion of a spinor contains sufficient information to exactly determine the (gauge dependent) eletromagnetic potential in which it is moving. Therefore, under these conditions it is possible to exactly calculate the *electromagnetic field* $F^{\mu\nu}$ in which it is moving.

The inversion was employed by Mickelsson [47] in tetrad variables and Radford [54] in spinor variables, to eliminate the potential from the coupled Maxwell Dirac equations. Both workers proceeded to investigate solutions to the coupled Maxwell Dirac equations that were spherically symmetrical with respect to gauge independent quantities. Mickelsson published a pair of coupled ODEs - Radford went on and obtained the numerical solution. The inversion itself was not new, having been known in mixed tensor variables (but not the tetrad) by Zhelnorovich [65], and also Kaempffer [37]. The inversion is implicit in the work of Takabayasi[58] since A_{μ} is contained in the definition of his quantity k_{μ}, which later appears in various dynamical equations.

It seems surprising, then, that in [1958], Eliezer[18] published a paper showing that if the four-spinor Dirac equation is written as a matrix equation acting on qA_{μ},

$$M^{\beta}_{\mu} qA^{\mu} = mc^2 \psi^{\beta} - i\hbar c \gamma^{\beta}_{\ \alpha\,\mu} \partial^{\mu} \psi^{\alpha} \tag{2.36}$$

where

$$\begin{aligned} M^{\beta}_{\mu} &= \gamma^{\beta}_{\ \alpha\,\mu} \psi^{\alpha} \\ M &\in M_4(\mathbb{C}). \end{aligned}$$

then it is possible to show that M has rank of only 3 and determinant therefore zero. Eliezer therefore concluded that (a) the Dirac equation *could not* be solved for the four-potential A_{μ} and (b) there must exist a complex consistency condition on the the right-hand side of equation (2.36) requiring that it lie in the rank 3 column space of M.

This paradox was addressed in our paper '*Solution of the Dirac equation for the 4 potential*' [10]. We showed that since $A^\mu \in \mathbb{R}^4$, $\psi^\alpha \in \mathbb{C}^4$, the correct approach is to view

$$M : \mathbb{R}^4 \longrightarrow \mathbb{C}^4$$

which can be represented as an 8×4 real matrix $M \in M_{8,4}(\mathbb{R})$ if we represent $\psi^\alpha \in \mathbb{R}^8$.

The question of being able to solve for A_μ is then not a question of the determinant, since M is not square, but a check on the rank of the column space of M over $\mathbb{R}$. We showed that providing that σ and ω are not both zero (which is equivalent to $\rho \neq 0$), then the rank of the column space of M over $\mathbb{R}$ is 4 and the equation can indeed be solved for A_μ. In addition there must exist 4 real consistency conditions that the right-hand side of (2.36) must satisfy, two of which are contained in Eliezer's single complex condition.

Because of the consistency conditions, the inversion for A_μ is not unique, since any linear combination of the 4 columns of the null space of M can be added to it without effect. Our paper demonstrated 2 different forms of inversion and derived the consistency conditions therefrom. Similar work had been done by Kaempffer[37], based on the work of Zhelnorovich[65].

A particularly succint expression of the consistency conditions were shown to be

$$\begin{array}{rl} \partial_\mu J^\mu = 0 & \text{(Conservation of current),} \\ \partial_\mu K^\mu = \frac{2mc}{\hbar}\omega & \text{(Partial conservation of axial current), and} \\ \overline{\psi}_c C\gamma^\mu \partial_\mu \psi = 0 & \text{(Eliezer's complex consistency condition),} \end{array}$$

where C is the charge conjugation matrix of the Dirac algebra.

One of the forms for the inversion allows A_μ to be calculated from the spinor components without needing to know the mass, providing that the pseudoscalar $\omega = \overline{\psi} i\gamma^5\psi$ is non-zero. It was shown in our paper that under these conditions, one could write the Dirac equation as

$$M' \begin{pmatrix} qA^0 \\ qA^1 \\ qA^2 \\ qA^3 \\ mc^2 \end{pmatrix} = -i\hbar c \gamma^\beta{}_{\alpha\,\mu} \partial^\mu \psi^\alpha$$

for a certain

$$M' : \mathbb{R}^5 \longrightarrow \mathbb{C}^4,$$

which can be inverted, providing $\omega \neq 0$, giving

$$\begin{aligned} qA^\mu &= \hbar c \, \frac{\left(\left(\partial_\mu \overline{\psi}\right) \gamma^5 \gamma^\mu \gamma^\nu \psi - \overline{\psi} \gamma^5 \gamma^\nu \gamma^\mu \partial_\mu \psi\right)}{2\omega} \\ mc^2 &= -\hbar c \, \frac{\left(\left(\partial_\mu \overline{\psi}\right) \gamma^5 \gamma^\mu \psi + \overline{\psi} \gamma^5 \gamma^\mu \partial_\mu \psi\right)}{2\omega} = -\hbar c \, \frac{\partial_\mu K^\mu}{2\omega}. \end{aligned}$$

Our paper also went on to show that similar methods allow the Dirac equation to be inverted for the potential in higher dimensions and for *anticommuting* as well as commuting wave functions.

2.4.3 Non-invertible self-consistent electromagnetic field

In contrast to Maxwell Dirac systems where the Dirac equation *can* be solved for A_μ, in the only publication of *exact massive* Maxwell Dirac solutions prior to the current work, Fushchich and Zhdanov[22], [20] published a family of exact solutions of the Maxwell Dirac equations where the Dirac equation could *not* be solved for A_μ. In fact their method relied on this fact: They establish a certain ansatz for Dirac 4 spinors that, in a suitable reference frame, can be written

$$\psi^\alpha = (\gamma^0 + \gamma^3) f(t - z, x, y) \chi_c^\alpha$$

where f is a complex scalar function

$$f : \mathbb{R}^3 \longrightarrow \mathbb{C}$$

and χ_c^α is a constant 4 spinor. They establish an ansatz for the potential of the form

$$A_\mu = (g(t - z, x, y), 0, 0, g(t - z, x, y)),$$

which is therefore a null vector. Upon substitution of these forms into the Dirac equation, the coupled term

$$A_\mu \gamma^\mu \psi^\alpha$$

is found to equal zero in all components. This enables the Dirac equation to be solved exactly for ψ, without needing to know the unknown potential. It is then possible to integrate Maxwell's equation for the electromagnetic field and potential, treating the current from the solved spinor as a source term, and obtaining a solution for A_μ of the required form.

In this case, it is clearly *not* possible to solve the Dirac equation for the potential, since A_μ is in the null space of the matrix $M^\beta_\mu = \gamma^\beta_{\alpha\ \mu} \psi^\alpha$ as described the previous section. But as expected, the condition for such inversion, $\rho \neq 0$, can be shown to be violated, by a simple calculation of the four-current vector, $J_\mu = \left(\overline{\chi}\chi \left|f\right|^2, 0, 0, \overline{\chi}\chi \left|f\right|^2\right)$, which is of zero length. The physical domain of these solutions is also limited by their functional dependence of $t - z$, a wave travelling at the speed of light.

2.5 The gauge independent total potential

The aim of this section is to show that as long as the external field can be prescribed in terms of a field tensor $F^{\mu\nu}$, rather than a gauge fixed four-potential, A^μ, then the Dirac equation can be expressed in tensor variables without gauge dependence. This is achieved by the definition of a *'gauge independent total potential'* $\mathfrak{B}_\nu$. This follows and is in the spirit of Takabayasi[58], Kaempffer[37], Mickelsson [47], Kijowksi and Rudolph[39]. We commence with a quick review of the differential properties of the four-potential, A^μ.

2.5.1 The role of the four-potential

The four-potential A_ν comprises a partial solution to Maxwell's equations, with an immediate consequence of reducing the number of degrees of freedom of the electromagnetic field from 6 down to 4. But since the original observable field tensor $F^{\mu\nu}$ is invariant under the gauge transformation

$$\begin{aligned} A_\mu &\longrightarrow A_\mu + \partial_\mu \lambda(x) \\ F_{\mu\nu} &\longrightarrow F_{\mu\nu} \end{aligned}$$

where $\lambda(x)$ is any scalar function of the coordinates, there is effectively a surplus degree of freedom in any given A_μ. This is called $U(1)$ gauge freedom, due to the corresponding transformation of the wave function.

We are at liberty to impose any constraint of convenience on A_μ that does not change the curl. This process is called gauge fixing. An example is the Lorenz gauge[2]

$$\partial^\mu A_\mu = 0.$$

With this gauge, the inhomogeneous Maxwell's equations take the simple form

$$\Box^2 A^\mu = \mu_0 \mathcal{J}^\mu.$$

While many of the studies of the Maxwell Dirac equations employ a gauge such as the Lorenz gauge, we will have no need for such a gauge, for reasons that will become clear.

In addition to being a useful device for solving equations in classical electrodynamics, the potential A_μ is found to be essential for writing the equations of quantum mechanics, such as the Dirac equation (1.1). But as previously presented, the ambiguity that exists in the four-potential carries over into the Dirac four-spinor.

$$\begin{aligned} A_\mu &\longrightarrow A_\mu + \partial_\mu \lambda(\mathbf{x}) \\ \psi &\longrightarrow e^{i\theta(x)}\psi \text{ where } \theta(x) = \frac{q\lambda(x)}{\hbar c}. \end{aligned}$$

We will demonstrate in this section that there is a way in the tensor version of the Dirac equation, inverted for the potential, to trade off the gauge freedom in the potential against the gauge freedom in the spinor.

2.5.2 The gauge independent total potential, $\mathfrak{B}_\nu$

Let us recap equation (2.35) for the Dirac equation as an inversion for A_τ in terms of bilinear current density tensors:

$$qA_\tau = -mc^2 \frac{J_\tau}{\rho e^{i\beta}} + \frac{\hbar c \partial_\nu \left(S^\nu_\tau + i^* S^\nu_\tau \right)}{2\rho e^{i\beta}} - \frac{\hbar c}{2} m^\nu \partial_\tau n_\nu + \frac{i\hbar c}{2} u^\nu \partial_\tau k_\nu \tag{2.37}$$

We have seen earlier that all of the tensor current densities used in this work are gauge invariant except for m_μ and n_μ. These are spacelike vectors that rotate,

[2]usually incorrectly called the Lorentz gauge

within the spin plane that they locally define, under a gauge transformation. They appear in only one term in the above equation, $m^\mu \partial_\tau n_\mu$. We are able to group this term together with the four-potential, which is also gauge dependent, to define the *gauge independent total potential*, $\mathfrak{B}_\tau$.

$$\mathfrak{B}_\tau \equiv qA_\tau + \frac{\hbar c}{2} m^\mu \partial_\tau n_\mu. \tag{2.38}$$

With this definition, we can rewrite (2.37) as a *gauge independent Dirac equation*:

$$\mathfrak{B}_\tau = -mc^2 \frac{J_\tau}{\rho e^{i\beta}} + \frac{\hbar c \partial_\nu \left(S_\tau^\nu + i^* S_\tau^\nu\right)}{2\rho e^{i\beta}} + \frac{i\hbar c}{2} u^\nu \partial_\tau k_\nu \ . \tag{2.39}$$

Since the RHS is gauge independent, it follows that $\mathfrak{B}_\nu$ is gauge independent.

The name of the gauge independent total potential arises from recognising that $\mathfrak{B}_\nu$ contains contributions from both the external potential and the particle itself, and drawing some inspiration from Bohm's 'Quantum Potential'[8] for the part intrinsic to the wavefunction.

2.5.3 Recovering $F_{\mu\nu}$ from $\mathfrak{B}_\nu$

From the definition (2.38), we see straight away that

$$\begin{aligned} qF_{\mu\nu} &= \operatorname*{curl}_\mu \mathfrak{B}_\nu - (\hbar c/2) \operatorname*{curl}_\mu Q_\nu \\ \text{where } \ Q_\nu &= \left(m^\mu \partial_\nu n_\mu\right). \end{aligned}$$

Since $F_{\mu\nu}$ and $\mathfrak{B}_\nu$ are gauge *independent*, it follows that the curl of Q_ν must also be gauge *independent*, even though m^μ and n^μ themselves are gauge *dependent*. The following derivation will show that this is indeed the case, at the same time providing a formula for $F_{\mu\nu}$ in terms of the gauge independent quantities, therefore removing the need to retain A_ν or any other gauge dependent quantities in our systems of equations. This derivation has been published by Takabayasi [58], Gliozzi [27], and Kijowski and Rudolph [39]. We will use the following identities that arise as an immediate consequence of the membership of $\mathbf{e}^\mu_{\ \alpha}$ in $SO(1,3)$:

$$\begin{aligned} \mathbf{e}^\mu_{\ \alpha} \mathbf{e}_\nu^{\ \alpha} &= \delta^\mu_\nu \\ \mathbf{e}^\mu_{\ \beta} \mathbf{e}_\mu^{\ \alpha} &= \delta^\alpha_\beta \end{aligned} \tag{2.40}$$

Differentiating the second,

$$\left(\partial_\nu \mathbf{e}^\mu_{\ \beta}\right) \mathbf{e}_\mu^{\ \alpha} + \mathbf{e}^\mu_{\ \beta} \left(\partial_\nu \mathbf{e}_\mu^{\ \alpha}\right) = 0,$$

therefore

$$\left(\partial_\nu \mathbf{e}^{\mu\beta}\right) \mathbf{e}_\mu^{\ \alpha} = -\mathbf{e}^{\mu\beta} \left(\partial_\nu \mathbf{e}_\mu^{\ \alpha}\right) \tag{2.41}$$

$$= -\left(\partial_\nu \mathbf{e}^{\ \mu\alpha}\right) \mathbf{e}_\mu^{\ \beta}. \tag{2.42}$$

Therefore this is antisymmetric in α and β. Its diagonal components must be zero:

$$\left(\partial_\nu \mathbf{e}^\mu_{\ \alpha}\right) \mathbf{e}_\mu^{\ \alpha} = 0, \quad \alpha = 0..3, \tag{2.43}$$

where the sum is *not over* α, the identity being true for each tetrad member separately. Now,

$$\begin{aligned}\underset{\mu}{\mathrm{curl}}\, Q_\nu &= \partial_\mu(m^\beta\partial_\nu n_\beta) - \partial_\nu(m^\beta\partial_\mu n_\beta)\\ &= (\partial_\mu m^\beta)(\partial_\nu n_\beta) + m^\beta\partial_\mu\partial_\nu n_\beta - (\partial_\nu m^\beta)(\partial_\mu n_\beta) - m^\beta\partial_\nu\partial_\mu n_\beta\\ &= (\partial_\nu n_\beta)(\partial_\mu m^\beta) - (\partial_\mu n_\beta)(\partial_\nu m^\beta)\end{aligned}$$

assuming differentiability. Using (2.40) to insert $\mathbf{1} = \mathbf{e}^\mu_{\ \beta}\mathbf{e}_\mu^{\ \alpha}$ into the middle of each term, the β - indexed vectors are projected onto the basis $e^\mu_{\ \alpha}$

$$\underset{\mu}{\mathrm{curl}}\, Q_\nu = \partial_\nu n_\beta \mathbf{e}^\beta_{\ \alpha}\mathbf{e}_\delta^{\ \alpha}\partial_\mu m^\delta - \partial_\mu n_\beta \mathbf{e}^\beta_{\ \alpha}\mathbf{e}_\delta^{\ \alpha}\partial_\nu m^\delta.$$

The expansion can be done explicitly in full over the index α using (2.6):

$$\begin{aligned}\underset{\mu}{\mathrm{curl}}\, Q_\nu = {} & \partial_\nu n_\beta u^\beta u_\delta\partial_\mu m^\delta - \partial_\nu n_\beta m^\beta m_\delta\partial_\mu m^\delta - \partial_\nu n_\beta n^\beta n_\delta\partial_\mu m^\delta - \partial_\nu n_\beta k^\beta k_\delta\partial_\mu m^\delta\\ & -\partial_\mu n_\beta u^\beta u_\delta\partial_\nu m^\delta + \partial_\mu n_\beta m^\beta m_\delta\partial_\nu m^\delta + \partial_\mu n_\beta n^\beta n_\delta\partial_\nu m^\delta + \partial_\mu n_\beta k^\beta k_\delta\partial_\nu m^\delta.\end{aligned}$$

By virtue of identity (2.43), the terms containing $m_\delta\partial_\mu m^\delta$, $\partial_\nu n_\beta n^\beta$ are zero. By use of identity (2.41), the other terms can be rearranged so that the derivatives are on u and k instead of m and n.

$$\begin{aligned}\underset{\mu}{\mathrm{curl}}\, Q_\nu &= \partial_\nu u_\beta n^\beta m_\delta\partial_\mu u^\delta - \partial_\nu k_\beta n^\beta m_\delta\partial_\mu k^\delta\\ &\quad -\partial_\mu u_\beta n^\beta m_\delta\partial_\nu u^\delta + \partial_\mu k_\beta n^\beta m_\delta\partial_\nu k^\delta\end{aligned}$$

By changing the β and δ labels on the last 2 terms, raising and lowering, the factor $n_\beta m_\delta - m_\beta n_\delta = \epsilon_{\beta\delta\alpha\kappa}u^\alpha k^\kappa$ can be extracted:

$$\begin{aligned}\underset{\mu}{\mathrm{curl}}\, Q_\nu &= (n_\beta m_\delta - m_\beta n_\delta)(\partial_\nu u^\beta\partial_\mu u^\delta - \partial_\nu k^\beta\partial_\mu k^\delta)\\ &= \epsilon_{\beta\delta\alpha\kappa}u^\alpha k^\kappa(\partial_\nu u^\beta\partial_\mu u^\delta - \partial_\nu k^\beta\partial_\mu k^\delta)\end{aligned}$$

That is,

$$qF_{\mu\nu} = \underset{\mu}{\mathrm{curl}}\, \mathfrak{B}_\nu - \frac{\hbar c}{2}\epsilon_{\beta\delta\rho\kappa}u^\rho k^\kappa(\partial_\nu u^\beta\partial_\mu u^\delta - \partial_\nu k^\beta\partial_\mu k^\delta) \tag{2.44}$$

which is solely in terms of observable tensors, therefore manifestly gauge invariant, as it should be. Therefore we have shown that we can use a modified Dirac tensor equation in terms of currents, in which the gauge *dependent* potential has been replaced by a gauge *independent* potential, and which uniquely defines the electromagnetic field via a gauge independent relation. Thus as long as the external field can be prescribed in terms of a field tensor $F^{\mu\nu}$, rather than the four-potential, A^μ, in a particular gauge, then the Dirac equation can be expressed in tensor variables without gauge dependence.

This approach has 'dressed' the gauge dependent potential with the gauge dependent term $\frac{\hbar c}{2q}m^\mu\partial_\nu n_\mu$ to arrive at a gauge independent quantity $\mathfrak{B}_\nu/q$ that effectively replaces the *external* A^μ. Takabayasi [58] on the other hand, starts by characterising $\frac{\hbar c}{2}m^\mu\partial_\nu n_\mu$ as one of the tensor *internal* degrees of freedom of the electron, and dresses it with A^μ/q in order to make it gauge independent, considering the combined quantity as a replacement for the *internal* degree of freedom.

Quantities equivalent to the above gauge independent total potential, $\mathfrak{B}_\nu$, are used by [58], [47] and [39]. Mickelsson does not actually define the quantity except to point out that (2.38) is a gauge invariant quantity. Kijowski and Rudolph[39] do have an equivalent definition, B_ν, which does not carry the factor q, and which they simply describe as a *gauge invariant covector field.* Takabayasi calls his corresponding quantity k_ν, and calls the relation (2.44) the "Second subsidiary condition".

2.5.4 Phase constraints on solutions

The previous section has shown how we can eliminate the gauge dependent tetrad legs, m^μ and n^μ from the system of equations, through the definition of a new vector quantity, $\mathfrak{B}_\nu$. This effectively eliminates the indeterminate spinor phase θ from the spinor degrees of freedom. However, it is well known from elementary quantum mechanics for a particle in an external potential, that bound energy eigenstates do not exist for all possible energies, but only those corresponding to modes of a continuous wave function that is required to satisfy certain boundary conditions. It is an interesting question as to whether a similar constraint exists in this system, and how to satisfy this constraint when we have effectively eliminated the phase from the equations.

This is a difficult question, which does not appear to have been addressed in the literature. The most important implications concern boundaries where $\rho = 0$, since under this condition, we cannot even define the tetrad. Since the condition $\rho = 0$ is often associated with nodal points, we can see again that this problem is connected to the emergence of quantum numbers. We might even find singular behaviour of the tetrad derivatives about these points. It is beyond the scope of the current work to pursue this question, or to answer the question as to which certain singular derivatives might be permitted and which not permitted. Within the context of the current study, we adopt the following position: Where we find candidate solutions that contain singular derivatives of tensor quantities at nodal points, then we require that it is possible to *globally reconstruct* a spinor function, in the manner described previously, that satisfies all the normal differentiability conditions for a spinor. A consistent reconstruction will impose constraints on the tetrad behaviour at those nodes, which should amount to quantisation conditions on the tensor solution. (As it turns out in the context of the current work, we discover only solutions *without* singular derivatives.)

2.6 Gauge independent Maxwell Dirac equations

2.6.1 Complete set of equations

We are rewriting the equations in terms of just the gauge independent quantities, $\{\rho, \beta, u^\mu, k^\mu\}$ and $\mathfrak{B}_\nu$. We adapt the Fierz identity (2.5) to tetrad variables using the duality of the spin plane (2.7),

$$(S_{\nu\mu} + i * S_{\nu\mu}) = \rho e^{i\beta}(\epsilon_{\nu\mu\alpha\beta} u^\alpha k^\beta + i(u_\nu k_\mu - k_\nu u_\mu)),$$

to rewrite (2.39) in terms of $\{\rho, e^{i\beta}, u^\mu, k^\mu\}$:

$$\boxed{\mathfrak{B}_\nu = -\frac{mc^2 u_\nu}{e^{i\beta}} + \frac{\hbar c \partial^\mu \left[\rho e^{i\beta}(\epsilon_{\nu\mu\alpha\beta} u^\alpha k^\beta + i(u_\nu k_\mu - k_\nu u_\mu))\right]}{2\rho e^{i\beta}} + \tfrac{i}{2}\hbar c u^\mu \partial_\nu k_\mu}. \tag{2.45}$$

We also showed in (2.44) that the the field tensor can be calculated in a gauge independent manner from the gauge independent total potential:

$$\boxed{qF_{\mu\nu} = \operatorname*{curl}_{\mu} \mathfrak{B}_\nu - \tfrac{\hbar c}{2}\epsilon_{\beta\delta\alpha\kappa} u^\alpha k^\kappa (\partial_\nu u^\beta \partial_\mu u^\delta - \partial_\nu k^\beta \partial_\mu k^\delta)} \tag{2.46}$$

The tetrad components are not all independent, and we have 3 algebraic constraints relating to the gauge independent members:

$$\boxed{\begin{aligned} u^\mu u_\mu &= 1 \\ k^\mu k_\mu &= -1 \\ k^\mu u_\mu &= 0 \end{aligned}} \tag{2.47}$$

Equations (2.45),(2.46) and (2.47) are applicable to problems concerning the Dirac electron field in the presence of a prescribed electromagnetic field. We can self-couple the system to form the gauge independent Maxwell-Dirac system through incorporating the inhomogeneous Maxwell's equation (1.3) for the self-field expressed in terms of the tetrad and current variables:

$$\boxed{\partial_\nu F^{\nu\mu} = \mu_0 q \rho u^\mu} \tag{2.48}$$

We will refer to the set of equations (2.45),(2.46), (2.47) and (2.48) as the *'Gauge independent Maxwell Dirac (M-D) equations'*.

Equations (2.45) are 8 nonlinear first-order partial differential equations (PDEs). Equations (2.46) are 6 nonlinear first-order PDEs. Equations (2.48) are 4 nonlinear first-order PDEs: The total number of first-order PDEs is 18. Equations (2.47) are 3 algebraic constraints on the components of u^u and k^μ, for out of the 8 components of these tetrad members, only 5 are independent. Obviously it would be possible to substitute some parameterisation, such as given by [60], for u^u and k^μ throughout all the equations, eliminating the need for these 3 algebraic constraints. However, this would increase the complexity of the other equations, and could not be done covariantly. The algebraic constraints are a necessary consequence of choosing to write the spinor equations in terms of tensor current variables. Parameterisation at this stage would not achieve anything towards the goal of finding group invariant solutions.

The number of unknowns (dependent variables) in the above set of equations is:

ρ	1
β	1
u_ν	4
k_ν	4
$\mathfrak{B}_\nu$	4
$F^{\nu\mu}$	6
Total	20

The system is overspecified, being 21 equations (18 first-order PDEs and 3 algebraic constraints) in only 20 unknowns! The redundancy in the set of equations is the same redunancy in the original Maxwell Dirac equations : The conservation of current $\partial^\nu (\rho u_\nu) = 0$ can be be derived quite simply from the Dirac equation by itself, but can also be derived from the inhomogeneous Maxwell's equations if it is assumed that the field tensor $F^{\nu\mu}$ derives from a potential A_ν.

2.6.2 Relativistic hydrodynamics of the Dirac matter

The preceding presentation contains elements from many different works, and we now take the opportunity to review those. The above equations are essentially equivalent to previous presentations of the Dirac theory in terms of gauge independent observable tensors [58], [59], [65], [37], [39]. For an early tensor version of the Dirac equation, see Whittaker [64]. Crawford [15] gives tensor equations, but not a complete gauge independent formulation. Equations with the same content were presented by [47], who was not primarily interested in gauge independence. We note that these equations represent the same content as the conventional Maxwell Dirac equations, except where $\rho = 0$, at which point they are no longer applicable.

By retaining the imaginary i through to the tensor formulation, equation (2.35) is a more compact form of the tensor expression of the Dirac equation than appears in [58], [59], [65], [37] or [47].

Takabayasi's major publication was titled *Relativistic Hydrodynamics of the Dirac Matter* [58]. In a subsequent publication, *Theory of the Dirac Field as a Continuous Assembly of Small Rotating Bodies* [59] he recognised the role of the tetrad in representing the Dirac field degrees of freedom. Takabayasi's work seemed unknown to Kaempffer [37] and only became known to Mickelsson [47] immediately prior to publication. For a hydrodynamic version of the Weyl equation (for a massless chargeless Dirac particle), see [6].

Kijowski and Rudolph's work is titled '*Spinor Electrodynamics in terms of gauge invariant quantities*'[39]. In a similar vein, Kaempffer describes his work as '*Spinor electrodynamics as a dynamics of currents*'[37]. He uses a less elegant expression for the solution of A_μ than (2.35), and instead of using (2.46), uses some difficult Fierz identities to find a different complicated expression to show that $F^{\nu\mu}$ can, in fact, be expressed in terms of the gauge independent variables. (Many equivalent expressions can be obtained by virtue of consistency conditions arising from the imaginary part of (2.35)). Kaempffer's form of the solution for A_μ does not lead him to be able define a gauge independent total potential $\mathfrak{B}_\nu$. But he goes on to point out that since $F^{\nu\mu}$ can be calculated from the tensor current variables, it might be possible to dispense with A_μ entirely. Kaempffer then suggests substituting this solution for $F^{\nu\mu}$ into Maxwell's equations, eliminating the electromagnetic field entirely, thus reformulating spinor electrodynamics entirely as a dynamics of currents. The resulting system is one vector set of differential equations (compatibility relations) of first-order and another set of third-order which are quintic in the currents. This appears to have dispensed with the potential in relativistic Quantum mechanics. But one is forced to conclude that the complicated expressions that ensue are not a good physical theory compared with the simplicity of the Dirac equation. There is the additional difficulty of resolving whether the resulting second-order equations return too many possible solutions.

Clearly the same idea can be used on the set of gauge independent equations presented here, substituting the real part of (2.45) into (2.46), to eliminate $\mathfrak{B}_\nu$, reducing the number of variables and the number of equations by 4, at the cost of introducing second-order differential equations. The further elimination of $F^{\nu\mu}$ by the substitution of (2.46) into (2.48) reduces the number of variables and the number of equations by 6. The resulting system consists of 3 first-order differential, 4 third-order differential and 3 algebraic equations in the 10 unknowns, $\rho, \beta, u^\mu, k^\mu$. But the 4 third-order differential equations are so complicated that it is not useful to write them down in their full form. For many purposes, such as surveying the prospects for numerical solution, it is more useful to retain the count in terms of variables and numbers of first-order equations. Nevertheless, analytical progress can be made by testing the solubility of the system in various symmetries. Later in this work, we will use study of group invariant solutions, where with the help of Mathematica, we find we can in fact work with the 10 equations in 10 unknowns. Note, in gauge dependent form, this elimination of A_ν and then $F^{\nu\mu}$ is essentially the same idea for simplifying the Maxwell Dirac equations that was used by Radford [54], working in spinor coordinates. (For a different point of view, note that when the Lorenz gauge is invoked and Maxwell's equations are written in terms of the potential, the Maxwell Dirac equations are described as 8 real first-order PDEs and 4 second-order PDEs in the 12 gauge *dependent* variables comprising the components of A_μ and the real and imaginary parts of ψ^α.)

2.6.3 Rearrangements of the equations

There are many different ways of writing equations equivalent to (2.45). Splitting (2.45) into real and imaginary parts yields the following:

$$\mathfrak{B}_\nu = -mc^2\cos(\beta)u_\nu + \frac{\hbar c}{2}(\partial^\mu(\ln\rho))\epsilon_{\nu\mu\alpha\beta}u^\alpha k^\beta - \frac{\hbar c}{2}(\partial^\mu\beta)(u_\nu k_\mu - k_\nu u_\mu) + \frac{\hbar c}{2}\partial^\mu(\epsilon_{\nu\mu\alpha\beta}u^\alpha k^\beta) \tag{2.49}$$

for the 'solution' of the four-vector $\mathfrak{B}_\nu$ in terms of tetrad and complex scalars, and

$$\begin{aligned} 0 &= -mc^2\sin(\beta)u_\nu + \frac{\hbar c}{2}\partial^\mu(\ln\rho)(u_\nu k_\mu - k_\nu u_\mu) + \frac{\hbar c}{2}(\partial^\mu\beta)\epsilon_{\nu\mu\alpha\beta}u^\alpha k^\beta \\ &\quad + \frac{\hbar c}{2}\partial^\mu(u_\nu k_\mu - k_\nu u_\mu) + \frac{\hbar c}{2}u^\mu\partial_\nu k_\mu \end{aligned} \tag{2.50}$$

for a four-vector statement of so-called 'consistency' or 'compatibility' conditions that exist between the tetrad and scalar components.

These forms are equivalent to those published by Mickelsson, who uses variables $a + ib$ instead of $\rho e^{i\beta}$. If the same split into real and imaginary parts is made to the earlier equation (2.35), retaining the rank 2 tensor $S^{\mu\nu}$, one obtains the forms published firstly by Zhelnorovich[65] and then by Kaempffer[37].

The author has a paper in preparation on the transformation via group operations between different forms of the Dirac equation in terms of the tetrad, and certain differential forms that appear. One such transformation, commencing with the Dirac equation as a tensor system of equations (2.35), is the transformation into 8 scalar equations by contraction with the members of the tetrad $\mathbf{e}^\mu_{\ \alpha}$. The equations so obtained are of the form published by Takabayasi [58]. It is beyond the scope of the current work to write down all 8 scalar equations, but two of them arise from contracting the consistency conditions (2.50) with u^μ and k^μ respectively. These are the well-known equations for the conservation of four-current and partial conservation of axial current:

$$\partial_\mu J^\mu = 0 \tag{2.51}$$

$$\partial_\mu K^\mu = -(2mc/\hbar)\,\rho\sin\beta \tag{2.52}$$

Later in this work when we are testing for solutions invariant under certain subgroups of the Poincaré group with transitive action, we will find it efficacious to check the validity of (2.51) and (2.52) as a quick check before testing the full set of equations.

In closing, we observe that (2.52) allows β in principle to be eliminated from the set of equations via

$$\beta = \arcsin\left[\frac{\hbar\partial^\mu\left(\rho k_\mu\right)}{-2mc\rho}\right].$$

If combined with the previous idea of eliminating $\mathfrak{B}_\nu$ and $F_{\mu\nu}$ then one could reduce the entire system to 4 × 3rd-order and 2 × 2nd-order equations, plus 3 algebraic equations in terms u^μ, k^μ and ρ. But this system is so complex that nothing is gained by writing it down.

2.7 Particle density and the fine structure constant

Note that the expression (2.35) for qA_μ in terms of local tensor and tetrad variables excluded the density ρ. From qA_μ we can calculate $q\partial_\nu F^{\nu\mu}$, which by

Maxwell's equations must equal $q^2\rho u^\nu/\varepsilon_0 c^2$. The M-D equations can be made dimensionless by using the Compton wavelength, $\lambda_c = h/mc$, as the length scale, λ_c/c as the time scale and the mass of the electron, m_e, as the mass scale. We can, if we choose, use the dimensionless variable

$$\varrho \equiv 4\pi\alpha\lambda_c^3\rho$$

in place of ρ, where α is the fine structure constant

$$\alpha = \frac{q^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137}.$$

It is easy to see that all the other appearances of ρ, as $\partial^\mu(\ln\rho)$ in (2.49) and (2.50), are invariant under $\rho \longrightarrow \varrho$. The outcome is that the fine structure constant no longer appears in the equations. Therefore, the M-D equations can be written in dimensionless units in which the fine structure constant has been factored out [54][11]. How are we to interpret this state of affairs? Firstly, if a given set of fields, (A_μ, ψ) or $(\mathfrak{B}_\nu, u_\nu, k_\nu, F_{\mu\nu}, \rho, \beta)$ as function of (x^μ) satisfies the Maxwell Dirac equations, then re-expressed in dimensionless variables it will satisfy the dimensionless equations, and therefore does not make sense as a series expansion in powers of α. Exactly the same considerations apply to the Maxwell-Dirac equations in standard spinor form. The conventional perturbation expansions in powers of α come about because the wave function is assumed to have unit probability density over all space, $\int \psi^\dagger\psi d^3x = \int \rho u_0 d^3x = 1$,which expressed in terms of these new dimensionless variables is

$$\int_{\text{all space}} \varrho u_0 d^3\left(\frac{x}{\lambda_c}\right) = 4\pi\alpha \approx 0.092. \tag{2.53}$$

So the fine structure constant emerges as a feature of the solution only if we can impose unit probability density on the solution. If we could impose unit probability density over some initial spacelike volume, then by the conservation of current, this will be maintained for all later times. But the question of interest is whether a localised field of unit probability density would remain local for all later times. When coupled with the Maxwell equation for self-field, the Dirac equation is nonlinear, and to impose such a constraint would quite possibly rule out any coupled solution at all. Most of the studies to find or prove the existence of coupled solutions have only been able to impose the weaker constraint - that the solutions should have asymptotically regular behaviour and finite integrated probability density, $\int \psi^\dagger\psi d^3x < \infty$. (One study, Lisi [43], imposes the unit probability constraint from the outset). Another study, [54], sought solutions that satisfied certain asymptotic properties for the field, and in so doing, found a unique solution that, in contrast to (2.53), had $\int_{\text{all space}} \varrho u_0 d^3\left(\frac{x}{\lambda_c}\right) \approx 11.4$. In defence of such studies, if an entire family of localised solutions were discovered, then it may very well be possible to select members of the family that do have unit probability density. Furthermore, we should not rule out the possibility that a certain special localised solution might be discovered for which fortuitiously (2.53) might be true.

Later in this thesis, a solution is obtained where the probability integral is unbounded, the local length of the four-current density ρ is constant through all space and time. The only sensible question about probability density in such a case is to calculate the volume that a single electron would occupy within the context of such a solution, in a manner analogous to calculating the incident frequency of electrons from a uniform beam modelled by a plane wave solution of the Dirac equation.

2.8 Parity symmetry

Let us consider the discrete symmetries of the Maxwell Dirac system. It is well known that QED phenomena are subject to the discrete CPT symmeries: If a certain configuration can exist, then that same configuration transformed by the discrete Charge conjugation, Parity or Time reversal operators can also exist. This section discusses how these operators transform the gauge independent field variables, in the context of solutions of the M-D equations. It is easy to check that the M-D system is invariant under the parity symmetry, P. P has an action on ψ and A such that if (ψ, A) is a solution on the M-D equations, then $(P\psi, PA)$ is also a solution. The action of P is well known:

$$\begin{aligned} P &: \psi \longrightarrow \gamma^0\psi \text{ (up to a phase)} \\ P &: (A_0, \mathbf{A}) \longrightarrow (A_0, -\mathbf{A}) \end{aligned}$$

from which it can be shown the gauge independent tensor quantities transform under P as :

$$\begin{aligned} P &: \mathbf{E} \longrightarrow -\mathbf{E} \\ P &: \mathbf{B} \longrightarrow +\mathbf{B} \end{aligned}$$

$$\begin{aligned} P &: \rho \longrightarrow \rho \text{ (unchanged)} \\ P &: (u_0, \mathbf{u}) \longrightarrow (u_0, -\mathbf{u}) \\ P &: (k_0, \mathbf{k}) \longrightarrow (-k_0, \mathbf{k}) \\ P &: \beta \longrightarrow -\beta \\ P &: (\mathfrak{B}_0, \mathfrak{B}_k) \longrightarrow (\mathfrak{B}_0, -\mathfrak{B}_k) \end{aligned}$$

It is easy to show that this transforms solutions of our equations (2.45),(2.46),(2.47) and (2.48) into other solutions.

2.9 Maxwell Dirac charge conjugation

Charge conjugation of the Dirac equation

If ψ is a solution to the Dirac equation in the presence of a four-potential A_μ,

$$qA_\mu\gamma^\mu\psi + mc^2\psi = i\hbar c\partial_\mu\gamma^\mu\psi$$

then there exists a discrete transformation, the *charge conjugation* operator,

$$\mathcal{C} : \psi \longrightarrow \psi_c$$

such that the *charge conjugate* spinor, ψ_c satisfies a transformed Dirac equation,

$$-qA_\mu \gamma^\mu \psi_c + mc^2 \psi_c = i\hbar c \partial_\mu \gamma^\mu \psi_c \ ,$$

where the sign of A_μ has been reversed. Therefore, if we also require that $\mathcal{C}$ has the following action on the potential,

$$\mathcal{C} : A^\mu \longrightarrow -A^\mu$$

then $(\mathcal{C}\psi, \mathcal{C}A^\mu)$ also satisfies the Dirac equation. Just as the *free* Dirac equation can be solved by "positive" and "negative" energy eigenstates, the Dirac equation with an attractive external potential has bound solutions that exhibit the kinematic characteristics of positive mass, while the Dirac equation with a repulsive external potential has bound solutions that exhibit the kinematic characteristics of negative mass (if the charge is considered to be negative). Such states are better interpreted as antiparticles of positive mass but opposite charge.

Field Theory

It is well known that the semiclassical theory suffers from a number of problems arising from the distinction between positive and negative energy states(ref [36], pages 61-63). Firstly, whereas plane wave solutions for the free particle can be decomposed into eigenstates with distinct positive or negative energy, it can be shown that an attempt to form a gaussian envelope of free solutions having limited spatial spread $d \sim \lambda_C = \frac{h}{mc}$ and with the spinor components representative of a positive energy solution at $t = 0$ requires contributions from negative energy solutions, and leads to jittery time evolution in the current J_μ. Secondly, it can be shown that in the presence of a potential step of height at least $2mc^2/q$, it is possible to have a wave function that varies continuously from a positive energy eigenstate to a negative energy eigenstate, but where the relationship between incident and reflected fluxes at the barrier is not physically sensible (Klein's paradox). Lastly, the charge current $\mathcal{J}_\mu = qJ_\mu$, derived from the probability current, $J_\mu = \overline{\psi}\gamma_\mu\psi$, does not change sign as expected under the spinor charge conjugation operation $\mathcal{C}$:$\psi \longrightarrow \psi_c = C\gamma^0\psi^*$(ref [36], page 86). In fact, it is straightforward to show that $\overline{\psi}\gamma^\mu\psi = \overline{\psi}_c\gamma^\mu\psi_c$. Therefore, for currents defined in terms of $\overline{\psi}\gamma^\mu\psi$,

$$\begin{aligned} \mathcal{C} &: J^\mu \longrightarrow J^\mu \text{ (unchanged)} \\ \mathcal{C} &: \mathcal{J}^\mu \longrightarrow \mathcal{J}^\mu \text{ (unchanged)} \end{aligned}$$

and so, because we have required $\mathcal{C} : A^\mu \longrightarrow -A^\mu$, from which it follows $\mathcal{C} : \partial_\nu F^{\mu\nu} \longrightarrow -\partial_\nu F^{\mu\nu}$, then Maxwell's equations in the form that we have used (1.3), *are not invariant under charge conjugation*[3]. As a consequence, the

[3]The $\mathcal{C}$ symmetry of the Dirac equation is preserved in the tensor version (2.35), by the action $\mathcal{C}$:$\beta \longrightarrow \beta + \pi, e^{i\beta} \longrightarrow -e^{i\beta}$. The action of $\mathcal{C}$ on the complete set of tensor quantities is summarised at the end of this chapter.

semiclassical M-D equations *are not invariant under charge conjugation*. The Dirac equation treats the spinor as if it can represent *either* an electron or a positron, whereas the Maxwell equation in the form given is biased to interpret the spinor as carrying charge only of the sign of q, normally that of an electron.

The theory of QED, which introduces *anticommuting* particle and antiparticle fields, resolves these issues. However, since this thesis concerns the *semiclassical* Maxwell Dirac equations, these problems will remain, and indeed, are borne out in the solutions that are obtained.

The negative charge, positive energy, eigenfunctions of the free Dirac equation are conventionally written in terms of a spinor u,

$$\psi = ue^{ip^\mu x_\mu}$$

where $p^\mu = (E/c, \mathbf{k})$ with $E > 0$ and $E^2 = m^2c^4 + k^2c^2$, while the positive charge, negative energy eigenfunctions are written in terms of a spinor v

$$\psi = ve^{-ip^\mu x_\mu}.$$

The constant spinors u and v each take values from a 2 dimensional spin subspace.

The full QED theory is physically consistent and requires that the postive 'energy' ψ^+ and negative 'energy' ψ^- fields are anticommuting operators and can be separated, so that their contributions to the Maxwell current can be assigned the correct signs [36],[46]. In principle, the current study could be pursued in the context of QED, using the expansion

$$\begin{aligned}
\psi &= \psi^+ + \psi^- \\
\psi^+ &= \kappa \sum_{\alpha=1,2} \int a_\alpha(p^\mu) u^\alpha(p^\mu) \exp\left(ip^\mu x_\mu/\hbar\right) d^3p \\
\psi^- &= \kappa \sum_{\alpha=1,2} \int b^\dagger_\alpha(p^\mu) v^\alpha(p^\mu) \exp\left(-ip^\mu x_\mu/\hbar\right) d^3p
\end{aligned} \tag{2.54}$$

where α enumerates the spin states, a and $b^\dagger$ are anticommuting particle and antiparticle creation operators, and κ incorporates various physical constants. In these circumstances the following expression can be used for the current arising from the combined electron-positron field:

$$\mathcal{J}_\mu = \frac{q}{2}\left[\overline{\psi^+}\gamma_\mu\psi^+ - \overline{\psi^-}\gamma_\mu\psi^-\right] \tag{2.55}$$

But as long we work with commuting wave functions, (which appears to be the convention within the large literature on the semiclassical Maxwell-Dirac equations), we cannot use this calculation for the current.

Nevertheless, we find a working hypothesis regarding the sign of $\cos\beta$. It is possible to show that for the free field solutions above,

Solution	$\cos\beta$
u (electron)	$+1$
v (positron)	-1

. (2.56)

We see there is an association between the angle β and the distinction between positive and negative energy states, at least in the free field case. The association of the factor $\cos\beta$ with positive and negative energy behaviour was noted by Takabayasi [58], who stated in his paper on the hydrodynamics of the Dirac matter, that

> The simple classical electrodynamics has no freedom of charge transformation, but our field has one ([only] in so far as $F_{\mu\nu}$ be external)...

Because the positron states that arise in the Dirac system exhibit the behaviour of negative mass, Takabayasi described these as "ass"-like behaviours. In chapter 3, a separate argument is given to justify the association of the factor $\cos\beta$ with electron/positron mixing, based on examining a classical tensor approximation of Dirac's equation where the $\hbar$ terms have been dropped, but the mass and potential terms retained. It is shown that where $\cos\beta$ is constant (or slowly varying),that classical approximation is consistent with a Lorentz force acting on a charged fluid carrying charge per unit mass of $\cos(\beta)q/m$.

In the current work, we will adopt the rule of thumb that we seek solutions where $\cos\beta \approx 1$. (Requiring $\cos\beta = 1$ may in some cases be too strict, since even bound atomic electrons are known to have some 'antiparticle' components.) Solutions where $\cos\beta$ is negative will definitely be considered as outside the scope of the semiclassical equations. We should also discourage solutions with large potential jumps, since we may encounter anomalies such as Klein's paradox.

Within the context of the semiclassical theory, inasmuch as the mixed positive and negative mass behaviour arises in the Dirac equation from the $m\cos\beta u^\mu$ term, we can speculate that the expression $q\cos\beta J^\mu$ for the Maxwell current could give the same or similar answer as (2.55), and we could alternatively use this term in Maxwell's equation to extend the scope of the equations.

However, it is beyond the scope of this thesis to deviate from the semiclassical equations already presented. Therefore, since the sign of q in Maxwell's equations is that of the electron, solutions can only be considered physically sensible where they comprise purely u. If we wish to find charge conjugation solutions, we consider both sets of equations rewritten with the opposite sign of q, but this is not the same as the original set of equations having charge conjugation symmetry.

Variation from positive to negative mass behaviour

As discussed, the semiclassical M-D equations cannot make physical sense for configurations that vary continuously from electron to positron states. As β varies continuously from 0 to π, leaving other current variables constant, the resulting spinor exhibits behaviour that changes from positive mass to negative mass, with factor $\cos\beta$. Meanwhile, the current in the Maxwell equation remains unchanged. We note that a number of the studies into the M-D system have not considered this anomaly, and therefore permit, at least in principle, continuous variations between positive and negative mass states, while not changing the sign of the Maxwell current. This sort of variation is exhibited in solutions published by Radford et al, [54],[11], who use the symbol $\chi = -\beta$. Those solutions demonstrate constantly increasing charge contained with increasing radius, even though β, which takes value 0 at ∞, goes through an infinite number of cycles of 2π as the radius goes to zero, therefore passing through shells of $\cos\beta < 0$, which would make more sense being recorded as opposite valued charge, and should, at least from a physical point of view, represent regions of *decreasing* total charge. In a similar way, Wakano[63] and Lisi [43] openly exploit 'attraction by a repulsive potential' in their search for localised Maxwell-Dirac solutions, thereby finding states that contain significant antiparticle contributions, even though the source current is always taken to be electron-signed. It is not known whether these problems enter into the existence proof for localised solutions to the Maxwell-Dirac equations[19].

Charge conjugation of tensor quantities

The normal charge conjugation $\mathcal{C}$:$\psi \longrightarrow \psi_c$, $A_\mu \longrightarrow -A_\mu$, produces the following transformations on the tensor quantities of our system:

$$\begin{aligned} \mathcal{C} &: A_\mu \longrightarrow -A_\mu \\ \mathcal{C} &: F^{\mu\nu} \longrightarrow -F^{\mu\nu} \end{aligned} \tag{2.57}$$

$$\begin{aligned} \mathcal{C} &: \rho \longrightarrow \rho \text{ (unchanged)} \\ \mathcal{C} &: u_\mu \longrightarrow u_\mu \\ \mathcal{C} &: k_\mu \longrightarrow -k_\mu \\ \mathcal{C} &: \beta \longrightarrow \pi + \beta \\ \mathcal{C} &: \mathfrak{B}_\mu \longrightarrow -\mathfrak{B}_\mu \end{aligned} \tag{2.58}$$

It is the conflict between transformations (2.57) and (2.58) that destroys intrinsic charge conjugation in Maxwell's equations for current $\overline{\psi}\gamma_\mu\psi$. It is clear that since C has failed, while P is preserved, then T will fail too. The remedy would be to go to anticommuting wave functions, giving current as in (2.55), requiring a corresponding change in u_μ.

Chapter 3

Self-field classical approximation

The aim of this chapter is to show that there is a useful classical approximation to the system presented in the previous chapter. This classical approximation is a nonlinear equation for the motion of a dilute classical electron gas with self-field, with a sharp delta function velocity distribution. This classical system is a useful base for studying the self-field problem, in particular the problem of self-repulsion. It is shown that a simple exact solution can be obtained for this system by inspection. This exact solution will provide a useful guide to future investigations.

3.1 The gauge independent Maxwell Dirac equations as $\hbar \longrightarrow 0$

To obtain the classical approximation, we will naively assume that we can let $\hbar \longrightarrow 0$ and simply drop all the terms containing $\hbar$ while still making use of the gauge independent vector field $\mathfrak{B}_\nu$ in the limit. Although this is a gross simplification, our goal in introducing this system is not to prove that it is an analytically valid limit, but to obtain a simpler system that approximates some of the attributes of the full Maxwell Dirac system.

Letting $\hbar \longrightarrow 0$ in equation (2.45) :

$$\mathfrak{B}_\nu \approx -mc^2 u_\nu e^{-i\beta} \ .$$

For the remainder of this chapter we will assume that the approximation holds exactly. Since $\mathfrak{B}_\nu$ is purely real,

$$e^{i\beta} = \pm 1.$$

That is, the pseudoscalar $\omega = \rho \sin\beta$ is eliminated from the system and we proceed with

$$\mathfrak{B}_\nu \pm mc^2 u_\nu = 0. \tag{3.1}$$

We will retain the choice of sign at this stage, then show in the next section that, on classical electrodynamical grounds, the only physically sensible choice is $\cos\beta = +1$. This is for reasons similar to those given in section 2.9 on Maxwell Dirac charge conjugation.

Note that with $\hbar \longrightarrow 0$, neither the consistency conditions, nor the axial current $k_\mu = K_\mu/\rho$, nor the spin current $S_{\mu\nu}$ have a role to play. The degrees of freedom of the electron field in this equation have reduced to just the relativistic velocity u_μ, which must still satisfy the constraint

$$u^\mu u_\mu = 1.$$

Letting $\hbar \longrightarrow 0$ in equation (2.46) for $F_{\mu\nu}$:

$$qF_{\mu\nu} \approx \underset{\mu}{\mathrm{curl}}\, \mathfrak{B}_\nu. \tag{3.2}$$

Once again, in this chapter, we will assume equality, although there may be cases, such as loops around singular points for the term $m^\nu \partial_\tau n_\nu$, where it is not appropriate to do so. If we combine (3.1) and (3.2), we obtain

$$qF_{\mu\nu} = \mp mc^2 \left(\partial_\mu u_\nu - \partial_\nu u_\mu\right)$$

which when contracted with u^ν yields

$$qF_{\mu\nu}u^\nu = \mp mc^2 \left(\partial_\mu u_\nu\right) u^\nu \pm mc^2 \left(\partial_\nu u_\mu\right) u^\nu.$$

Since $u_\nu u^\nu = 1$, as shown in (2.43) the first term disappears, giving:

$$qF_{\mu\nu}u^\nu = \pm mc^2 \left(\partial_\nu u_\mu\right) u^\nu. \tag{3.3}$$

This is the relativistic equation of motion for a compressible fluid with fluid velocity u^ν carrying charge density $\pm q/m$ times its matter density, acted on by the Lorentz force arising from an external electromagnetic field. There are no internal 'pressures', all internal forces being mediated through the field $F_{\mu\nu}$. Bialynicki-Birula [4],[5] refers to such a substance as a relativistic charged 'dust'. Takabayasi[58] noticed this Lorentz force term acting on the fluid that he introduced in his hydrodynamical description of the Dirac matter. This is akin to relativistic motion of the fluid in the presence of $\mathfrak{B}_\nu$ viewed as a 'conservative potential' . We therefore uphold the classification of the sign of the charge carrying species given in (2.56).

3.2 Self-field coupled classical equations

Now consider the same charged fluid as being the one and only source for the electromagnetic field in which it moves. As in (2.48), we take Maxwell's equations for the self-field to be

$$\partial_\nu F^{\nu\mu} = \mu_0 q\rho u^\mu \tag{3.4}$$

which in terms of the gauge independent potential, using the approximation (3.2), is

$$-\Box^2 \mathfrak{B}_\mu + \partial_\nu \partial^\mu \mathfrak{B}_\mu = \mu_0 q^2 \rho u_\mu. \tag{3.5}$$

Despite the similarity of equation (3.5) to Maxwell's equations expressed in terms of the potential A_μ, we need to emphasize that conceptually, the $\mathfrak{B}_\mu$ in this equation is *not quite* the same as qA_μ. The $\mathfrak{B}_\mu$ here was reached from the gauge independent total four-potential by dropping the terms involving $\hbar$. It is the same $\mathfrak{B}_\mu$ that is shown in equation (3.1) to be equal, in the limit as $\hbar \longrightarrow 0$, to $-mc^2u_\nu$. Therefore, we are *not* free to apply gauge transformations to it, and *cannot* simplify (3.5) using the Lorenz gauge.

Nevertheless, if we did consider $\mathfrak{B}_\mu$ as a substitute for qA_μ, then (3.5) would represent the content of Maxwell's equations for the field in the presence of a charged fluid of density ρ, while (3.1), as shown in (3.3), is consistent with the equation for a Lorentz force law acting on the same fluid.

We will use the substitution $\mathfrak{B}_\mu = -mc^2u_\mu$ from (3.1). We choose the negative sign because otherwise the kinematic behaviour under the Lorentz force law as described in (3.3) is that of an oppositely charged species than the one sourcing the current. This upholds, from a classical point of view, the argument given in section 2.9 that in the self-field coupled equations, we need to choose $\cos\beta = 1$ (or at least close to 1).

We therefore obtain the single equation

$$-\Box^2 u_\nu + \partial_\nu \partial^\mu u_\mu = -\frac{\mu_0 q^2}{mc^2}\rho u_\nu \tag{3.6}$$

for the motion of the charged fluid in its own self field, under this approximation. The equation is not as linear as it appears: The relativistic velocity u_μ only has 3 independent degrees of freedom because it must satisfy the algebraic relation

$$u^\mu u_\mu = 1.$$

The density ρ is one of the unknown dependent variables, and is free to vary from point to point. Thus in total, there are 5 unknowns, 4 second-order differential equations and 1 algebraic equation. The right-hand side of this equation is proportional to the current J^μ, and it is easy to verify by inspection that the continuity equation for the conservation of current $\partial_\mu(\rho u_\nu) = 0$ is satisfied.

Remarks

1. Note that we can eliminate ρ covariantly by contraction with u^ν

$$\frac{\mu_0 q^2}{mc^2}\rho = u^\nu \Box^2 u_\nu - u^\nu \partial_\nu \partial^\mu u_\mu$$

   or we can eliminate both ρ and u^0 non-covariantly:

$$\begin{aligned} u^0 &= \sqrt{1+\mathbf{u}^2} \\ \frac{\mu_0 q^2}{mc^2}\rho &= \frac{\Box^2 u_0 - \partial_0 \partial^\mu u_\mu}{u^0} \end{aligned} .$$

2. Equations (3.3) and (3.4) are the equations upon which the Vlasov Maxwell equations for a dilute relativistic plasma are based [35]. (The Vlasov Maxwell equations actually account for a *distribution* of velocities amongst the particles at any given point, and allow for more than one particle

species, but the underlying dynamical principles are the same.) The limiting equations (3.1) and (3.2) gave (3.5) which was coupled into (3.6). However, they were also used to deduce (3.3), and so it follows that any solution to the coupled equation (3.6) also satisfies the Vlasov-Maxwell type coupled equations (3.3) and (3.4). However, we cannot prove the converse.

3. We note that Mauser and Selberg , [45], demonstrate analytically that in the limit as $\hbar/c \longrightarrow 0$, the Maxwell Dirac equations converge to the Vlasov-Poisson equations. However, analytical convergence to the Vlasov-Maxwell equations as $\hbar \longrightarrow 0$ is described in that work as a conjecture.

3.3 A simplification and a solution

Let us assume that there exist solutions to (3.6) that satisfy the very strong constraint

$$\rho(x) = \rho_i \text{ , a constant.}$$

Then from the continuity equation $\partial_\mu (\rho_i u^\mu) = 0$ we obtain:

$$\partial_\mu u^\mu = 0.$$

Therefore (3.6) becomes

$$-\Box^2 u_\nu = \frac{\mu_0 q^2}{mc^2} \rho_i u_\nu. \tag{3.7}$$

Under this simplication, the equation has the same form as the free Klein Gordon equation, given here in dimensionless units:

$$\left(\Box^2 + m^2\right) \phi(x) = 0,$$

although u_ν has 4 real components while ϕ is one complex variable. This is also componentwise a 4-dimensional version of the Helmholtz equation

$$\nabla^2 u = k^2 u$$

applied individually to the components of u^ν. Even though the Klein Gordon equation and the Helmholtz equation are linear and have general solutions in terms of Fourier transforms, the required relationship $u^\nu u_\nu = 1$ between the components prevents the linear addition of solutions to form new solutions. Any solution

$$u_\nu = \mathrm{Re} \int\limits_{\substack{p^\mu p_\mu = \frac{\mu_0 q^2}{mc^2}\rho_i \\ p^\mu \in \mathbb{C}^4}} e^{ip^\mu x_\mu} f_\nu(p^\mu) d^3p$$

must satisfy a difficult relationship $u^\nu u_\nu = 1$ between the integrals of the four distributions, f_ν, required to be valid at all points x. The algebraic constraint is additional to the differential equations (3.7), and in the form given, the quantity $u^\nu u_\nu$ is not conserved by any differential relation. Given these difficulties, and the limited applicability of the strong constraint, $\rho(x) = \rho_i$, we will not pursue the question of *general* solutions to (3.7), although we now turn to a *particular* solution.

Two primitive componentwise trial solutions of the equation (3.7) are

$$\begin{aligned} u &= A\sin\kappa t + B\cos\kappa t \text{ (sinusoidal time dependence)} \\ \text{and } u &= A\sinh\kappa y + B\cosh\kappa y \text{ (exponential space dependence).} \end{aligned}$$

We find no combination of the sinusoidal time dependent solutions can satisfy the constraint $u^\nu u_\nu = 1$. However, the following combination of space dependent solutions does satisfy the constraint:

$$\begin{aligned} u_t &= \cosh\kappa y \\ u_y &= 0, \quad u_z = 0 \\ u_z &= \sinh\kappa y \end{aligned}$$

where

$$\kappa^2 = \frac{\mu_0 q^2}{mc^2}\rho_i.$$

Therefore

$$\rho_i = \frac{mc^2\kappa^2}{\mu_0 q^2}$$

and

$$J_\nu = \frac{mc^2\kappa^2}{\mu_0 q^2}(\cosh(\kappa y), 0, 0, \sinh(\kappa y)).$$

The gauge independent total potential is

$$\begin{aligned} \mathfrak{B}_\nu &= -mc^2(\cosh(\kappa y), 0, 0, \sinh(\kappa y)) \\ &= -(\mu_0 q^2\rho_i/\kappa^2)(\cosh(\kappa y), 0, 0, \sinh(\kappa y)), \end{aligned}$$

from which the electromagnetic field is calculated:

$$\begin{aligned} \mathbf{E} &= (0, (c\mu_0 q\rho_i/\kappa)\sinh(\kappa y), 0) \\ \mathbf{B} &= ((\mu_0 q\rho_i/\kappa)\cosh(\kappa y), 0, 0). \end{aligned}$$

This solution describes uniform $x - z$ sheets of equal 'at rest' charge density, stacked in the $y-$direction, each sheet sliding in the z direction relative to the one beneath. It is a relativistic charged analogue of laminar hydrodynamic stream flow. The rate of change of the three-current with increasing y is constant in a relativistic sense. The parameter κ represents the amount of an incremental boost per increase in y, that acting as an infinitesimal Lorentz transformation on a sheet gives it identical four-current to the one above. The sheet at $y = 0$ is at rest in the frame of the problem. The functional dependence of the three-current, which points in the positive and negative z directions, on $\sinh(\kappa y)$, represents uniform integrated accumulation of relativistic boosts.

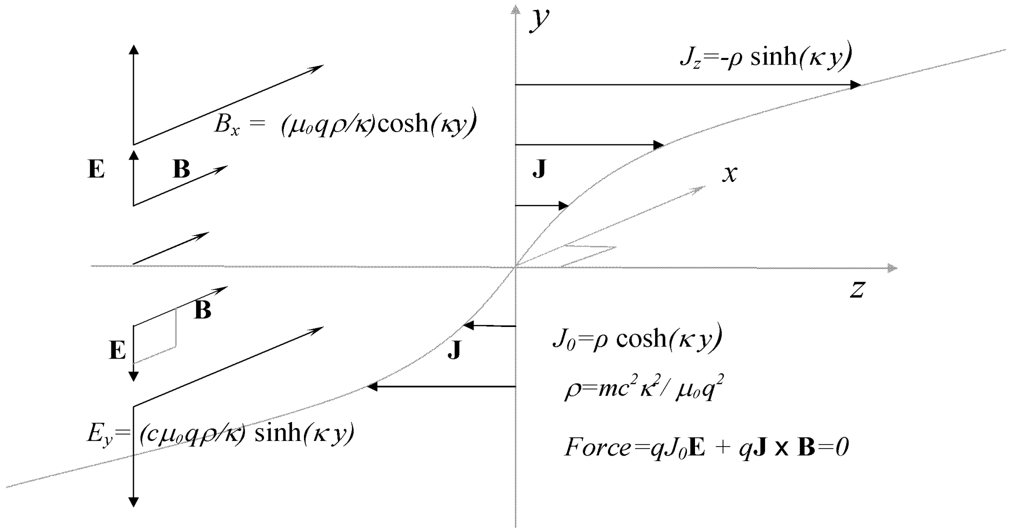


Figure 3.1: Relativistic laminar electron current with self-sourced electromagnetic field as a solution to the self-field classical approximation. To an observer at rest, the laminar current flow is consistent with the Lorentz force law.

The electric field, which points in the positive and negative $y-$directions, is zero on the sheet $y = 0$, and has dependence $(c\mu_0 q\rho_i/\kappa)\sinh(\kappa y)$ on y. This is seen to uphold Gauss's law for the amount of static charge $q\rho_i \cosh(\kappa y)$ in a layer between y_1 and y_2 equalling the integrated flux of electric field through the $x - z$ surfaces at y_1 and y_2. The magnetic field, which points strictly in the positive $x-$direction, equals $(\mu_0 q\rho_i/\kappa)$ on the sheet $y = 0$, then increases as $(\mu_0 q\rho_i/\kappa)\cosh(\kappa y)$ away from $y = 0$. This is seen to uphold Ampere's law for the amount of current flowing through a rectangular $x - y$ region equalling the path integral of $\mathbf{B}.d\mathbf{l}$ around the rectangle. (Since $\mathbf{B}$ is x-pointing, then the only contributions to the path integral are the top and bottom $x-$edges, so the 3-current must be proportional to the difference between the value of B_x at the top and the bottom of this rectangle.)

Even though q is conventionally taken as negative, we have chosen to show the electric field in the figure pointing *away* from the $x - z$ plane, since its effect as a self-field acting on the same charge q is *repulsive*. However, the solution shows that the flow is *not* deflected by the electric field, the reason being that the magnetic field produces an *equal and opposite* force $q\mathbf{J} \times \mathbf{B}$ on the current. This is a simple steady-state solution, albeit unbounded, to a classical nonlinear self-field problem. All of the terms in these classical approximation equations came from the full M-D equations, so we can expect some of the same phenomena, plus additional 'quantum' terms, to exhibit in the full equations. As will be seen later in this thesis, this solution provides a very useful model and important case in the study of group invariant solutions of the M-D equations.

Chapter 4

The method of group invariant solutions

This chapter is a brief review of the method of group invariant solutions of PDEs. It is not practical to give a complete development of the theory here. Ovsiannikov's book [51] is a good introduction. Other staged approaches with explicit coordinate based expositions are [7] and the volumes by Ibragimov [34], [35]. Olver's book[50] is a more coordinate independent approach, being expressed in terms of arbitrary smooth manifolds. All of these works introduce Lie theory in the context of groups of transformations acting on systems of equations, rather than the classical matrix group presentation. (It will be assumed that the reader already has some familiarity with the theory of Lie groups). The idea of determining the symmetry group of a set of equations applies to both ODEs as well as PDEs. What differs is how that group is used to find solutions[1]. This chapter is concerned with the application of the theory to systems of PDEs, for which we need the definition of a *group invariant solution.*[2] Each group invariant solution corresponds to a subgroup S of the maximal symmetry group. The subgroup is used to reduce the entire set of equations. The group invariant solutions can be expressed in the *reduced system* as functions of a reduced number of independent variables, the new variables being *invariants* of the action of S. In the original system, the group invariant solutions can be written in terms of to-be-determined functions of the invariants, but with known functional dependence on the remaining degrees of freedom.

4.1 Preliminary definitions

Consider a group of transformations G acting on a manifold E. We write

$$g \cdot x$$

for the *action* of a group element g on a point x in E.

[1] The application to ODEs looks at whether the symmetry groups so obtained are *solvable.*

[2] There do exist group invariant solutions of ODEs as well as solutions invariant under the actions of discrete group transformations, but these will not arise in the current work.

Definition 4.1 Let G be a local group of transformations acting on a manifold E. A submanifold $\Omega \subset E$ is called G**-invariant**, and G is called a **symmetry group** *of* Ω if whenever $x \in \Omega$ and $g \in G$, then $g \cdot x \in \Omega$ (where $g \cdot x$ is defined).

Definition 4.2 An **orbit** $\mathcal{O}_S$ of a local transformation group S is a minimal nonempty S-invariant submanifold of the manifold E. In other words, $\mathcal{O}_S \subset E$ is an orbit provided it satisfies the conditions:

1. If $x \in \mathcal{O}_S$, $g \in S$ and $g \cdot x$ is defined, then $g \cdot x \in \mathcal{O}_S$.
2. If $\widetilde{\mathcal{O}} \subset \mathcal{O}_S$, and $\widetilde{\mathcal{O}}$ satisfies (1) then either $\widetilde{\mathcal{O}} = \mathcal{O}_S$ or $\widetilde{\mathcal{O}}$ is empty.

We sometimes write $\mathcal{O}_{S,x}$ for the $S-$orbit that passes through x.

Definition 4.3 A function $F : E \longrightarrow \mathbb{R}$ is an **invariant** of a group G of transformations if for all $x \in E$ and all $g \in G$ such that $g \cdot x$ is defined, $F(g \cdot x) = F(x)$.

In general, there will a number of independent invariants $\xi_l = F_l(x)$ of a group of transformations. Ovsiannikov refers collectively to the *universal invariant* $\xi = \{\xi\}$ of a group of transformation. Any other function $G(\xi)$ will also be an invariant.

The **dimension** s of an orbit $\mathcal{O}_G$ is its dimension as a real manifold. The **order** l of a Lie group G is the number of linearly independent generators in the Lie algebra $\mathfrak{g}$ (over $\mathbb{R}$). Necessarily, $s \leq l$. If $s < l$ at the point x, then, in general, the orbit covers the neighbourhood of the point x repeatedly. That is, some points can be obtained from x by multiple transformations of the group. Other transformations of the group will leave the particular point x unchanged.

The set $G_x = \{g \in G | g \cdot x = x\}$ is a subgroup of G, called the **isotropy group** of G at x.

Definition 4.4 A group action is called **transitive** if there is only one orbit, namely the manifold itself.

That is, G is transitive if any point can be obtained from any other point by a transformation in G. (A local definition can be given where the group G is called *transitive at the point* $x \in E$ *(on the set* E*)* if the orbit of the point x is an open set in E.) A group which is not transitive is called *intransitive.*

Definition 4.5 The group G is said to be *simply transitive* at the point $x \in E$ if it is transitive and the dimension of the orbit equals the order of G; $s = l$. That is to say, there is locally a one-to-one mapping $G \longrightarrow E$ associated with the orbit.

4.2 Symmetry group of a system of equations

In essence, a symmetry group of a system of differential equations is a group that converts every solution of the equations into another solution of the same equations. Formally, suppose we are considering a system Δ of differential equations in p independent variables $x = \{x^1, ..., x^p\}$ and q dependent variables $u = \{u^1, ..., u^q\}$

$$\Delta_l(x_i, u_\alpha, \frac{\partial u_\alpha}{\partial x_i}, \frac{\partial^2 u_\alpha}{\partial x_i \partial x_j}...) = 0 \qquad (\Delta)$$

Consider the space E of p independent variables $\{x_i\}$, adjoined by direct product with the space U of q dependent variables $\{u_\alpha\}$. If $u_\alpha = f_\alpha(x_i)$ is a solution of Δ over a domain Ω, then f traces out a *graph* in $E \times U$:

$$\Gamma_f = \{(x, f(x)) : x \in \Omega\} \subset E \times U$$

$E \times U$ is a $p + q$ dimensional manifold, and the graph Γ_f is a p dimensional sub-manifold. It is possible to consider groups of transformations acting on $E \times U$.

Definition 4.6 Let Δ be a system of differential equations. A **symmetry group** of the system Δ is a local group of transformations G acting on an open subset M of the space $E \times U$ of independent and dependent variables of the system with the property that whenever $u = f(x)$ is a solution of Δ, and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system.

The terms *symmetry groups of differential equations*, *groups admitted by differential equations, Lie groups of transformations admitted by differential equations, invariance groups of differential equations* and even *equivalence group of a set of differential equations* are used interchangeably in the literature. The procedure for calculating symmetry groups involves *prolongation* to the *jet space,* $E \times U \times \mathrm{pr}^n U$, where $\mathrm{pr}^n U$ is the Cartesian space of derivatives of $\{u_\alpha\}$ with respect to x_i up to the order n of the set of equations. The set of equations Δ defines a manifold in this space and the method identifies the local Lie group under which the manifold is invariant. It is beyond the scope of this review to cover that algorithm here. The method of group invariant solutions requires finding the *largest* symmetry group, G_M that is, the *maximal invariance group* or *full symmetry group.*

The invariance group of a set of differential equations is usually expressed in terms of a set of *Lie generators* or *symmetry operators*, X_A, expressed over a basis $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial u_\alpha}$

$$X_A = \xi^i_A(x, u)\frac{\partial}{\partial x_i} + \chi^\alpha_A(x, u)\frac{\partial}{\partial u_\alpha}$$

The group G acts *projectably* or is *fibre-preserving* on $M = E \times U$ if all the transformations in G take the form

$$(x', u') = g \cdot (x, u) = (\Xi(x), \Phi(x, u))$$

The criteria of having a projectable action is necessary for the algorithm to be carried out in the usual manner, although non-projectable actions can still be dealt with.

4.3 Group invariant solutions

Now consider S, a non-trivial subgroup of G_M.

Definition 4.7 A *group invariant solution* f of the differential equation with respect to S is a solution whose *graph* is *unchanged* by the action of S. That is, the graph Γ_f is an $S-$invariant submanifold of $E \times U$.

A rigorous global geometric setting for group invariant solutions to exist is the *quotient manifold* under the subgroup of transformations. Formally, it is necessary to determine whether a general manifold generated by the group is *non-singular*. Assuming that the invariant manifold is non-singular, then it is possible to form the quotient manifold and derive a reduced set of equations over it. The reduced set of equations is known as the *reduced system* or the *factor system*. The theory provides that, subject to these conditions, the reduced system exists and is in terms of a reduced number of independent variables (invariants). Furthermore, there exists an $S-$group invariant solution if and only if there exists a corresponding solution to the reduced equations. This is Ovsiannikov's *theorem of conditional existence of the invariant $S-$solutions* [51].[3]

The formal statement of the theorem relies on some conditions of non-singularity of action, which are too lengthy to give here. The reader is referred to the existence theorem in section 19.6 of Ovsiannikov's book[51], or Olver's theorems 3.36 and 3.41 [50]. Without citing all the prerequisites, we quote Olver's theorem 3.41:

> Let S be a local group of transformations acting regularly and transversally on $M \subset E \times U$ with globally defined independent invariants ξ, whereby $M/S \subset Y \times V$. Let Δ be a system of partial differential equations defined over M for which S is a symmetry group. Then there is a reduced system of differential equations Δ/S over M/S, with the property that any S-invariant function $u = f(x)$ on M corresponding to a well-defined function $v = h(\xi)$ on M/S will be a solution to Δ if and only if its representative h is a solution to Δ/S.

The *rank* r of an invariant $S-$solution is the number of independent variables in the factor system Δ/S.

The reduced system possesses the property that it requires solution for a number of unknown functions that depend upon a smaller number of *independent*

[3] Lie proved a version of this theorem in one of his late papers [42]. With the rediscovery of this theorem, Ovsiannikov established modern recognition of the method of group invariant solutions.

variables than the original system Δ. The number of these is equal to the rank $r = p - s$ of the unknown invariant $S-$solutions (where s is the dimension of the orbit of S projected onto E). The rank can have, generally speaking, values $r = 0, 1, ...p-1$. This number is determined by the subgroup S. When the rank is zero, the reduced system is a system of algebraic (not differential) equations, and when the rank is one it is a system of ordinary differential equations.

Assuming all actions are regular, we can write

$$\begin{aligned} p &= s + r \\ l &= s + d \end{aligned}$$

where p is the number of independent variables x, s is the dimension of the orbit of S projected onto E , r is the rank of the $S-$invariant solution (=the number of invariants on which it depends), l is the order of the Lie group S, and d is the order of the isotropy group of S acting on the point $x \in E$. The way in which d reduces the number of unknown functions will be discussed below in the context of Poincaré subgroup invariant fields.

4.4 Classification of invariant solutions

We can use tranformations $g \in G_M$ (where G_M is the maximal symmetry group of Δ) to transform $S-$invariant solutions into other solutions. These are said to be *nonessentially different* solutions and are in fact $gSg^{-1}-$invariant solutions. That is, nonessentially different solutions correspond to conjugate subgroups of G. Therefore we find it convenient to classify the subalgebras of G according to conjugacy classes and then use the same classification schema for group invariant solutions of Δ. Whereas the subalgebras are often classified according to order, the group invariant solutions are more usefully classified according to rank. (For very simple systems, this may amount to the same thing.)

Definition 4.9 An *optimal system* of rank s group-invariant solutions to a system of differential equations is a collection of solutions $u = f(x)$ with the following properties:

1. Each solution is a function of s parameters.
2. Each solution in the list is invariant under some symmetry group of the system of differential equations
3. If $u = \widetilde{f}(x)$ is any other solution invariant under an $s-$parameter symmetry group, then there is a further symmetry g of the system which maps $\widetilde{f}$ to a solution $f = g \cdot \widetilde{f}$ on the list.

This is called an optimal system of rank-s group invariant solutions, also known as an optimal system of s-parameter group invariant solutions.

An invariant $S-$solution u of rank r is called *degenerate* if there is a subgroup $S' \subset G, S \subset S'$, such that u is also an invariant S' solution of rank $r' < r$.

Chapter 5

Maxwell Dirac symmetry group

The previous chapter gave a general introduction to the theory of group invariant solutions. Historically, the invariance groups of various important physical systems were discovered without reference to the theory of group invariant solutions. When the equations being studied have manifest Lorentz covariance, as the M-D equations do, then the maximal symmetry group will be a representation of the Poincaré group with extensions. The dependent variables in the equations are tensor or spinor valued fields. The task of identifying the unknown dependence on invariant coordinates and known dependence on the other parametric coordinates can be considerably simplified. The forms of the various tensors and spinors comprising the group invariant solutions are called *group invariant fields,* and can be written for each subgroup in a general form suitable for a wide range of physical problems.

This chapter commences by citing the symmetry groups of the M-D and classical self-field equations. We present a general method for constructing group invariant fields. In particular, we cite the generators and optimal system of subalgebras of the Poincaré group The prospects for applying the method of group invariant solutions to the M-D equations are discussed, concluding with a course of action focussing on the optimal system of rank-0 group invariant solutions.

5.1 Manifest Lorentz covariance

Lorentz covariant equations are written in terms of Lorentz *scalars* and *tensors* of various ranks, with various covariant derivatives and contractions. The Lorentz transformation rules concern the application of Lorentz transformations, $\Lambda^{\mu}_{\ \nu}$ according to the rank of the tensors. It is established by well-known elementary identities within the tensor formalism, that contractions and expansions of Lorentz tensors, including covariant derivatives, are also Lorentz tensors. We also require that differential equations expressing physical laws have no dependence on a particular origin. That is to say, they are not only form invariant under Lorentz transformations, but also under Poincaré transformations. If a given physical equation is known to have Lorentz covariance, then the dependent variables are tensors of various ranks as functions of the space time coordinates, $x = x^{\mu}$, $\{u\} = \{\sigma(x), V^{\mu}(x), ..., T^{\mu\nu}(x), ...\}$. The maximal symmetry group G_M

of that equation must contain as subgroup the Poincaré group $\mathcal{P}$, whose action on the variables is given by

$$\begin{aligned} g \cdot (x, u) &= g \cdot (x^\mu, \sigma, V^\mu, ..., T^{\mu\nu}, ...) \\ &= (\Lambda^\mu_{\ \nu} x^\nu + a^\mu, \sigma, \Lambda^\mu_{\ \nu} V^\nu, ..., \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \omega} T^{\rho\omega}, ...) \end{aligned} \quad (5.1)$$

for $g = (\Lambda, a)$ belonging to the Poincaré group, where $\Lambda^\mu{}_\nu$ is a 4×4 matrix corresponding to a Lorentz transformation on x^μ, and a^μ is a global spacetime translation.

The condition that a field ϕ be *invariant* under the action of a subgroup S of the full symmetry group acting is simply $g \cdot \phi = \phi$ for all $g \in S$. Considering then all one-parameter subgroups of the form $g(t) = \exp(tX)$ for generators X belonging to the Lie algebra $\mathfrak{s}$ of S, we have

$$\mathcal{L}_X \cdot \phi = 0 \quad (5.2)$$

where

$$\mathcal{L}_X \cdot \phi = \lim_{t \to 0} \left(\frac{g(t) \cdot \phi - \phi}{t} \right)$$

is the *Lie derivative* of ϕ along the direction of the flow determined by $g(t)$.

5.2 Symmetry group of the Maxwell Dirac equations

The symmetry group of the M-D equations is well known to be the direct product of the Poincaré group $\mathcal{P}$ and the Abelian gauge group[35]. Since we have eliminated the gauge dependent variables from the coupled system in chapter 2, the gauge independent M-D equations have invariance simply under $\mathcal{P}$.

If we allow $m = 0$, then the coupled system has the *conformal* group as a symmetry group [22][21], which is also a symmetry group of the vacuum Maxwell equations and free, massless Dirac equation (Weyl equation) separately[35]. The full symmetry group of the massless M-D equations in standard form is the conformal group extended by the Abelian gauge group, and by the global, coordinate frame independent, transformations

$$\psi \longrightarrow e^{\alpha\gamma^5}\psi.$$

However, if we require that all solutions satisfy unit probability density, as discussed in chapter 2, then the extensions to conformal symmetry cannot be permitted. This accounts for conflicting claims that the symmetry group of the massless equations is instead the *Poincaré* group extended by the Abelian gauge group and γ^5 transformations [35][62]. In the gauge independent M-D equations given in this thesis, the γ^5 transformations affect only the β variable via the action

$$\beta(x) \longrightarrow \beta(x) + 2\alpha.$$

However, the current work will be limited to $m \neq 0$, and massless solutions will not be pursued any further in the current study.

This work also looks at the self-field classical approximation given in chapter 3. Those equations have not only Poincaré invariance, but also a dilatation symmetry. Under dilatations

$$x \longrightarrow \lambda x$$

there exist transformed solutions where

$$\rho \longrightarrow \frac{1}{\lambda^2}\rho.$$

The Poincaré group extended by dilatations is known as the similitude, Weyl or extended Poincaré group, and is a subgroup of the conformal group[52][53]. This is consistent with the known symmetry group of the Vlasov Maxwell equations [40], that the self-field classical approximation solutions satisfy for discrete velocity distribution. In the context of the current study, the self-field classical approximation system solutions are only introduced as a pattern or model for pursuing solutions to the M-D equations. In this study they are studied only under the same Poincaré subgroups that are applied to the Maxwell Dirac equations, but none of the dilatation extensions.

5.3 Global method for constructing group invariant fields

Because most of our generators and their exponentiated actions are well known, and because of a lot of commonality between different S, in most cases we find it convenient to use the 'global' method for constructing group invariant fields [3],[2]. Many of the results will come by inspection. Consider an orbit $\mathcal{O}_{S,x_0}$ passing through a convenient point x_0, generated by transformations (5.1) where $g = (\Lambda, a) \in S \subset \mathcal{P}$. Suppose that the field ϕ is known at p_0:$\phi_0 = \phi(p_0)$, and that p_1 is any other point on the orbit. Then in order to know $\phi(p_1)$, one merely needs to know $g = (\Lambda, a) \in S \mid g \cdot p_0 = p_{1,}$ for then $\phi(p_1) = D(\Lambda) \cdot \phi(p_0)$. In fact, one only need know Λ. Therefore, it is sufficient to find constants $\alpha, \beta, \gamma, \delta$ such that for g given by the product

$$g = \exp(\alpha X_A)\exp(\beta X_B)\exp(\gamma X_C)\exp(\delta X_D),$$

we have

$$g \cdot p_0 = p_{1,}$$

where $X_A..X_D$ are generators of the Lie algebra of S, which can be written in the form $X_N = B_N + P_N$, where B_N is a generator of the Lorentz group, and P_N is a generator of translations. The required Lorentz transformation is then given by

$$\Lambda = \exp(\alpha B_A)\exp(\beta B_B)\exp(\gamma B_C)\exp(\delta B_D)$$

and the vector at an arbitrary point on the orbit is given by

$$V^\mu(x) = \Lambda V(p_0) = \Lambda \begin{pmatrix} A(\xi) \\ B(\xi) \\ C(\xi) \\ D(\xi) \end{pmatrix},$$

where the components of V at p_0, $A(\xi), B(\xi), C(\xi), D(\xi)$, are unknown functions of the invariants. This method is used in the calculations given in Appendix B in order to work out the values of vector fields at arbitrary points on orbits relative to the vector field at some convenient point.

5.4 Poincaré generators and subalgebras

The reference providing the subgroup structure of the Poincaré group is Patera, Winternitz and Zassenhas [52], hereafter referred to as PWZ. The Poincaré group is the semidirect product of the Lorentz group with the Abelian group of 4 dimensional translations, $T(4)$.

$$\mathcal{P} = \mathcal{L} \ltimes T(4)$$

The generators of the Lorentz group are the boosts, $\mathbf{K}_i$, and the rotations, $\mathbf{J}_i$. The additional translational generators of $\mathcal{P}$ are denoted $\mathbf{P}_\mu$. The infinitesimal transformations effected by the generators of $\mathcal{P}$ on the vector space $\mathbb{R}^4$ are by definition:

$$\begin{aligned} 1+\epsilon\mathbf{J}_i : \left(x^0, x^k\right) &\longrightarrow \left(x^0, x^k + \epsilon\varepsilon^k_{ij}\mathbf{J}_i x_j\right) \\ 1+\epsilon\mathbf{K}_i : \left(x^0, x^k\right) &\longrightarrow \left(x^0 + \epsilon x_i, x^k + \epsilon\delta^k_i x_0\right) \\ 1+\epsilon\mathbf{P}_\nu : \quad x^\mu &\longrightarrow x^\mu + \epsilon\delta^\mu_\nu . \end{aligned} \tag{5.3}$$

The commutation relations of the Poincaré group are:

$\frac{1}{2}[,]$	$\mathbf{K}_x$	$\mathbf{K}_y$	$\mathbf{K}_z$	$\mathbf{J}_x$	$\mathbf{J}_y$	$\mathbf{J}_z$	$\mathbf{P}_t$	$\mathbf{P}_x$	$\mathbf{P}_y$	$\mathbf{P}_z$
$\mathbf{K}_x$	0	$-\mathbf{J}_z$	$+\mathbf{J}_y$	0	$-\mathbf{K}_z$	$+\mathbf{K}_y$	$+\mathbf{P}_x$	$+\mathbf{P}_t$	0	0
$\mathbf{K}_y$	$+\mathbf{J}_z$	0	$-\mathbf{J}_x$	$+\mathbf{K}_z$	0	$-\mathbf{K}_x$	$+\mathbf{P}_y$	0	$+\mathbf{P}_t$	0
$\mathbf{K}_z$	$-\mathbf{J}_y$	$+\mathbf{J}_x$	0	$-\mathbf{K}_y$	$+\mathbf{K}_x$	0	$+\mathbf{P}_z$	0	0	$+\mathbf{P}_t$
$\mathbf{J}_x$	0	$-\mathbf{K}_z$	$+\mathbf{K}_y$	0	$+\mathbf{J}_z$	$-\mathbf{J}_y$	0	0	$+\mathbf{P}_z$	$-\mathbf{P}_y$
$\mathbf{J}_y$	$+\mathbf{K}_z$	0	$-\mathbf{K}_x$	$-\mathbf{J}_z$	0	$+\mathbf{J}_x$	0	$-\mathbf{P}_z$	0	$+\mathbf{P}_x$
$\mathbf{J}_z$	$-\mathbf{K}_y$	$+\mathbf{K}_x$	0	$+\mathbf{J}_y$	$-\mathbf{J}_x$	0	0	$+\mathbf{P}_y$	$-\mathbf{P}_x$	0

Note that the first 6 columns give the structure of the Lorentz group, $\mathcal{L}$. As part of the study of group invariant solutions, it is necessary to be able to classify all the subgroups of $\mathcal{P}$ up to conjugacy classes. PWZ's classification of subgroups of $\mathcal{P}$ is pegged on a labelling F_1 to F_{15} of the known classes of non-conjugate subgroups of $\mathcal{L}$.

Subgroups of the Lorentz group (from [52])

PWZ Label	Generators	Comment
F_1	$\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z, \mathbf{K}_x, \mathbf{K}_y, \mathbf{K}_z$	Entire Lorentz group $\mathcal{L}$, $SL_2(\mathbb{C})$, $SO(1,3)$
F_2	$\mathbf{J}_z, \mathbf{K}_z, \mathbf{J}_y + \mathbf{K}_x, \mathbf{J}_x - \mathbf{K}_y$	Borel subgroup of $\mathcal{L}$
F_3	$\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z$	$SU(2)$, $SO(3)$
F_4	$\mathbf{J}_z, \mathbf{K}_x, \mathbf{K}_y$	$SU(1,1)$, $SO(1,2)$
F_5	$\cos\phi \mathbf{J}_z + \sin\phi \mathbf{K}_z$, $\mathbf{J}_y + \mathbf{K}_x, \mathbf{J}_x - \mathbf{K}_y$	'Screw $S(3)$'
F_6	$\mathbf{J}_z, \mathbf{J}_y + \mathbf{K}_x, \mathbf{J}_x - \mathbf{K}_y$	Rotations semidirect with F_{10}
F_7	$\mathbf{K}_z, \mathbf{J}_y + \mathbf{K}_x, \mathbf{J}_x - \mathbf{K}_y$	Dilatations semidirect with F_{10}
F_8	$\mathbf{K}_z, \mathbf{J}_y + \mathbf{K}_x$	1-d Conformal $C(1)$
F_9	$\mathbf{J}_z, \mathbf{K}_z$	$\cong$ 2-d Abelian, $T_{Cylinder}$
F_{10}	$\mathbf{J}_y + \mathbf{K}_x, \mathbf{J}_x - \mathbf{K}_y$	$\cong$ 2-d Abelian, T_{Plane}
F_{11}	$\cos\phi\, \mathbf{J}_z + \sin\phi\, \mathbf{K}_z$	'Screw $S(1)$'
F_{12}	$\mathbf{J}_z$	$O(2)$
F_{13}	$\mathbf{K}_z$	$O(1,1)$
F_{14}	$\mathbf{J}_y + \mathbf{K}_x$	1-d Euclidean $E(1)$
F_{15}	0	Trivial group, $\{1\}$

PWZ separate the subgroups of $\mathcal{P}$ into *splitting* subgroups [subalgebras] and *non-splitting* subgroups [subalgebras]. Splitting subgroups $P_{n,m}$ are of the form $F_n \ltimes T_m$ where $F_n \subseteq \mathcal{L}$ is one of the previous Lie subgroups of the Lorentz group, and $T_m \subseteq T(4)$. Non-splitting subgroups $\widetilde{P}_{n,m}$ are of the form $\widetilde{F}_n \ltimes \widetilde{T}_m$ where again $\widetilde{T}_m \subseteq T(4)$ but now $\widetilde{F}_n$ is not a simple subalgebra of $\mathcal{P}$, and comprises generators which are linear combinations. The index n indicates that $\widetilde{F}_n$ is closely related to F_n whereas the index m provides cumulative listing for both $P_{n,m}$ and $\widetilde{P}_{n,m}$.

The complete table of subalgebras, in which there are 92 splitting and 66 non-splitting subalgebras of the Poincaré group, is provided in appendix A. It is an amalgamation of PWZ's tables III and IV, showing not only the generators, but also for our purposes, not the dimension of the Lie algebra, but the dimension s of the orbit of the action in $\mathbb{R}^4$ of each subgroup S. We show the order of the isotropy group S_x at a general non-singular point, which equals the order of S (the number of linearly independent generators), less the dimension of the orbit. In the case of transitive groups ($d = 4$), we list any simply transitive subgroups. The purpose for this will become clear in the final section of this chapter. In the case of some of the non-splitting groups, PWZ have identified isomorphisms, at the group level, with certain splitting groups. However, despite these isomorphisms, these subgroups are not conjugate to one another, and their action on $\mathbb{R}^4$ and behaviour as invariance groups is quite distinct.

5.5 Prospects for group invariant solutions of the Maxwell Dirac equations

One must concede that the number of equations and independent variables in the (gauge invariant) M-D system is large (20), and that the number of conjugacy classes of subgroups of $\mathcal{P}$ is also large (158). A typical exercise in the application of the theory of group invariant solutions would find *all* subgroups S that generate $p-1$ dimensional orbits in the space of p independent variables. For each S, the reduced equations, a set of ODEs, would be written, in terms of the new dependent variables as functions of a new independent variable, ξ. These systems of ODEs would then be solved, where possible, typically making use of the theory of *solvable Lie groups of ODEs.* For an example, see Güngör [29], who seeks solutions of a non-linear Klein Gordon equation in 2+1 dimensions. That system, which has the 2+1 similitude group as invariance group, has one complex dependent variable in 3 independent variables. Güngör finds 16 subgroups that reduce the equations to ODEs, in each case a single second-order ODE.

However, for the four dimensional gauge independent M-D system, there are 66 classes of subgroups generating $p-1$ dimensional orbits. We will assume in the discussion that follows that we are using the idea, put forward in chapter 2, of using just the 10 unknowns, $\rho, \beta, u^\mu, k^\mu$, having eliminated $\mathfrak{B}^\mu$ and $F^{\mu\nu}$ by (2.45) and (2.46). Each reduced system of ODEs will in general contain 3 first-order differential equations, 4 third-order differential equations and 3 algebraic equations. It would be completely unthinkable to pursue solutions without some additional constraints.

Some of these systems have isotropy groups providing such constraints. In general, a $1-$dimensional isotropy group is sufficient to reduce the number of unknown components in a four-vector from 4 down to 2. For example, if there is a one dimensional isotropy group S_x generated by $\mathbf{J}_z$, then an invariant vector at x must be of the form $(A, 0, 0, D)$. A single constraint like this would immediately reduce the number of unknown components of u^μ and k^μ to just 4 in total. In many cases, the 3 algebraic relations (2.47) concerning u^μ and k^μ could then be invoked to represent u^μ and k^μ and terms of just one parameter, α. In the example of the isotropy group generated by $\mathbf{J}_z$, we could have $u^\mu = (\cosh\alpha, 0, 0, \sinh\alpha)$ and $k^\mu = (\sinh\alpha, 0, 0, \cosh\alpha)$ (at the point x). The resulting system would consist of just 3 unknowns, ρ,β,α, and all but 3 of the differential equations would become redundant It is not clear what mixture of first and third-order these 3 linked ODEs would be. Note in the spherical M-D system, where such a constraint applies, Mickelsson[47] and Radford[54] imposed an additional non-group theoretic 'static current' assumption, which amounted to setting α to zero[1]. The subsequent system was one first-order and one third-order ordinary differential equation. Even so, this system could not be solved explicitly, only numerically.

The isotropy groups can impose very restrictive conditions on spinors and

[1]Incidentally this assumption can be justified on the grounds that there cannot be steady state current with source at the origin.

constrained vector fields: The algebraic relations on u^μ and k^μ require that u^μ always be timelike, while k^μ always be spacelike. But any subgroup with local isotropy group that contains a boost generator, say $\mathbf{K}_z$, will require that $u^0 = 0$ if u^μ is an invariant field. This is not possible for u^μ timelike, so there can be no group invariant solutions to the gauge independent M-D equations in such cases. Likewise, any subgroup with local isotropy group that contains PWZ's generators $B_3 \equiv -\mathbf{J}_y - \mathbf{K}_x$ or $B_4 \equiv +\mathbf{J}_x - \mathbf{K}_y$ will require that $u^0 + u^3 = 0$ if u^μ is an invariant field This is also incompatible with u^μ timelike, so there can be no group invariant solutions to the gauge independent M-D equations in these cases either. As a final 'no go' example, consider $SO(3)$ as local isotropy group. This will require $k^\mu = (k^0, 0, 0, 0)$ if it is an invariant field. But this is not possible for k^μ spacelike, so there can be no group invariant solutions to the gauge independent M-D equations in such cases.

Every isotropy group necessarily contains a generator with Lorentz component. But no Dirac spinor field in standard gauge *dependent* form ψ^α can exist where there is a Lorentz isotropy constraint, seen at a glance by the representation of the degrees of freedom of the Dirac spinor by the tetrad and other currents. Therefore, non-null-current *spinors* that are group invariant solutions of the standard *gauge dependent* M-D equations must be group invariant fields with no isotropy constraint, requiring the solution of 4 first-order and 4 third-order DEs for 8 spinor variables. The advantage of the gauge independent M-D equations in the context of group invariant solutions is that we are able to employ subgroups with an isotropy constraint, reducing the number of variables and equations to just 3, as described above.

Of the 65 classes of subgroups generating 3-d orbits, 39 have trivial isotropy group, 20 have $1-$d isotropy group, 2 have $2-$d isotropy group and 4 have $3-$d isotropy group. A programme of reducing the M-D equations to tractable systems of ODEs should therefore commence with those subgroups that generate 3-d orbits and have non-trivial isotropy group. A number of these cases should immediately show no solution where the isotropy group violates the aforementioned contraints on u^μ and k^μ.

5.6 Algebraic systems from transitive actions

A different approach is suggested by the solution to the self-field classical approximation that was presented at the end of chapter 3. Recollect that solution was of the form

$$\begin{aligned} u^\mu &= (\cosh \kappa y, 0, 0, \sinh \kappa y) \\ \rho &= \frac{mc^2\kappa^2}{\mu_0 q^2}. \end{aligned}$$

Note this solution is invariant under the Poincaré subgroup generated by $\{\kappa\mathbf{K}_x + \mathbf{P}_y, \mathbf{P}_t, \mathbf{P}_x, \mathbf{P}_z\}$. This is equivalent to PWZ's Poincaré subgroup $\widetilde{P}_{13,10}$, if we set $\kappa = -\frac{2}{a}$. What stands out is that the orbit of this group is *all of* $\mathbb{R}^4$, which is a *transitive* action. By the theory of group invariant solutions, the reduced equations for the self-field classical approximation under this symmetry will be in terms of 4-4=0 independent variables, that is, a set of *algebraic equations* amongst the dependent variables. The actual set of equations is given in section 6.5.2.

One of the very few references to the use of the theory of group invariant solutions to reduce a complicated set of PDEs to a set of algebraic equations is Ovskiannov's book [51](p 254). For systems with smaller numbers of variables, and with groups with smaller numbers of generators, or for scalar fields, it would not produce very interesting solutions. But for large vector systems such as the system presented here, it is a worthwhile technique.

The remainder of the current study will consist of using those subgroups of the Poincaré group that act *transitively* on $\mathbb{R}^4$ to reduce the gauge invariant Maxwell Dirac equations to a set of 10 *algebraic* equations in 10 unknowns.

Since this task is considerably easier than solving systems of *differential* equations, we are quite prepared to look at systems that have no isotropy constraints. (The case described above has no isotropy constraint.) These are the simply transitive groups and have exactly 4 generators. In fact, for general transitive systems that do have isotropy constraints, we first check to see if they contain a simply transitive subgroup. This would be a 4 dimensional subalgebra. The solution of the system with the isotropy constraint, if it exists, will then be a special case of the solution of the simply transitive system. It turns out that all subgroups of the Poincaré group that generate a 4 dimensional orbit do possess a simply transitive subgroup, so we only need to find solutions of the simply transitive systems. It is for this reason that the table of subgroups given in the appendix has a column giving any simply transitive subgroups.

There are 45 classes of transitive subgroups of $\mathcal{P}$, 19 of which are simply transitive, and are subgroups of the remainder. We will now consider the reduced classical self-field approximation and gauge independent M-D equations under each of these 19.

Chapter 6

Rank 0 group invariant solutions

At the end of chapter 5 we saw how subgroups of $\mathcal{P}$ that act *transitively* on $\mathbb{R}^4$ lead to a reduced set of equations that are a set of *algebraic* equations. We will now investigate such algebraic systems for both the full gauge independent M-D equations and self-field classical approximation, under the 19 equivalence classes of simply transitive subgroups of $\mathcal{P}$ (one of which, $T(4)$, has trivial action on vectors). Scalar quantities invariant under transitive action are constant. These solutions, if they exist, have constant scalar density, are clearly non-localised, and the probability cannot be normalised by integrating over all space. In this chapter, we present some demonstration cases, but the majority of the calculations have been presented in Appendix B.

Only one case, $\widetilde{P}_{13,10}$, yields a non-trivial solution, for both the gauge independent M-D case and the self-field classical approximation. It is not entirely surprising that the same case that supports a classical solution should also support a full M-D solution. The classical solution, presented in chapter 3, exhibited a balance between the self-repulsive electric field, and a Lorentz force on the current arising from a consistent magnetic field. Since self-repulsion in some form can also be expected in the M-D equations, it is interesting to see whether some modification of the classical solution will solve the M-D equations. We find that indeed this is the case. This chapter presents that solution, and discusses its dependence upon the group parameter, $\kappa \equiv \frac{2}{a}$. We use the technique of spinor reconstruction to determine the spinor consistent with the gauge independent tensor quantities that were found to satisfy the gauge independent M-D equations.

6.1 Constant scalar simplification

Lorentz scalars and pseudoscalars, ρ and β, will be constant on orbits generated by generators of the Lorentz group, and therefore in the case of these transitive subgroups, *constant over all spacetime*. Hence their derivatives disappear and we can simplify the gauge independent Maxwell Dirac equations. Assuming ρ and β are constant, and using the definition (2.7) of the spin plane, $H_{\mu\nu}$, equation

(2.49) becomes

$$\begin{aligned}\mathfrak{B}_\nu &= -mc^2\cos(\beta)u_\nu + \frac{\hbar c}{2}\partial^\mu(\epsilon_{\nu\mu\alpha\beta}u^\alpha k^\beta) \\ &= -mc^2\cos(\beta)u_\nu + \frac{\hbar c}{2}\partial^\mu H_{\mu\nu} \end{aligned} \tag{6.1}$$

and equation (2.50) becomes

$$\begin{aligned} 0 &= -mc^2\sin(\beta)u_\nu + \frac{\hbar c}{2}\partial^\mu(u_\nu k_\mu - k_\nu u_\mu) + \frac{\hbar c}{2}u^\mu\partial_\nu k_\mu \\ &= -mc^2\sin(\beta)u_\nu + \frac{\hbar c}{2}\partial^\mu\left({}^*H_{\mu\nu}\right) + \frac{\hbar c}{2}u^\mu\partial_\nu k_\mu. \end{aligned} \tag{6.2}$$

The expression for the electromagnetic field tensor in terms of $\mathfrak{B}_\nu$ can also be written to include the spin plane

$$qF^{\nu\mu} = \underset{\mu}{curl}\mathfrak{B}_\nu - \frac{\hbar c}{2}H_{\beta\delta}(\partial_\nu u^\beta\partial_\mu u^\delta - \partial_\nu k^\beta\partial_\mu k^\delta). \tag{6.3}$$

The remaining equations are:

$$\begin{aligned} u^\mu u_\mu &= 1 \\ k^\mu k_\mu &= -1 \\ k^\mu u_\mu &= 0 \end{aligned} \tag{6.4}$$

$$\partial_\nu F^{\nu\mu} = \mu_0 q\rho u^\mu \tag{6.5}$$

The corresponding simplification of the self-field classical approximation equations for transitive group invariant solutions is that given by (3.7).

6.2 Method

For each subgroup S, we first determine the form of a group invariant vector field using the global method described in chapter 5. The task is simplified by the fact that in the transitive cases, there are no invariants to be determined and the orbits fill all of $\mathbb{R}^4$. We find a convenient point, p_0, that can easily be transformed to any other (non-singular) point x by $g(x) \in S$ given as a sequence of explicit transformations. The Lorentz part $\Lambda(x)$ of the composite transformation $g(x)$ is determined. We define the values of a general vector field at the convenient point in terms of a set of constants $V^\mu(p_0) = (A, B, C, D)$, and calculate the form of the general vector field at an arbitrary point as $V^\mu(x) = \Lambda(x)V^\mu(p_0)$. It is useful at this point to calculate the forms of $\partial_\mu V^\mu(x)$ and $\Box^2 V^\mu(x)$ for the general vector. These invariant forms are of general interest for a wide range of relativistic problems.

We now turn to the equations of interest. We define the values of the unknown vector fields at the convenient point in terms of a set of constants

$u^\mu(p_0) = (u_A, u_B, u_C, u_D)$, $k^\mu(p_0) = (k_A, k_B, k_C, k_D)$ for use in the respective invariant forms. The invariant forms are then to be substituted into the equations (6.1),(6.2),(6.3),(6.4) and (6.5) As a consequence of the theory of group invariant solutions, what emerges is a set of algebraic equations. The calculations are exceedingly long and tedious, and were performed using Mathematica.

It was found that a useful check could be done simply by checking the validity of the continuity equation, $\partial_\mu J^\mu = 0$ for $J^\mu(x)$ having the required group invariant form, in terms of its components at the convenient point, $J^\mu(p_0) = (J_A, J_B, J_C, J_D)$. This equation provided a relation between J_A, J_B, J_C, J_D that in a number of cases showed that J^μ could not be timelike. This left $J^\mu J_\mu = 0$ as the only possibility, but our gauge independent formulation of the M-D system forbids this, so we cannot accept it as a solution to our system.

As a second quick check, we calculate $\partial_\mu(\rho k^\mu) + 2m\rho \sin\beta$. The consistency conditions (2.52) require that this be zero. This gives an algebraic relation between k_A, k_B, k_C, k_D, ρ and β.

We calculate the group invariant form of $\mathfrak{B}_\nu$ using (6.1) and substitute it into (6.3) to calculate $F_{\mu\nu}$, then substitute that into (6.5). This gives a four-vector equation in terms of group invariant vector forms based on ρ, β, (u_A,u_B,u_C,u_D), (k_A,k_B,k_C,k_D) that derives from both the Dirac equation and Maxwell's equations. Since we are dealing with transitive groups, this will reduce to 4 algebraic equations.

Finally we calculate the full consistency conditions using (6.2). Reading off coefficients in the simplified expressions yields another set of 4 algebraic relations between the constants ρ, β, (u_A,u_B,u_C,u_D), (k_A,k_B,k_C,k_D). The previous 2 scalar consistency conditions are contained within these algebraic relations. In total to this point we only have 7 *independent* algebraic relations between the 10 quantities.

Finally, we have the 3 tetrad relations (6.4) bringing the total to 10 algebraic relations in 10 unknowns. The solution, if possible, of the resulting set of algebraic equations, is done manually.

6.3 Table of results

The forms of a group invariant vector relating to each of the following simply transitive subgroups are given in Appendix B. We also give the form of the D'Alembertian and the divergence of the same invariant vector. In all but one case, $\widetilde{P}_{13,10}$, conditions were found that showed that current could not be timelike, ruling out a solution to the M-D system. Similarly, no case supported a classical solution with timelike current apart from $\widetilde{P}_{13,10}$.

The case $P_{15,1}$ has constant and uniform action on vectors, therefore all derivatives of vector and tensor quantities are zero. Therefore, from Maxwell's equations the current is zero, and from the relation between $\mathfrak{B}^\mu$ and $F^{\mu\nu}$, the electromagnetic field is also zero. The only $P_{15,0}$ group invariant solution is the zero solution.

$$\rho = 0$$

We call this condition '*Zero current from zero derivatives*'.

The other demonstration cases $\widetilde{P}_{12,13}$ ('*Zero current by Gauss's law*'), $P_{13,2}$ ('*Zero current from anti-timelike continuity condition*'), and $\widetilde{P}_{13,10}$ (exact solution) are given below.

PWZ label	**Reason for success or failure**
$P_{5,3}$	Zero current from anti-timelike continuity condition
$P_{7,4}$	"" ""
$\widetilde{P}_{7,7}$	"" ""
$P_{8,4}$	"" ""
$P_{8,6}$	"" ""
$\widetilde{P}_{8,11}$	"" ""
$\widetilde{P}_{8,13}$	"" ""
$\widetilde{P}_{10,7}$	A detailed calculation to Maxwell's equations found zero current from an anti-timelike coupled condition.
$P_{11,2}$	Zero current from anti-timelike continuity
$\widetilde{P}_{12,11/12}$	Zero current by Gauss's law
$\widetilde{P}_{12,13}$	Zero current by Gauss's law
$\widetilde{P}_{12,14}$	Zero current by Gauss's law
$P_{13,2}$	Zero current from anti-timelike continuity
$\widetilde{P}_{13,10}$	**Solution obtained (see section below)**
$\widetilde{P}_{14,10}$	Zero current by Gauss's law
$\widetilde{P}_{14,11/12}$	A detailed calculation to Maxwell's equations found zero current from an anti-timelike coupled condition.
$P_{15,1}$	Trivial uniform case: Zero field from zero derivatives

Any calculations that went through to Maxwell's equations were very lengthy. $\widetilde{P}_{10,7}$ was especially lengthy. It is interesting to note that in only 2 cases, $P_{5,3}$ and $\widetilde{P}_{10,7}$, was the general form of $\text{curl}\,(m^\mu \partial_\nu n_\mu) \neq 0$.

6.4 Demonstration cases

6.4.1 $\widetilde{P}_{12,13}$

We are investigating the algebraic system that arises as the reduced equations when the Maxwell Dirac equations are reduced by considering invariance under the subgroup $\widetilde{P}_{12,13}$ which is generated by $2\mathbf{J}_z + a\mathbf{P}_t, \mathbf{P}_z, \mathbf{P}_x, \mathbf{P}_z$ (or using the definitions of PWZ, $B_1 + a\,(X_1 + X_4)$,$X_1 - X_4, X_2, X_3$).

We choose $p_0 = (0,0,0,0)$ as the convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A,B,C,D)$. The global method, described in chapter 5 and with many additional examples in the appendix, gives the form of the invariant vector field at a general point as

$$V^\mu(x) = \begin{pmatrix} A \\ \cos(2t/a)\,B - \sin(2t/a)\,C \\ \sin(2t/a)\,B + \cos(2t/a)\,C \\ D \end{pmatrix} \tag{6.6}$$

from which we have

$$\partial_\mu V^\mu(x) = 0,$$

and

$$\Box^2 V^\mu(x) = -\frac{4}{\alpha^2}\begin{pmatrix} 0 \\ \cos(2t/a)\,B - \sin(2t/a)\,C \\ \sin(2t/a)\,B + \cos(2t/a)\,C \\ 0 \end{pmatrix}. \tag{6.7}$$

Reduced classical equations:

Let u_μ take the form of the group invariant vector field V_μ and so $u_\mu(p_0) = (A, B, C, D)$. Checking (3.7) using using the forms (6.6) and (6.7) leads to

$$\begin{aligned} A &= 0 \\ D &= 0. \end{aligned}$$

But

$$u^\mu u_\mu = A^2 - B^2 - C^2 - D^2 = 1$$

and therefore, no solution for u_μ is possible in this case.

Reduced gauge independent M-D equations

In Mathematica, the general form of vectors (6.6) has been substituted for u^μ, k^μ, into the real part of the inversion for $\mathfrak{B}^\mu$ (6.1) and subsequently into the calculation of the electromagnetic field tensor $F^{\mu\nu}$ from $\mathfrak{B}^\mu$,u^μ and k^μ (6.3). From this, the divergence of $F^{\mu\nu}$ has been calculated. This calculation has given a zero result for the zeroth component for all x:

$$\partial_\mu F^{\mu 0} = 0. \tag{6.8}$$

Therefore, the inhomogeneous Maxwell's equations (6.5) require

$$J_0 = 0.$$

But

$$\rho^2 = J_0^2 - J_1^2 - J_2^2 - J_3^2$$

and so $J_1 = 0$, $J_2 = 0$, $J_3 = 0$ and $\rho = 0$. We do not have a solution because the gauge independent M-D equations require that $\rho \neq 0$. We call this '*zero current by Gauss's Law*'.

6.4.2 $P_{13,2}$

We are investigating the algebraic system that arises as the reduced equations when the Maxwell Dirac equations are reduced by considering invariance under the subgroup $P_{13,2}$, which generated by $2\mathbf{K}_z, \frac{1}{2}(\mathbf{P}_t - \mathbf{P}_z), \mathbf{P}_x$,$\mathbf{P}_y$ (or using the definitions of PWZ, B_2, X_1, X_2, X_3).

We choose $p_0 = (R, 0, 0, R)$ as the convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. The form of a group invariant vector at any point x^μ with the same sign of $t+z$ is:

$$V^\mu(x^\mu) = \begin{pmatrix} (A-D)\frac{R}{t+z} + (A+D)\frac{t+z}{4R} \\ B \\ C \\ (-A+D)\frac{R}{t+z} + (A+D)\frac{t+z}{4R} \end{pmatrix}, \tag{6.9}$$

from which we have

$$\partial_\mu V^\mu = (A+D)\frac{1}{2R} \tag{6.10}$$

and

$$\Box^2 V^\mu(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6.11}$$

Both the Maxwell Dirac and the self-field classical approximation equations require $\partial^\mu J_\mu = 0$ for the four-current, J_μ, which if we apply the form of the general vector V^μ to J^μ and use (6.10) leads to the requirement that

$$A + D = 0. \tag{6.12}$$

But

$$\begin{aligned} \rho^2 &= A^2 - B^2 - C^2 - D^2 \\ &= -B^2 - C^2 \end{aligned}$$

by requirement (6.12). Therefore

$$\begin{aligned} \rho &= 0, \\ B &= 0 \\ C &= 0, \end{aligned}$$

We do not have a solution because the gauge independent M-D equations require that $\rho \neq 0$. . We will refer to these conditions as '*zero current from anti-timelike continuity condition*'.

6.5 Exact solution: $\widetilde{P}_{13,10}$

6.5.1 $\widetilde{P}_{13,10}$ invariant forms

We are investigating the algebraic system that arises as the reduced equations when the Maxwell Dirac equations are reduced by considering invariance under the subgroup $\widetilde{P}_{13,10}$ which generated by $2\mathbf{K}_z - a\mathbf{P}_y, \mathbf{P}_t, \mathbf{P}_x, \mathbf{P}_z$ (or using the definitions of PWZ, $B_2 + aX_2, X_1, X_3, X_4$). Set

$$\kappa \equiv \frac{2}{a}.$$

The first generator can now be written

$$\kappa \mathbf{K}_z - \mathbf{P}_y.$$

We choose $p_0 = (0,0,0,0)$ as the convenient point, and a general vector at the convenient point $V^\mu(p_0) = (V_a, V_b, V_c, V_d)$. Then the form of an invariant vector under the action of this subgroup is:

$$V^\mu(y) = \begin{pmatrix} V_a \cosh \kappa y - V_d \sinh \kappa y \\ V_b \\ V_c \\ -V_a \sinh \kappa y + V_d \cosh \kappa y \end{pmatrix}. \tag{6.13}$$

The forms of the divergence and the D'Alembertian of the invariant vector field are

$$\partial_\mu V^\mu = 0 \tag{6.14}$$

$$\Box^2 V^\mu = \begin{pmatrix} -\kappa^2(V_a \cosh \kappa y - V_d \sinh \kappa y) \\ 0 \\ 0 \\ -\kappa^2(-V_a \sinh \kappa y + V_d \cosh \kappa y) \end{pmatrix}. \tag{6.15}$$

6.5.2 Classical Solution

This has been discussed at the end of chapter 5, since the solution at the end of chapter 3 was noted to be of this form. We can now demonstrate the discovery of that group invariant solution by a constructive process: Substitution of the above forms for u^μ into the self-field classical approximation equations (3.7) yields the following set of algebraic equations:

$$\begin{aligned} \kappa^2 \begin{pmatrix} u_a \\ 0 \\ 0 \\ u_d \end{pmatrix} &= \frac{\mu_0 q^2 \rho}{mc^2} \begin{pmatrix} u_a \\ u_b \\ u_c \\ u_d \end{pmatrix} \\ 1 &= u_a^2 - u_b^2 - u_c^2 - u_d^2 \end{aligned}$$

The solution is

$$\begin{aligned} u_a &= \cosh A \\ u_b &= 0 \\ u_c &= 0 \\ u_d &= \sinh A, \qquad \text{for some constant } A, \end{aligned}$$

and

$$\rho = \frac{mc^2 \kappa^2}{\mu_0 q^2}.$$

The choice $A = 0$ is the particular solution given at the end of chapter 3. Any other value of A is the same solution under the transformation of the origin

$$y \longrightarrow y - \frac{A}{\kappa},$$

and is therefore a group invariant solution under a group conjugate to $\widetilde{P}_{13,10}$. In any set of equations between vectors invariant under $\widetilde{P}_{13,10}$, it is always possible to make such a transformation of the origin so that *one* of the *timelike* invariant vectors is of this simpler general form,

$$V_1^\mu(y) = \begin{pmatrix} V_a \cosh \kappa y \\ V_b \\ V_c \\ -V_a \sinh \kappa y \end{pmatrix}. \tag{6.16}$$

We will use this fact to simplify the form of u_ν in the next section which considers the M-D equations.

6.5.3 Maxwell Dirac Solution

As discussed in the previous paragraph, we can change the origin of our $\widetilde{P}_{13,10}$ invariant vector fields so that *one* of them has the the simpler form (6.16),

$$u^\mu(y) = \begin{pmatrix} u_a \cosh \kappa y \\ u_b \\ u_c \\ -u_a \sinh \kappa y \end{pmatrix},$$

but we must use the general form (6.13) for the remaining vector fields:

$$k^\mu(y) = \begin{pmatrix} k_a \cosh \kappa y - k_d \sinh \kappa y \\ k_b \\ k_c \\ -k_a \sinh \kappa y + k_d \cosh \kappa y \end{pmatrix}$$

It is profitable to first consider the scalar equations for the conservation of current (2.51)

$$\partial_\mu J^\mu = \partial_\mu \left(\rho u^\mu\right) = 0$$

and the partial conservation of axial current (2.52)

$$\hbar c \partial_\mu K^\mu = \hbar c \partial_\mu \left(\rho k^\mu\right) = -2mc^2 \rho \sin \beta.$$

By (6.14), the first is automatically satisfied. Our study requires $\rho \neq 0$ (otherwise the tetrad variables and inversion cannot be used), and so from substituting (6.14) into the second, we require

$$\begin{aligned} 0 &= -2mc^2 \rho \sin \beta. \\ &\Longrightarrow \sin \beta = 0 \end{aligned} \tag{6.17}$$

which in turn gives

$$\cos \beta = \pm 1.$$

We have given reasons in section 2.9 for requiring that $\cos\beta$ be equal to or close to 1 in the semiclassical Maxwell-Dirac equations. We must rule out $\cos\beta = -1$, since this implies a positron field, whereas the source current for Maxwell's equations, derived from commuting wavefunctions, is only electron-signed. (Nevertheless, note from the defining gauge invariant Maxwell Dirac equations (2.49) and (2.50), that $\sin\beta$ enters only the consistency conditions, whereas $\cos\beta$ enters the real part of the inversion on the mass term. This is the only place in the coupled equations where the mass appears. Therefore, we could if we chose, investigate the $\cos\beta = -1$ alternative in this case by changing the sign of the mass term in the final solution.)

The spin plane and its dual

$$\begin{aligned} H^{\mu}_{\ \nu} &\equiv \epsilon^{\nu}_{\ \mu\alpha\beta} u^{\alpha} k^{\beta} \\ *H^{\mu}_{\ \nu} &\equiv (u^{v} k_{\mu} - k^{\nu} u_{\mu}) \end{aligned}$$

have components

$$\begin{aligned} {}^*H^{0}_{\ 1} &= {}^*H^{1}_{\ 0} = H^{2}_{\ 3} = -H^{3}_{\ 2} = (-u_b k_a + u_a k_b)\cosh\kappa y + u_b k_d \sinh\kappa y \\ {}^*H^{0}_{\ 2} &= {}^*H^{2}_{\ 0} = -H^{1}_{\ 3} = H^{3}_{\ 1} = (u_a k_c - u_c k_a)\cosh\kappa y + u_c k_d \sinh\kappa y \\ {}^*H^{0}_{\ 3} &= {}^*H^{3}_{\ 0} = H^{2}_{\ 1} = -H^{1}_{\ 2} = u_a k_d \\ {}^*H^{1}_{\ 2} &= -{}^*H^{2}_{\ 1} = -H^{0}_{\ 3} = -H^{3}_{\ 0} = u_c k_b - u_b k_c \\ {}^*H^{1}_{\ 3} &= -{}^*H^{3}_{\ 1} = H^{2}_{\ 0} = H^{0}_{\ 2} = (u_b k_a - u_a k_b)\sinh\kappa y - u_b k_d \cosh\kappa y \\ {}^*H^{2}_{\ 3} &= -{}^*H^{3}_{\ 2} = -H^{0}_{\ 1} = -H^{1}_{\ 0} = (-u_a k_c + u_c k_a)\sinh\kappa y - u_c k_d \cosh\kappa y. \end{aligned}$$

The inversion (6.1) for $\mathfrak{B}^{\nu}$ gives

$$\mathfrak{B}^{\nu} = \begin{pmatrix} -mc^2 u_a \cosh\kappa y + \frac{\hbar c}{2}\kappa\left[\cosh\kappa y(u_a k_b - u_b k_a) + u_b k_d \sinh\kappa y\right] \\ -mc^2 u_b \\ -mc^2 u_c \\ +mc^2 u_a \sinh\kappa y + \frac{\hbar c}{2}\kappa\left[-\sinh\kappa y(u_a k_b - u_b k_a) - u_b k_d \cosh\kappa y\right] \end{pmatrix}.$$

In the calculation (6.3) of the field tensor,

$$qF_{\mu\nu} = \operatorname*{curl}_{\mu}\mathfrak{B}_{\nu} - \hbar c\epsilon_{\beta\delta\alpha\kappa} u^{\alpha} k^{\kappa} (\partial_{\nu} u^{\beta} \partial_{\mu} u^{\delta} - \partial_{\nu} k^{\beta} \partial_{\mu} k^{\delta},$$

the sum in the second term would reverse sign under exchange of β, δ and is therefore antisymmetric with respect to μ,ν. But since the only derivatives that are non-zero are ∂_y, such antisymmetric sums of products of derivatives must all be zero. Therefore, the only contribution to $qF_{\mu\nu}$ in this case is from $\operatorname*{curl}_{\mu}\mathfrak{B}_{\nu}$. We find

$$\begin{aligned} qF^{0}_{\ 2} &= qE_y = \kappa\left(-mc^2 u_a \sinh\kappa y + \frac{\hbar c}{2}\kappa\left(u_b k_d \cosh\kappa y + (u_a k_b - u_b k_a)\sinh\kappa y\right)\right) \\ qF^{2}_{\ 3} &= qB_x = \kappa\left(+mc^2 u_a \cosh\kappa y + \frac{\hbar c}{2}\kappa\left(-u_b k_d \sinh\kappa y + (-u_a k_b + u_b k_a)\cosh\kappa y\right)\right) \\ E_x &= 0, E_z = 0, B_y = 0, B_z = 0. \end{aligned}$$

Maxwell's equations give

$$\partial^\mu F^\nu_{\ \mu} = \frac{1}{q}\begin{pmatrix} +mc^2\kappa^2 u_a \cosh\kappa y + \frac{\hbar c}{2}\kappa^3\left[(u_b k_a - u_a k_b)\cosh\kappa y - u_b k_d \sinh\kappa y\right] \\ 0 \\ 0 \\ -mc^2\kappa^2 u_a \sinh\kappa y + \frac{\hbar c}{2}\kappa^3\left[(-u_b k_a + u_a k_b)\sinh\kappa y + u_b k_d \cosh\kappa y\right] \end{pmatrix}.$$

Since this is required to equal $\mu_0 q\rho u^\mu$, we arrive at the 4 algebraic equations

$$\mu_0 q^2 \rho u_a = +mc^2\kappa^2 u_a + \frac{\hbar c}{2}\kappa^3(u_b k_a - u_a k_b) \tag{6.18}$$

$$\mu_0 q^2 \rho u_b = 0 \tag{6.19}$$

$$\mu_0 q^2 \rho u_c = 0 \tag{6.20}$$

$$0 = \frac{1}{2}\kappa^3 u_b k_d. \tag{6.21}$$

The four-vector form of consistency conditions gives

$$\begin{aligned} 0 &= -mc^2\sin(\beta)u_\nu + \frac{\hbar c}{2}\partial^\mu(u_\nu k_\mu - k_\nu u_\mu) + \frac{\hbar c}{2}u^\mu\partial_\nu k_\mu \\ &= \begin{pmatrix} \frac{\hbar c}{2}\kappa\left[(-k_c u_a + u_c k_a)\sinh\kappa y - u_c k_d \cosh\kappa y\right] \\ 0 \\ \frac{\hbar c}{2}\kappa u_a k_d \\ \frac{\hbar c}{2}\kappa\left[(k_c u_a - u_c k_a)\cosh\kappa y + u_c k_d \sinh\kappa y\right] \end{pmatrix}. \end{aligned}$$

Since this is required to equal 0 in each component, we arrive at the 3 algebraic equations

$$-k_c u_a + u_c k_a = 0 \tag{6.22}$$

$$u_c k_d = 0 \tag{6.23}$$

$$u_a k_d = 0. \tag{6.24}$$

The orthogonality relations

$$1 = u_a^2 - u_b^2 - u_c^2 \tag{6.25}$$

$$-1 = k_a^2 - k_b^2 - k_c^2 - k_d^2 \tag{6.26}$$

$$0 = u_a k_a - u_b k_b - u_c k_c \tag{6.27}$$

must also be observed.

We require for any solution that $\rho > 0$ and $u_a \geq 1$. One therefore deduces $u_b = 0$, $u_c = 0$, $k_d = 0$ from (6.19), (6.20), (6.24) respectively. Then back substituting these results, $k_a = 0, k_c = 0$ from (6.27) and (6.22). The orthogonality relations then give $u_a = +1$, $k_b = \pm 1$. The final solution is

$$\begin{aligned} (u_a, u_b, u_c, 0) &= (1, 0, 0, 0) \\ (k_a, k_b, k_c, k_d) &= (0, \pm 1, 0, 0) \\ \beta &= 0 \\ \rho &= \frac{\kappa^2}{\mu_0 q^2}\left(mc^2 \mp \frac{\hbar c}{2}\kappa\right), \end{aligned}$$

where the sign in ρ is opposite to the sign of k_b. The choice in the sign of k_b represents a selection in the direction of the axial current, and makes a difference in the physical characteristics of the solution. We find that for the physical quantities in this solution, a parity transformation followed by a rotation of π about the $y-$axis has the same effect on all quantities as making the substitutions $\kappa \longrightarrow -\kappa$, $k_b \longrightarrow -k_b$. Therefore, exercising the choice in the solution of the sign in k_b, leaving κ unchanged, amounts to the same as choosing the negative value of κ, leaving k_b unchanged, followed by a parity transformation. Since κ is a continuous parameter that we are free to set, we already have this choice available to us in the resulting expressions. Therefore for the remainder of this derivation we will arbitrarily select $k_b = +1$ and so $\rho = \frac{\kappa^2}{\mu_0 q^2}\left(mc^2 - \frac{\hbar c}{2}\kappa\right)$. Substituting these values back into the invariant vector forms gives

$$u^\mu(y) = \begin{pmatrix} \cosh \kappa y \\ 0 \\ 0 \\ -\sinh \kappa y \end{pmatrix}, \qquad J^\mu(y) = \rho u^\mu(y)$$

and

$$k^\mu(y) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad K^\mu(y) = \rho k^\mu(y).$$

The four-velocity is exactly as in the classical case, while the axial current is fixed x-pointing.

6.5.4 Explicit spinor reconstruction

We use Crawford's formula[14] as presented in section 2.1.6 for the reconstruction of a spinor from its bilinear currents;

$$\psi = \frac{\exp(-i\lambda)}{2\sqrt{\bar{\eta} Z \eta}} Z\eta,$$

where λ is an arbitrary phase, η is any spinor (with $\sqrt{\bar{\eta} Z \eta} \neq 0$) and

$$Z = \left(\sigma + i\omega\gamma_5 + J_\mu \gamma^\mu\right)\left(\sigma + i\omega\gamma_5\right)^{-1}\left(\sigma + i\omega\gamma_5 + K_\nu \gamma_5 \gamma^\nu\right).$$

We use the γ^5 diagonal representation of the Dirac algebra given in chapter 1. For the currents discovered above, Crawford's expansion becomes:

$$\begin{aligned} Z &= \left(\rho + \rho\cosh(\kappa y)\gamma^0 - \rho\sinh(\kappa y)\gamma^3\right)(\rho)^{-1}\left(\rho + \rho\gamma_5\gamma^1\right) \\ &= \rho(1 + \cosh(\kappa y)\gamma^0 - \sinh(\kappa y)\gamma^3)(1 + \gamma_5\gamma^1). \end{aligned} \tag{6.28}$$

We posit the reference spinor, $\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ at all points.

It follows that providing $\rho = 0$, which is outside the domain of interest anyway, then $\sqrt{\bar{\eta}Z\eta} \neq 0$ for all x^μ and the reconstruction is

$$\psi = \frac{1}{2}\sqrt{\rho}\begin{pmatrix} +\exp\left(+\frac{\kappa y}{2}\right) \\ -\exp\left(-\frac{\kappa y}{2}\right) \\ -\exp\left(-\frac{\kappa y}{2}\right) \\ +\exp\left(+\frac{\kappa y}{2}\right) \end{pmatrix}. \tag{6.29}$$

where we have set $\lambda(x^\mu) = 0$, which fixes the gauge. We will now show that this spinor, and the corresponding potential A_μ, do in fact satisfy Dirac's equation.

In order to reconstruct A_μ in this gauge from the definition of $\mathfrak{B}_\nu$ (2.38), we need to know $\frac{1}{2}m^\mu\partial_\nu n_\mu$. A simple calculation gives $n_\mu = (0, 0, 1, 0)$, and so $\frac{1}{2}m^\mu\partial_\nu n_\mu = 0$. Therefore, in this gauge

$$qA_\nu = \mathfrak{B}_\nu = \begin{pmatrix} (-mc^2 + \frac{\hbar c}{2}\kappa)\cosh\kappa y \\ 0 \\ 0 \\ (+mc^2 - \frac{\hbar c}{2}\kappa)\sinh\kappa y \end{pmatrix}. \tag{6.30}$$

We are now in a position to check the entire Dirac equation. We substitute into and expand the Dirac equation, dividing by the factor $\frac{1}{2}\sqrt{\rho}$. We find that the the Dirac equation for the reconstructed spinor requires:

$$\begin{aligned}
\left(-mc^2 + \frac{\hbar c\kappa}{2}\right)e^{-\frac{1}{2}\kappa y}\left(\cosh\kappa y + \sinh\kappa y\right) + mc^2 e^{\frac{1}{2}\kappa y} - \frac{\hbar c\kappa}{2}e^{\frac{1}{2}\kappa y} &= 0 \\
\left(-mc^2 + \frac{\hbar c\kappa}{2}\right)e^{\frac{1}{2}\kappa y}\left(-\cosh\kappa y + \sinh\kappa y\right) - mc^2 e^{-\frac{1}{2}\kappa y} + \frac{\hbar c\kappa}{2}e^{-\frac{1}{2}\kappa y} &= 0 \\
\left(-mc^2 + \frac{\hbar c\kappa}{2}\right)e^{\frac{1}{2}\kappa y}\left(-\cosh\kappa y + \sinh\kappa y\right) - mc^2 e^{-\frac{1}{2}\kappa y} + \frac{\hbar c\kappa}{2}e^{-\frac{1}{2}\kappa y} &= 0 \\
\left(-mc^2 + \frac{\hbar c\kappa}{2}\right)e^{-\frac{1}{2}\kappa y}\left(\cosh\kappa y + \sinh\kappa y\right) + mc^2 e^{\frac{1}{2}\kappa y} - \frac{\hbar c\kappa}{2}e^{\frac{1}{2}\kappa y} &= 0
\end{aligned}$$

all of which are immediately seen to be correct.

6.5.5 Representation independent reconstruction

Even though (6.28) is independent of the representation of the γ matrices, the reference spinor η is not. Note from (6.29) that in the γ^5-diagonal representation, ψ can be written

$$\psi = \frac{\sqrt{\rho}}{2}(\phi_+ e^{+\frac{1}{2}\kappa y} + \phi_- e^{-\frac{1}{2}\kappa y}) \tag{6.31}$$

where ϕ_+ and ϕ_- are constant spinors. It follows that the form (6.31) will hold in any representation and the easiest way to determine sufficient conditions on ϕ_+ and ϕ_- is to substitute this form back into the Dirac equation, with the potential as given by (6.30). The resulting equation is required to hold for all y, and for a continuous range of values of κ. This gives the following set of conditions on

ϕ_+ and ϕ_- :

$$\begin{aligned}(1 - i\gamma^2)\phi_+ &= 0\\ (\gamma^0 - \gamma^3)\phi_+ &= 0\\ (1 + i\gamma^2)\phi_- &= 0\\ (\gamma^0 + \gamma^3)\phi_- &= 0\\ \phi_{-0} &= \frac{1}{2}(\gamma^0 + \gamma^3)\phi_{+0}\\ \phi_{+0} &= \frac{1}{2}(\gamma^0 - \gamma^3)\phi_{-0},\end{aligned}$$

which may be summarised :

$$(1 - i\gamma^2)\phi_+ = 0 \tag{6.32}$$

$$(1 - \gamma^0\gamma^3)\phi_+ = 0 \tag{6.33}$$

$$\phi_- = \gamma^0\phi_+ = \gamma^3\phi_+.$$

Finally, let us define a new constant spinor, scaled to have unit probability density per unit volume:

$$\phi \equiv \frac{1}{\sqrt{2}}\phi_+$$

$$\overline{\phi}\gamma^0\phi = 1. \tag{6.34}$$

Then providing ϕ satisfies (6.32),(6.33) and (6.34) in a certain representation of the Dirac algebra, the reconstructed spinor in that representation may be written

$$\psi = \sqrt{\frac{\rho}{2}}\left(e^{+\frac{1}{2}\kappa y} + \gamma^3 e^{-\frac{1}{2}\kappa y}\right)\phi. \tag{6.35}$$

Clearly ϕ is indeterminate up to a phase. In the γ^5-diagonal representation used here, we can choose

$$\phi_\alpha = (\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}),$$

while in the 'Standard' Dirac algebra representation, we can choose

$$\phi_\alpha = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}).$$

6.5.6 Physical interpretation

Let us review the entire range of physical quantities in this solution:

$$\begin{aligned}\cos\beta &= 1\\ \rho &= \frac{\kappa^2}{\mu_0 q^2}\left(mc^2 - \frac{1}{2}\kappa\hbar c\right)\end{aligned}$$

$$u^\mu(y) = \begin{pmatrix}\cosh\kappa y\\ 0\\ 0\\ -\sinh\kappa y\end{pmatrix}, \quad J^\mu(y) = \rho u^\mu(y)$$

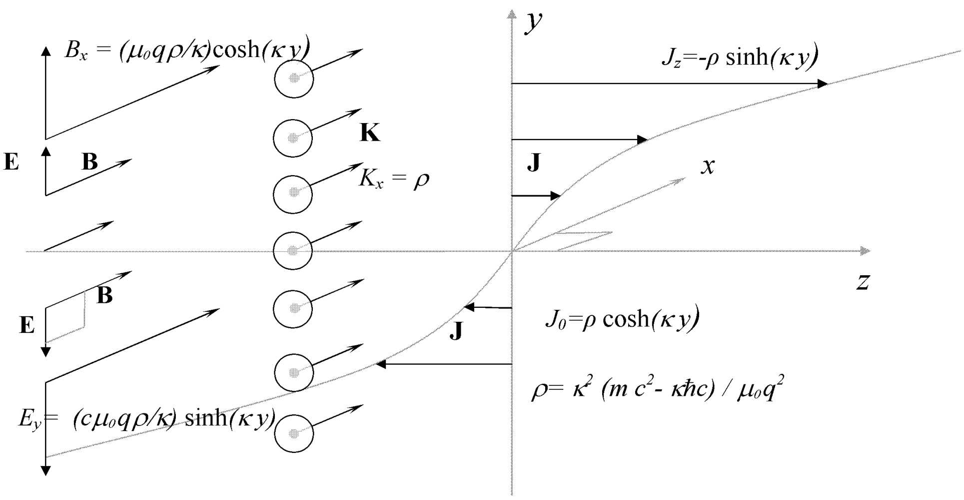


Figure 6.1: $\widetilde{P}_{13,10}$ group invariant solution to the gauge independent Maxwell Dirac equations, comprising relativistic laminar electron current with self-sourced electromagnetic field. The axial current can be either parallel or anti-parallel to the magnetic field, which is x pointing.

$$k^\mu(y) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad K^\mu(y) = \rho k^\mu(y).$$

The gauge independent total potential is

$$\mathfrak{B}^\nu = \begin{pmatrix} (-mc^2 + \frac{1}{2}\kappa\hbar c)\cosh\kappa y \\ 0 \\ 0 \\ (+mc^2 - \frac{1}{2}\kappa\hbar c)\sinh\kappa y \end{pmatrix} = -\frac{\mu_0 q^2\rho}{\kappa^2}\begin{pmatrix} \cosh\kappa y \\ 0 \\ 0 \\ -\sinh\kappa y \end{pmatrix}.$$

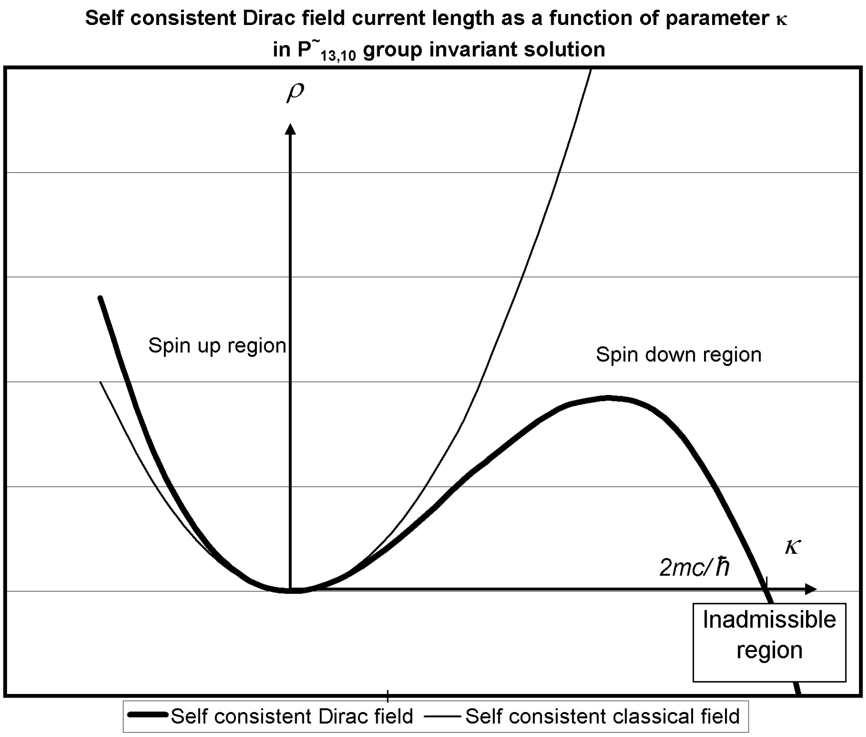
The only non-zero components of the electromagnetic field are:

$$\begin{aligned} E_y &= (c\mu_0 q\rho/\kappa)\sinh\kappa y \\ B_x &= (\mu_0 q\rho/\kappa)\cosh\kappa y. \end{aligned}$$

This is coincident with the only non-zero components of the spin plane:

$$\begin{aligned} H_2^0 &= -\sinh\kappa y \\ H_2^3 &= -\cosh\kappa y. \end{aligned}$$

We see that the relativistic fluid four-velocity u^μ has the identical laminar stream flow as the classical case given in chapter 3, whereas the length (and



therefore the respective densities) of the self-consistent four-current is changed relative to the classical value by a 'quantum' factor $1 \mp \kappa\hbar/mc$. The electric and magnetic fields are sourced by the four-current in accordance with Maxwell's equations in exactly the same way as for the classical solution, described fully in chapter 3. What differs is the kinematic response to the field, in accordance with the quantum terms. In quantum mechanics, we cannot talk about a Lorentz *force law*, except in a limited or approximate sense. Rather, we need to think in terms of the potential. In accordance with the Dirac equation, the kinematic response of the Dirac matter field to the electromagnetic potential is given not only by the mass current term (which was shown in chapter 3 to act like a fluid in response to a Lorentz force if $\cos\beta = 1$), but also the $\hbar$ (quantum) terms. In the gauge independent Maxwell Dirac equations, the quantum terms are higher degree expressions of currents and their derivatives. In the self-consistent solution presented here, these quantum terms either add or subtract to the mass current response, depending on the sign of k^1. Therefore the electromagnetic field required to maintain laminar flow is either increased or reduced relative to the classical case. As a consequence, the self-consistent charge density to source this required field is either increased or reduced.

Magnetic moment of the electron

Noticing the alignment of the axial current with the magnetic field, one is tempted to speculate that the intrinsic magnetic dipole of the electron participates in the dynamic balance with the conventional forces in a positive or negative fashion, depending on its sense. But we should not use the language of forces. The associated second-order Klein-Gordon-like equation incorporating the magnetic moment of the electron (obtained in the well-known way by applying a second differential operator, $(i\hbar c\partial_\mu\gamma^\mu - qA_\mu\gamma^\mu + mc^2)$, to the Dirac equation), is [1]:

$$\left[(i\hbar c\partial_\mu - qA_\mu)(i\hbar c\partial^\mu - qA^\mu) - m^2c^4 - \frac{1}{2}q\hbar cF^{\mu\nu}\sigma_{\mu\nu}\right]\psi = 0.$$

Even though ψ is still a Dirac spinor in this equation, all of the operators, except for the last, are proportional to the identity matrix of the Dirac algebra, hence the similarity to the Klein Gordon equation. This above equation derives from the Dirac equation and is therefore true for the coupled solution, but ψ should not be viewed as a eigenvalue of an operator, since the potential terms that would normally comprise the operator are in this case a part of the solution. Nevertheless, the equation is still valid in its own right, and the conventional interpretation that the last term contains within it the coupling with what might be called the intrinsic magnetic moment of the electron is still valid. We make substitutions from the solution (6.29) and (6.30) at $y = 0$. At this point, A^μ has only the zeroth component and $F^{\mu\nu}$ has only F^{23} component, which is the $x-$pointing magnetic field. Also we note that for this form of ψ (6.29), $\sigma_{23}\psi = -\psi$, showing, as expected, that the spin is $x-$oriented. Therefore, *for these fields, at this point,* we have the following equality:

$$(qA_0)^2 - \frac{1}{2}q\hbar cB_x = m^2c^4 - \frac{\kappa^2\hbar^2c^2}{4}. \tag{6.36}$$

We can think of the final negative term on the RHS as arising from the square of the momentum, $-\frac{\kappa^2\hbar^2c^2}{4} = p^2c^2$, which has imaginary values

$$\mathbf{p} = (0, \pm\frac{i\,\kappa\hbar}{2}, 0),$$

leading us to discuss tunnelling in the next subsection. Relation (6.36) arises from and remains consistent with the same equation that is used to demonstrate that in the non relativistic limit, where qA_0 and mc^2 are the dominant terms, the electron exhibits magnetic moment of

$$\mu_e = g_S\left(\frac{e}{2m}\right)S.$$

S is the spin, $S = \pm\frac{1}{2}$, and $g_s = 2$. Therefore, we can see that the coupling of the intrinsic magnetic moment of the electron with the magnetic field, in this case self-field, is having an important affect on the behaviour of the electron field.

Even though the self-field has entirely been accounted for in this problem, and in fact is the *only* field, *no* fine structure corrections to g_s were expected because in the problem as formulated, as mentioned in section 2.7, in the absence of a normalisation condition, the fine structure constant can be absorbed into the dimensionless variables, and no meaning can be ascribed to corrections in terms of powers of α^2.

[1] Equation 2-73 of [36].

Tunnelling

Consider for the remainder of this subsection the Dirac equation alone, without the self-field coupling, for a spinor in a prescribed electromagnetic potential. The appearance of the real exponential terms in (6.29) shows that we have a 'tunnelling' configuration for the electron, as if under a potential barrier. We will show that this is indeed the case, starting by considering the graft of the above solution to a 'free stationary' particle solution along the surface $y = 0$. Having fixed the gauge in (6.29) such that there is no time dependence, we match this to our 'free stationary' electron by specifying a constant potential of depth $A_0 = -mc^2/q$, $\mathbf{A} = 0$ for all $y < 0$. We then have the free stationary plane wave solution for the electron in the region $y < 0$:

$$\psi = u_1 e^{iEt/\hbar}$$

where u is a constant spinor which must satisfy the following condition for 'postive energy' solutions in the rest frame:

$$(1 - \gamma^0)u_1 = 0$$

and where $E = 0$ as a result of our choice of gauge. (The regular form in which $E = mc^2$ for a stationary particle would be reached under the gauge transformation $\psi \longrightarrow \psi e^{-imc^2t/\hbar}$ applied to (6.29) as well as the 'free' particle.) In the γ^5-diagonal representation used here, u_1 has the general form

$$u_1 = \begin{pmatrix} \alpha \\ \beta \\ -\alpha \\ -\beta \end{pmatrix}$$

where the coefficients α and β span the possible spin states. We note that at $y = 0$, we can match this free stationary spinor to (6.29), componentwise, and in the derivative, by the choice

$$\begin{aligned} \alpha &= \frac{\sqrt{\rho}}{2} \\ \beta &= -\frac{\sqrt{\rho}}{2}. \end{aligned}$$

This characterises a free stationary state with spin $\frac{1}{2}$ (up) in the $x-$direction, which does match the spin of (6.29).

We now consider a second 'free' particle, in the region $y > \Delta$. Thus we are considering a slice of our self-consistent solution confined to the region $0 < y < \Delta$. We will set this second 'free' particle to have momentum of $\mathbf{p} = -mc \sinh \kappa\Delta$ in the negative $z-$direction, and match the 'energy', $E = 0$, of the other regions by imposing a constant background potential of

$$\begin{aligned} A_\mu &= -(1/q)\left(\sqrt{m^2c^4 + p^2c^2}, 0, 0, pc\right) \\ &= -(1/q)mc^2 \left(\cosh \kappa\Delta, 0, 0, -\sinh \kappa\Delta\right). \end{aligned}$$

This potential and particle motion is nothing more than the 'free stationary' particle prescribed for the region $y < 0$, subject to a Lorentz boost corresponding to four-velocity $(\cosh \kappa\Delta, 0, 0, -\sinh \kappa\Delta)$.

The corresponding wave function for a positive energy particle is given by

$$\psi = u_2 e^{iEt/\hbar}$$

(with $E = 0$ in this gauge), which can be matched, up to first derivatives, with the self-consistent solution along the surface $y = \Delta$, by the choice

$$u_2 = \frac{\sqrt{\rho}}{2}\begin{pmatrix} +e^{+\kappa\Delta/2} \\ -e^{-\kappa\Delta/2} \\ -e^{-\kappa\Delta/2} \\ +e^{+\kappa\Delta/2} \end{pmatrix}.$$

This match up is nothing more than applying the previously mentioned Lorentz boost to the match up that we achieved on the surface $y = 0$. Once again, this is a particle with spin $\frac{1}{2}$ (up) in the $x-$direction, and would appear in the regular form of a free plane wave solution of momenum $\mathbf{p}$ if subject to the gauge transformation $\psi \longrightarrow \psi e^{-i(Et+pz)/\hbar}$.

Thus we have shown that it is possible to use the self consistent Maxwell Dirac solutions derived in this section to graft together two regions with different constant negative potential, both supporting plane wave solutions of the usual form, one in which the electron field is stationary, the other in which it has momentum in the $z-$direction. Therefore, the exponential dependence of the spinor (6.29) is consistent with an electron of 'positive' energy and negative charge, tunnelling through a barrier that separates two regions of electron field with different momentum along the barrier wall, and is therefore quite a reasonable physical scenario. Within the barrier, the momenum is imaginary, although the *current* varies continuously from matching the at rest current along $y = 0$, to matching the three-current of $J = \rho(0, 0, -\sinh \kappa y)$ along $y = \Delta$. Whereas real exponential spinor functions are normally of little interest, except at the boundaries of potential wells, the solution that has emerged in this problem has shown that they can be used to represent relativistic laminar electron current flow, with or without self field.

In the case where $\kappa = 2mc/\hbar$, $\widetilde{P}_{13,10}$-group invariant laminar electron current can be sustained *without any electromagnetic field*, although if we require a self-consistent solution, this is impossible on account of the self-field. One way of interpreting the zero that occurs in ρ for the self-consistent problem at this value, is to form the view that this represents a free field solution with infinitesimally dilute density. This zero in the self consistent density in the quantum self-field problem does not appear in the self-field classical approximation. (See figure 6.2.) Whereas the classical relativistic laminar electron current always requires an electromagnetic field, in this case self-sourced, in order to maintain uniform motion, in the case of relativistic laminar Dirac electron current there is a particular value of the parameter κ at which *no* electromagnetic field is required in order to maintain uniform motion, and so we see that there *are* quantum mechanical effects, that can 'overcome' the classical self-repulsion problem. Note that $2mc/\hbar = 2/(2\pi\lambda_C)$, where λ_C is the Compton wavelength. The factor of 2π comes about from the convention in the usual definition of λ_C, while the factor of 2 arises from the step down from vector spatial dependence to spinor spatial dependence.

Chapter 7

Conclusions

The semiclassical M-D equations were rewritten in a gauge independent way in terms of a set of tensor variables, comprising a normalised relativistic fluid velocity u^μ, a normalised axial vector k^μ, the invariant length of the current vector, ρ, the argument β of the complex number $\sigma + i\omega = \rho e^{i\beta}$, a gauge independent total potential $\mathfrak{B}^\mu$, and the standard electromagnetic field tensor $F^{\mu\nu}$. This brought together work from [58], [47], and [39]. This system is mathematically equivalent to the original M-D equations, providing that $\rho \neq 0$. (Because we have eliminated the 'phase' from the spinor, in general it is necessary to show that a well behaved Dirac spinor function, perhaps with modes can in fact be reconstructed globally from the solutions, so that phase matching and quantisation are still respected.) The resulting set of equations was still very complicated, comprising 17 nonlinear first-order equations and 3 algebraic equations in 20 variables. This formulation allows elimination of the electromagnetic field, reducing the entire system to 3 nonlinear first-order differential equations, 4 nonlinear third-order differential equations, and 3 algebraic equations in 10 variables. The equations are amenable to the study of group invariant solutions, by substituting in the various forms of vector fields invariant under subgroups of the Poincaré group, $\mathcal{P}$.

We showed that by naively dropping the terms that contain $\hbar$ from the gauge independent M-D equations, there exists a classical approximation to the self-field equations for a single particle. It was shown that these equations are a particular case of more general equations that can be written down for the motion of a relativistic charged fluid or dust with no internal pressures, with sharply defined fluid velocity, and with self-field. Our priority was *not* to study nor justify these classical equations in their own right. Their main purpose was to provide a model for finding solutions to the full Maxwell Dirac equations. These self-field classical approximation equations are nonlinear and not easy to solve. Nevertheless, one solution was obtained by inspection that proved a useful model for solutions to the full Maxwell Dirac equations. In the current work, where 19 different group invariant fields were tested as self-field solutions, the only case that supported a solution of the full M-D equations was also the only case that supported solutions of the self-field classical approximation, so the model has proved its merit. As shown at the end of chapter 6, that M-D solution is a quantum 'correction' to the corresponding classical self-field solution.

We applied the theory of group invariant solutions in an unusual manner, by investigating only simply transitive subgroups of $\mathcal{P}$. The method of group invariant solutions is normally used to obtain an optimal set of solutions of rank 1, that is, solutions where the reduced equations are ODEs. Due to the large number of variables and equations, and the large numbers of subgroups of the Poincaré group, the current work has only progressed as far as rank 0, that is, all solutions where the reduced equations are sets of algebraic equations. Of the 158 distinct conjugacy classes of subgroups of $\mathcal{P}$, 45 are transitive, generating 4 dimensional orbits. It is sufficient to consider rank-0 group invariant solutions for the 19 of these that are simply transitive, and are subgroups of the remainder. The full calculations of the reduced equations were quite complicated, and were done in Mathematica. It was found that only one case, the class of subgroups conjugate to the exponentiation of $\{\kappa \mathbf{K}_z + \mathbf{P}_y, \mathbf{P}_t, \mathbf{P}_x, \mathbf{P}_z\}$, where κ is a parameter, permitted a solution to the resulting set of algebraic equations. In the parlance of Ovsiannikov, we do have an *optimal system of group invariant solutions of rank* 0, although it is a very depleted set. For 1 class of subgroups the invariant solution is that published in chapter 6, and for the other 44 classes of transitive subgroups there is no invariant solution. (Strictly speaking, the zero solution in which all quantities are zero, is a degenerate solution of all the subgroups, but since it is not consistent with the definition of the tetrad in the gauge independent M-D equations, we do not give it.)

7.1 Prospects for more group invariant solutions

There are 65 classes of subgroups of $\mathcal{P}$ whose action generates 3-d orbits in $\mathbb{R}^4$. These will reduce the M-D system to a system of 17 ODEs and 3 algebraic equations in up to 20 unknowns. Therefore the task of producing an optimal set of solutions of rank 1 is gargantuan. Given the difficulties in the fourth-order systems addressed by Radford, Radford and Booth, and the system of Das, we would not expect many of these systems to be capable of exact solution. We already know at least three that do support exact solutions, $\widetilde{P}_{13,11}$, $\widetilde{P}_{13,12}$, and a group conjugate to $\widetilde{P}_{15,2}$ (which amounts to variations in one spatial direction only) since the $\widetilde{P}_{13,10}$ invariant solution given in chapter 6 is also invariant under these subgroups.

The most viable way forward for looking at rank-1 systems would be to employ only those subgroups that have isotropy constraints, therefore reducing the number of variables. (This method is open to us because of the gauge independent formulation of the M-D system. Had the M-D system been left in spinor coordinates, it would not be possible to support a group invariant spinor field under any non-trivial isotropy group.) As discussed in section 5.5, the u^μ, k^μ pair can only be sustained under a single *rotation type* isotropy constraint. The way forward then is to find which of the classes of subgroups that generate 3 d orbits have 1 d isotropy groups with a rotation type constraint. These systems would be in terms of 3 variables and would be 3 nonlinear ODEs of mixed first and third-order. It would be hoped that some other additional simplifying principle could be employed. One of these cases has already been studied in the literature: The spherical system, studied by Radford with the simplifying principle of static current.

Another suggestion is to do a full study of rank-1 group invariant solutions to the self-field *classical approximation* before attempting the M-D system. Even this would be a large undertaking, looking at the same set of 65 subgroups of $\mathcal{P}$, let alone extensions to the similitude group.

On the other hand, this work has demonstrated that for large systems of vector equations invariant under a large symmetry group, such as the Poincaré group, interesting results can be obtained from *transitive* subgroups, giving algebraic equations and rank-0 solutions. We note that there are at least 2 additional applications of this method that have not yet been attempted: Firstly, to the subgroups of the similitude group, applied to the self-field classical approximation, and secondly, to the subgroups of the conformal group applied to the *massless* Maxwell Dirac equations.

7.2 Exact solution

The exact $\widetilde{P}_{13,10}$-group invariant solution was described in chapter 6 fully both in terms of tensor quantities and reconstructed in terms of spinor quantities. It has quite a simple form, and represents laminar current flow, static at $y = 0$, increasing negative $z-$pointing velocity with increasing y, increasing positive $z-$pointing velocity with large negative values of y. This Dirac field is consistent with a positive and negative $y-$pointing electric field of strength proportional to $\sinh \kappa y$ and an $x-$pointing magnetic field of strength proportional to $\cosh \kappa y$. The axial current, which is to say the spin, aligns with the magnetic field. This solution is stationary, but is certainly not localised.

It was shown that in a certain gauge, the solution for the spinor is a superposition of $y-$exponential Dirac spinor functions. By taking a layer of solution, say between $y = 0$ and $y = \Delta$, the solution was shown to be consistent with tunnelling of an electron field between a region with $y < 0$ where an electron wave function in a constant potential is 'at rest', to a second region with $y > \Delta$ where the electron wave function in a constant potential has real momentum in the $z-$direction along the wall. Although the scalar length of current vector is constant, the probability density J_0 and the vector quantities have cosh and sinh dependence on y.

This solution does appear to have certain theoretical value. It is, to the author's knowledge, the first exact closed-form solution to the coupled M-D equations for non-null current. Because of its surprisingly simple form, it is possible to see at a glance how the different terms in the Dirac and Maxwell equations balance, and it is hoped that this will inspire other exact solutions to the M-D equations. The solution is shown to be consistent with the normal dynamical expectations of single-electron theory. The balance between the magnetic and electric fields acting on the electron in this solution may provide useful physical insights and new ideas for constructing self-consistent electromagnetic fields for the electron. The self consistent electromagnetic field in both the classical and Maxwell Dirac systems has an important magnetic component. The field in both systems is sourced in the same way, but the quantum mechanical electron field has a different kinematic response to the self-field. The difference was shown to arise from the alignment of the intrinsic magnetic moment of the electron with the magnetic field. When $\kappa = 2mc/\hbar = 2/(2\pi\lambda_C)$, where λ_C is the Compton wavelength, the electromagnetic field required to maintain laminar flow of the Dirac field is zero, the flow being a solution of the free Dirac equation. (At this value, the coupled equations require that $\rho = 0$, in order that there be no field sources.) However, at the same value of κ for the classical field, an electromagnetic field is still required in order to maintain laminar flow. This shows that the alignment of the intrinsic magnetic moment with the magnetic part of the self-field plays an important role in self-consistent Dirac field configurations.

7.3 The limitations of commuting wave-functions

In chapter 2, we reviewed the discrete symmetries of the M-D equations, and showed that if we assume commuting wave functions, which is the convention in the literature for the semi-classical Maxwell-Dirac equations, the Maxwell equations with Dirac source current $q\overline{\psi}\gamma^\mu\psi$ are not invariant under the usual charge conjugation $\psi \longrightarrow \psi_c$, $A_\mu \longrightarrow -A_\mu$. Regardless of the stated sign of q, the Dirac equation treats the electron as being able to behave as either an electron or a positron, whereas the Maxwell source current $q\overline{\psi}\gamma^\mu\psi$ assumes that the particle strictly has the sign of q. The Dirac equation in terms of tensor currents shows that, as observed by Takabayasi[58], the positive or negative mass behaviour is associated with the factor $\cos\beta$ preceding mu^μ , in dynamic balance with qA^μ and the derivative terms. When it came to examining the solutions found in chapter 6, we considered it prudent to rule out those solutions of the coupled Maxwell Dirac equations where $\cos\beta < 0$, since these represent the physically unreasonable situation of negative mass Dirac behaviour (a positron) providing negative charge source for the Maxwell equation. We find that by the same reasoning, we need to question the solutions published by Wakano[63], Lisi[43], Radford and Booth[54][11], since these have exploited to some extent ‘attraction by a repulsive potential’, showing continuous variation into regions where the spinor is exhibiting negative mass behaviour but continuing as always the same sign source current for the Maxwell equations. If we are to continue to work with the Maxwell Dirac equations in the traditional form, then we seem to need to accept that they are limited to states where the particle is predominantly an electron, without mixing positron states. This observation has grounds not only in the charge asymmetry of the equations, but also in unacceptable variation from electron to positron states such as that exhibited by the Klein paradox [36]. These problems demonstrate the flaws in the semiclassical theory.

It may be possible to find a partial remedy while retaining commuting wave functions: We propose investigating the idea of putting the same factor $\cos\beta$ in front of the Dirac current in the Maxwell equation,

$$\begin{aligned}\partial_\mu F^{\mu\nu} &= \mu_0 q \cos\beta \rho u^\mu \\ &= \mu_0 q \sigma u^\mu .\end{aligned}$$

It seems quite reasonable that the same scalar that is associated with the mass in the tensor formulation of the Dirac equation should be associated with the charge in the Maxwell equation. This expression for the current would make the alternative coupled Maxwell Dirac equations support charge conjugate solutions without having to change of sign of q. The exact solution discovered in chapter 6 would in fact also satisfy this alternative formulation, since that solution has $\cos\beta = 1$. Whereas the original formulation of the Maxwell Dirac equations does not produce the charge conjugate solution without changing the sign of q, the alternative formulation produces the charge conjugate solutions without change of the original equations. This modification to the Dirac current in the Maxwell equation is expected to have very small effect on atomic problems, where the spinor is almost entirely 'electron like'.

Ultimately, however, the work given here should be extended beyond the semi-classical M-D equations, to anticommuting wave functions, using the decomposition (2.54). This will probably require modifications to the tensor expressions given in chapter 2.

7.4 Tensor and tetrad descriptions

The gauge independent tensor description of the Maxwell Dirac system does not have the mathematical simplicity or elegance of the standard formulation. Nevertheless, it does help visualise the Dirac field in terms of 'observable currents', scalars, and the spin plane, interacting with the electric and magnetic fields. The current work has demonstrated that, complicated as the resulting system is, it is capable of solution. As demonstrated in the current work, it removes the complication of the Abelian gauge field from the search for solutions, which enabled the straightforward application of the method of group invariant solutions using group invariant vector fields. Another advantage of tensor systems, not exploited here, is that they are easier to adapt to curved space time than spinor equations. Mickelsson[48], Reiffler and Morris[57] have investigated unified electro-weak-grativational Lagrangians based on the tetrad description of spin-half-matter.

A solution to the gauge independent M-D equations can be used to reconstruct a solution to the original set of equations in spinor quantities using Crawford's reconstruction formula. A concrete demonstration of the usefulness of that reconstruction was given in chapter 6, where the exact solution referred to above was written in terms of spinor quantities in a particular gauge.

In the author's opinion, an important outstanding problem with the gauge independent tensor formulation of the Dirac field is to identify the conditions surrounding singular points and points where $\rho = 0$ in the spinor field. The requirement to be able to reconstruct a spinor in a continuous neighbourhood surrounding such points should lead to quantisation conditions on the tensor variables. It is the author's intention to address this matter in a subsequent study.

Appendix A

Classification of Poincaré subgroups

The following set of generators of the Poincaré group, $\mathcal{P}$, and the labels and classification of subalgebras of $\mathcal{P}$ up to conjugacy classes in the table that follows, come from PWZ [52]. We have calculated the dimension s of the orbit generated in $\mathbb{R}^4$, the order d of the isotropy group at a non-singular point, and have indicated for those subgroups with transitive action ($s = 4$), whether they contain a simply transitive subgroup ($d = 0$). The relevance of these features is discussed in the last section of chapter 5.

PWZ Lorentz generators	PWZ Translational generators
$B_1 \equiv 2\mathbf{J}_z$	
$B_2 \equiv -2\mathbf{K}_z$	$X_1 \equiv \frac{1}{2}(\mathbf{P}_t - \mathbf{P}_z)$
$B_3 \equiv -\mathbf{J}_y - \mathbf{K}_x$	$X_2 \equiv \mathbf{P}_y$
$B_4 \equiv +\mathbf{J}_x - \mathbf{K}_y$	$X_3 \equiv -\mathbf{P}_x$
$B_5 \equiv \mathbf{J}_y - \mathbf{K}_x$	$X_4 \equiv \frac{1}{2}(\mathbf{P}_t + \mathbf{P}_z)$
$B_6 \equiv \mathbf{J}_x + \mathbf{K}_y$	

The commutation relations in terms of these generators are:

$\frac{1}{2}[,]$	B_1	B_2	B_3	B_4	B_5	B_6
B_1	0	0	$2B_4$	$-2B_3$	$-2B_6$	$2B_5$
B_2	0	0	$2B_3$	$2B_4$	$-2B_5$	$-2B_6$
B_3	$-2B_4$	$-2B_3$	0	0	B_2	B_1
B_4	$2B_3$	$-2B_4$	0	0	B_1	$-B_2$
B_5	$2B_6$	$2B_5$	$-B_2$	$-B_1$	0	0
B_6	$-2B_5$	$2B_6$	$-B_1$	B_2	0	0
X_1	0	$-2X_1$	0	0	$-X_3$	$-X_2$
X_2	$-2X_3$	0	0	$2X_1$	0	$-2X_4$
X_3	$2X_2$	0	$-2X_1$	0	$-2X_4$	0
X_4	0	$2X_4$	$-X_3$	X_2	0	0

PWZ Label	Generators with Lorentz components	Pure translational generators	dim orbit $=s$	dim isotropy group $=d$	simply transitive subgroups
$P_{1,1}$	$B_1, .., B_6$	X_1, X_2, X_3, X_4	4	6	$T(4)$, +all below
$P_{1,2}$	$B_1, .., B_6$	–	3	3	–
$P_{2,1}$	B_1, B_2, B_3, B_4	X_1, X_2, X_3, X_4	4	4	$T(4)$, +all below
$P_{2,2}$	B_1, B_2, B_3, B_4	X_1, X_2, X_3	4	3	$P_{13,2}$ $P_{7,4}$
$P_{2,3}$	B_1, B_2, B_3, B_4	X_1	4	1	$P_{7,4}$
$P_{2,4}$	B_1, B_2, B_3, B_4	–	3	1	–
$P_{3,1}$	$B_1, B_3 - B_5, B_4 + B_6$	X_1, X_2, X_3, X_4	4	3	$T(4)$
$P_{3,2}$	$B_1, B_3 - B_5, B_4 + B_6$	$X_1 - X_4, X_2, X_3$	3	3	–
$P_{3,3}$	$B_1, B_3 - B_5, B_4 + B_6$	$X_1 + X_4$	3	1	–
$P_{3,4}$	$B_1, B_3 - B_5, B_4 + B_6$	–	2	1	–
$P_{4,1}$	$B_1, B_3 + B_5, B_4 - B_6$	X_1, X_2, X_3, X_4	4	3	$T(4)$
$P_{4,2}$	$B_1, B_3 + B_5, B_4 - B_6$	$X_1 + X_4, X_2, X_3$	3	3	–
$P_{4,3}$	$B_1, B_3 + B_5, B_4 - B_6$	$X_1 - X_4$	3	1	–
$P_{4,4}$	$B_1, B_3 + B_5, B_4 - B_6$	–	2	1	–
$P_{5,1}$	$\cos\phi B_1 + \sin\phi B_2$, B_3, B_4	X_1, X_2, X_3, X_4	4	3	$T(4)$ $P_{11,2}$ $P_{5,3}$
$P_{5,2}$	$\cos\phi B_1 + \sin\phi B_2$, B_3, B_4	X_1, X_2, X_3	4	2	$P_{11,2}$ $P_{5,3}$
$P_{5,3}$	$\cos\phi B_1 + \sin\phi B_2$, B_3, B_4	X_1	4	0	Self
$P_{5,4}$	$\cos\phi B_1 + \sin\phi B_2$, B_3, B_4	–	3	0	–
$P_{6,1}$	B_1, B_3, B_4	X_1, X_2, X_3, X_4	4	3	$T(4)$
$P_{6,2}$	B_1, B_3, B_4	X_1, X_2, X_3	3	3	–
$P_{6,3}$	B_1, B_3, B_4	X_1	3	1	–
$P_{6,4}$	B_1, B_3, B_4	–	2	1	–
$\widetilde{P}_{6,5}$	$B_1 + X_4, B_3, B_4$	X_1, X_2, X_3	4	2	$\widetilde{P}_{12,11}$
$\widetilde{P}_{6,6}$	$B_1 - X_4, B_3, B_4$	X_1, X_2, X_3	4	2	$\widetilde{P}_{12,12}$
$\widetilde{P}_{6,7}$	$B_1, B_3 + X_2, B_4 + X_3$	X_1	3	1	–
$\widetilde{P}_{6,8}$	$B_1, B_3 - X_2, B_4 - X_3$	X_1	3	1	–
$\widetilde{P}_{6,9}$	$B_1 + X_1, B_3, B_4$	–	3	0	–
$\widetilde{P}_{6,10}$	$B_1 - X_1, B_3, B_4$	–	3	0	–

PWZ Label	Generators with Lorentz components	Pure translational generators	dim orbit $=s$	dim isotropy group $=d$	simply transitive subgroups
$P_{7,1}$	B_2, B_3, B_4	X_1, X_2, X_3, X_4	4	3	$T(4)$,$P_{7,4}$ $P_{8,4}$,$P_{13,2}$
$P_{7,2}$	B_2, B_3, B_4	X_1, X_2, X_3	4	2	$P_{7,4}$ $P_{8,4}$,$P_{13,2}$
$P_{7,3}$	B_2, B_3, B_4	X_1, X_2	4	1	$P_{7,4}, P_{8,4}$
$P_{7,4}$	B_2, B_3, B_4	X_1	4	0	Self
$P_{7,5}$	B_2, B_3, B_4	–	3	0	–
$\widetilde{P}_{7,6}$	$B_2 + aX_3, B_3, B_4$	X_1, X_2	4	1	$\widetilde{P}_{8,11}, \widetilde{P}_{7,7}$
$\widetilde{P}_{7,7}$	$B_2 + aX_3, B_3, B_4$	X_1	4	0	Self
$P_{8,1}$	B_2, B_3	X_1, X_2, X_3, X_4	4	2	$T(4)$ $P_{8,4}$,$P_{13,2}$
$P_{8,2}$	B_2, B_3	X_1, X_2, X_3	4	1	$P_{8,4}$,$P_{13,2}$
$P_{8,3}$	B_2, B_3	X_1, X_3, X_4	3	2	–
$P_{8,4}$	B_2, B_3	X_1, X_2	4	0	Self
$P_{8,5}$	B_2, B_3	X_1, X_3	3	1	–
$P_{8,6}$	B_2, B_3	$X_1, X_2 + bX_3$	4	0	Self
$P_{8,7}$	B_2, B_3	X_1	3	0	–
$P_{8,8}$	B_2, B_3	X_2	3	0	–
$P_{8,9}$	B_2, B_3	–	2	0	–
$\widetilde{P}_{8,10}$	$B_2 + aX_2, B_3$	X_1, X_3, X_4	4	1	$\widetilde{P}_{13,10}$
$\widetilde{P}_{8,11}$	$B_2 + aX_3, B_3$	X_1, X_2	4	0	Self
$\widetilde{P}_{8,12}$	$B_2 + aX_2, B_3$	X_1, X_3	3	1	–
$\widetilde{P}_{8,13}$	$B_2 + aX_2, B_3$	$X_1, X_2 + bX_3$	4	0	Self
$\widetilde{P}_{8,14}$	$B_2 + aX_3, B_3$	X_1	3	0	–
$\widetilde{P}_{8,15}$	$B_2 + aX_2, B_3$	X_1	3	0	–
$\widetilde{P}_{8,16}$	$B_2 + aX_2 + bX_3, B_3$	X_1	3	0	–
$\widetilde{P}_{8,17}$	$B_2 + aX_2, B_3$	–	2	0	–
$P_{9,1}$	B_1, B_2	X_1, X_2, X_3, X_4	4	2	$T(4)$,$P_{13,2}$
$P_{9,2}$	B_1, B_2	X_1, X_2, X_3	4	1	$P_{13,2}$
$P_{9,3}$	B_1, B_2	X_2, X_3	3	1	–
$P_{9,4}$	B_1, B_2	X_1, X_4	3	1	–
$P_{9,5}$	B_1, B_2	X_1	3	0	–
$P_{9,6}$	B_1, B_2	–	2	0	–

PWZ Label	Generators with Lorentz components	Pure translational generators	dim orbit $=s$	dim isotropy group $=d$	simply transitive subgroups
$P_{10,1}$	B_3, B_4	X_1, X_2, X_3, X_4	4	2	$T(4)$
$P_{10,2}$	B_3, B_4	X_1, X_2, X_3	3	2	–
$P_{10,3}$	B_3, B_4	X_1, X_2	3	1	–
$P_{10,4}$	B_3, B_4	X_1	3	0	–
$P_{10,5}$	B_3, B_4	–	2	0	–
$\widetilde{P}_{10,6}$	$B_3 + X_4, B_4$	X_1, X_2, X_3	4	1	$\widetilde{P}_{14,10}$
$\widetilde{P}_{10,7}$	$B_3 + X_4, B_4 + bX_3$	X_1, X_2	4	0	Self
$\widetilde{P}_{10,8}$	$B_3 + X_4, B_4$	X_1, X_2	3	1	–
$\widetilde{P}_{10,9}$	$B_3, B_4 + X_3$	X_1, X_2	3	1	–
$\widetilde{P}_{10,10}$	$B_3, B_4 - X_3$	X_1, X_2	3	1	–
$\widetilde{P}_{10,11}$	$B_3, B_4 + X_2$	X_1	3	0	–
$\widetilde{P}_{10,12}$	$B_3 + X_2$, $B_4 + aX_2 + X_3$	X_1	3	0	–
$\widetilde{P}_{10,13}$	$B_3 - X_2$, $B_4 + aX_2 - X_3$	X_1	3	0	–
$\widetilde{P}_{10,14}$	$B_3 + X_2, B_4 + X_3$	X_1	3	0	–
$\widetilde{P}_{10,15}$	$B_3 - X_2, B_4 - X_3$	X_1	3	0	–
$\widetilde{P}_{10,16}$	$B_3, B_4 + X_2$	–	2	0	–
$P_{11,1}$	$\cos\phi B_1 + \sin\phi B_2$	X_1, X_2, X_3, X_4	4	1	$T(4), P_{11,2}$
$P_{11,2}$	$\cos\phi B_1 + \sin\phi B_2$	X_1, X_2, X_3	4	0	Self
$P_{11,3}$	$\cos\phi B_1 + \sin\phi B_2$	X_2, X_3	3	0	–
$P_{11,4}$	$\cos\phi B_1 + \sin\phi B_2$	X_1, X_4	3	0	–
$P_{11,5}$	$\cos\phi B_1 + \sin\phi B_2$	X_1	2	0	–
$P_{11,6}$	$\cos\phi B_1 + \sin\phi B_2$	–	1	0	–
$P_{12,1}$	B_1	X_1, X_2, X_3, X_4	4	1	$T(4)$
$P_{12,2}$	B_1	X_1, X_2, X_3	3	1	–
$P_{12,3}$	B_1	$X_1 - X_4, X_2, X_3$	3	1	–
$P_{12,4}$	B_1	$X_1 + X_4, X_2, X_3$	3	1	–
$P_{12,5}$	B_1	X_1, X_4	3	0	–
$P_{12,6}$	B_1	X_2, X_3	2	1	–
$P_{12,7}$	B_1	X_1	2	0	–
$P_{12,8}$	B_1	$X_1 - X_4$	2	0	–
$P_{12,9}$	B_1	$X_1 + X_4$	2	0	–
$P_{12,10}$	B_1	–	1	0	–

PWZ Label	Generators with Lorentz components	Pure translational generators	dim orbit $=s$	dim isotropy group $=d$	simply transitive subgroups
$\widetilde{P}_{12,11}$	$B_1 + X_4$	X_1, X_2, X_3	4	0	Self
$\widetilde{P}_{12,12}$	$B_1 - X_4$	X_1, X_2, X_3	4	0	Self
$\widetilde{P}_{12,13}$	$B_1 + a\,(X_1 + X_4)$	$X_1 - X_4, X_2, X_3$	4	0	Self
$\widetilde{P}_{12,14}$	$B_1 + b(X_1 - X_4)$	$X_1 + X_4, X_2, X_3$	4	0	Self
$\widetilde{P}_{12,15}$	$B_1 + X_4$	X_2, X_3	3	0	–
$\widetilde{P}_{12,16}$	$B_1 - X_4$	X_2, X_3	3	0	–
$\widetilde{P}_{12,17}$	$B_1 + a\,(X_1 + X_4)$	X_2, X_3	3	0	–
$\widetilde{P}_{12,18}$	$B_1 + b(X_1 - X_4)$	X_2, X_3	3	0	–
$\widetilde{P}_{12,19}$	$B_1 + X_4$	X_1	2	0	–
$\widetilde{P}_{12,20}$	$B_1 - X_4$	X_1	2	0	–
$\widetilde{P}_{12,21}$	$B_1 + a(X_1 + X_4)$	$X_1 - X_4$	2	0	–
$\widetilde{P}_{12,22}$	$B_1 + b(X_1 - X_4)$	$X_1 + X_4$	2	0	–
$\widetilde{P}_{12,23}$	$B_1 + X_4$	–	1	0	–
$\widetilde{P}_{12,24}$	$B_1 - X_4$	–	1	0	–
$\widetilde{P}_{12,25}$	$B_1 + a(X_1 + X_4)$	–	1	0	–
$\widetilde{P}_{12,26}$	$B_1 + b(X_1 - X_4)$	–	1	0	–
$P_{13,1}$	B_2	X_1, X_2, X_3, X_4	4	1	$T(4), P_{13,2}$
$P_{13,2}$	B_2	X_1, X_2, X_3	4	0	Self
$P_{13,3}$	B_2	X_1, X_3, X_4	3	1	–
$P_{13,4}$	B_2	X_1, X_4	2	1	–
$P_{13,5}$	B_2	X_1, X_2	3	0	–
$P_{13,6}$	B_2	X_2, X_3	3	0	–
$P_{13,7}$	B_2	X_1	2	0	–
$P_{13,8}$	B_2	X_2	2	0	–
$P_{13,9}$	B_2	–	1	0	–
$\widetilde{P}_{13,10}$	$B_2 + aX_2$	X_1, X_3, X_4	4	0	Self
$\widetilde{P}_{13,11}$	$B_2 + aX_2$	X_1, X_3	3	0	–
$\widetilde{P}_{13,12}$	$B_2 + aX_2$	X_1, X_4	3	0	–
$\widetilde{P}_{13,13}$	$B_2 + aX_2$	X_1	2	0	–
$\widetilde{P}_{13,14}$	$B_2 + aX_2$	X_3	2	0	–
$\widetilde{P}_{13,15}$	$B_2 + aX_2$	–	1	0	–
$P_{14,1}$	B_3	X_1, X_2, X_3, X_4	4	1	$T(4)$
$P_{14,2}$	B_3	X_1, X_2, X_3	3	1	–

PWZ Label	Generators with Lorentz components	Pure translational generators	dim orbit $=s$	dim isotropy group $=d$	simply transitive subgroups
$P_{14,3}$	B_3	X_1, X_3, X_4	3	1	–
$P_{14,4}$	B_3	X_1, X_2	3	0	–
$P_{14,5}$	B_3	X_1, X_3	2	1	–
$P_{14,6}$	B_3	$X_1, X_2 + bX_3$	3	0	–
$P_{14,7}$	B_3	X_1	2	0	–
$P_{14,8}$	B_3	X_2	2	0	–
$P_{14,9}$	B_3	–	1	0	–
$\widetilde{P}_{14,10}$	$B_3 + X_4$	X_1, X_2, X_3	4	0	Self
$\widetilde{P}_{14,11}$	$B_3 + X_2$	X_1, X_3, X_4	4	0	Self
$\widetilde{P}_{14,12}$	$B_3 - X_2$	X_1, X_3, X_4	4	0	Self
$\widetilde{P}_{14,13}$	$B_3 + X_4$	X_1, X_2	3	0	–
$\widetilde{P}_{14,14}$	$B_3 + X_4$	X_1, X_3	3	0	–
$\widetilde{P}_{14,15}$	$B_3 + X_2$	X_1, X_3	3	0	–
$\widetilde{P}_{14,16}$	$B_3 - X_2$	X_1, X_3	3	0	–
$\widetilde{P}_{14,17}$	$B_3 + X_4$	$X_1, X_2 + bX_3$	3	0	–
$\widetilde{P}_{14,18}$	$B_3 + X_2$	$X_1, X_2 + bX_3$	3	0	–
$\widetilde{P}_{14,19}$	$B_3 - X_2$	$X_1, X_2 + bX_3$	3	0	–
$\widetilde{P}_{14,20}$	$B_3 + X_4$	X_1	2	0	–
$\widetilde{P}_{14,21}$	$B_3 + X_2$	X_1	2	0	–
$\widetilde{P}_{14,22}$	$B_3 - X_2$	X_1	2	0	–
$\widetilde{P}_{14,23}$	$B_3 + X_4$	X_2	2	0	–
$\widetilde{P}_{14,24}$	$B_3 + X_4$	–	1	0	–
$\widetilde{P}_{14,25}$	$B_3 + X_2$	–	1	0	–
$\widetilde{P}_{14,26}$	$B_3 - X_2$	–	1	0	–
$P_{15,1} = T(4)$	–	X_1, X_2, X_3, X_4	4	0	Self
$P_{15,2}$	–	X_1, X_2, X_3	3	0	–
$P_{15,3}$	–	$X_2, X_3, X_1 - X_4$	3	0	–
$P_{15,4}$	–	$X_1 + X_4, X_2, X_3$	3	0	–
$P_{15,5}$	–	X_1, X_2	2	0	–
$P_{15,6}$	–	X_1, X_4	2	0	–
$P_{15,7}$	–	X_2, X_3	2	0	–
$P_{15,8}$	–	X_1	1	0	–
$P_{15,9}$	–	$X_1 + X_4$	1	0	–
$P_{15,10}$	–	$X_1 - X_4$	1	0	–
$P_{15,11} = \{1\}$	–	–	0	0	–

Appendix B

Calculations

B.1 Useful identitites for Poincaré transformations

B.1.1 Light-cone coordinates

We will have use in some cases for light-cone coordinates:

$$\begin{aligned} l_+ &\equiv t + z \\ l_+ &\equiv t - z \end{aligned}$$

It is easy to show that for any function of light cone coordinates,

$$\begin{aligned} \Box^2 f(l_+) &= 0 \\ \Box^2 f(l_-) &= 0. \end{aligned} \tag{B.1}$$

B.1.2 Exponentiation of various Poincaré generators

The definitions of the PWZ's generators $B_1..B_6, X_1..X_4$ that are referred to in this appendix are given at the beginning of appendix A. We have trivially from the definitions of the translation generators (5.3):

$$\begin{aligned} \exp(\alpha X_1) \cdot x &= x + \left(\frac{1}{2}\alpha, 0, 0, -\frac{1}{2}\alpha\right) \\ \exp(\alpha X_2) \cdot x &= x + (0, 0, \alpha, 0) \\ \exp(\alpha X_3) \cdot x &= x + (0, -\alpha, 0, 0) \\ \exp(\alpha X_4) \cdot x &= x + \left(\frac{1}{2}\alpha, 0, 0, \frac{1}{2}\alpha\right). \end{aligned}$$

We need to emphasize that the above notation for the action of the group member on the point in the space, $g \cdot x$, does *not* represent a *matrix* operation on x. g is simply a member of a group of transformations acting that can act on x. Only when g is a pure Lorentz group member can the action be written in terms of a matrix multiplication. Neither is the notation $\exp(\alpha X_1)$ intended to convey matrix exponentiation (although it is possible to find a representation for $\mathcal{P}$ in which all the generators can be written as matrices [52]). In what follows, when the action is performed via matrix multiplication, it will be obvious, because a full 4×4 matrix will be given. In other caes, where the group member contains translational action, it is necessary to calculate the action without using matrices. Fortunately, as shown by the above, this is often trivial.

The representation of the generators $B_1..B_6$ as matrix operators on $x = x^\mu$ are

$$
\begin{aligned}
B_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} \\
B_3 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; B_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
B_5 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; B_6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{aligned}
$$

from which we get

$$
\exp \alpha B_1 \cdot x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & -\sin 2\alpha & 0 \\ 0 & \sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x. \tag{B.2}
$$

Therefore, t and z (or alternatively, l_+ and l_-) are invariant under B_1.

$$
\exp \alpha B_2 \cdot x = \begin{pmatrix} \cosh 2\alpha & 0 & 0 & -\sinh 2\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh 2\alpha & 0 & 0 & \cosh 2\alpha \end{pmatrix} x. \tag{B.3}
$$

Therefore, x and y are invariant under B_2.

To calculate the exponentiated action of B_3 we use the fact that $(B_3)^3 = 0$ in the expansion:

$$
\begin{aligned}
\exp \alpha B_3 &= 1 + \alpha B_3 + \frac{1}{2}\alpha^2 B_3^2 + \frac{1}{6}\alpha^3 B_3^3 + ... \\
&= 1 + \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ -\alpha & 0 & 0 & -\alpha \\ 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha^2/2 & 0 & 0 & \alpha^2/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha^2/2 & 0 & 0 & -\alpha^2/2 \end{pmatrix} + 0 + 0 + ...,
\end{aligned}
$$

so

$$\exp \alpha B_3 \cdot x = \begin{pmatrix} 1+\alpha^2/2 & -\alpha & 0 & \alpha^2/2 \\ -\alpha & 1 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \\ -\alpha^2/2 & \alpha & 0 & 1-\alpha^2/2 \end{pmatrix} x \tag{B.4}$$

$$= \begin{pmatrix} t + (\alpha^2/2)\, l_+ - \alpha x \\ x - \alpha l_+ \\ y \\ z - (\alpha^2/2)\, l_+ + \alpha x \end{pmatrix}. \tag{B.5}$$

Therefore, l_+ and y are invariant under B_3.

By a similar calculation,

$$\exp \beta B_4 \cdot x = \begin{pmatrix} 1+\beta^2/2 & 0 & -\beta & \beta^2/2 \\ 0 & 1 & 0 & 0 \\ -\beta & 0 & 1 & -\beta \\ -\beta^2/2 & 0 & \beta & 1-\beta^2/2 \end{pmatrix} x \tag{B.6}$$

$$= \begin{pmatrix} t + \left(\beta^2/2\right) l_+ - \beta y \\ x \\ y - \beta l_+ \\ z - \left(\beta^2/2\right) l_+ + \beta y \end{pmatrix}. \tag{B.7}$$

Therefore, l_+ and x are invariant under B_4.

We use that fact that if $[G_1, G_2] = 0$ then $\exp(G_1 + G_2) = \exp G_1 \exp G_2 = \exp G_2 \exp G_1$, to give the following identities:

$$\exp \alpha(B_1 \pm X_4) = \exp(\pm\alpha X_4)\exp(\alpha B_1) \tag{B.8}$$
$$\exp \alpha(B_1 + a(X_1 \pm X_4)) = \exp(\alpha a(X_1 \pm X_4))\exp(\alpha B_1) \tag{B.9}$$
$$\exp \beta(B_2 + aX_3) = \exp(a\beta X_3)\exp(\beta B_2) \tag{B.10}$$
$$\exp \alpha(B_3 \pm X_2) = \exp(\pm\alpha X_2)\exp(\alpha B_3) \tag{B.11}$$
$$\exp \alpha(B_4 + bX_3) = \exp(\alpha B_4)\exp(\alpha b X_3) \tag{B.12}$$
$$\exp(\alpha B_3 + \beta B_4) = \exp(\alpha B_3)\exp(\beta B_4) \tag{B.13}$$
$$\exp(\alpha B_1 + \beta B_2) = \exp(\alpha B_1)\exp(\beta B_2). \tag{B.14}$$

We will also need a result concerning the orbit passing through the origin that is generated by $(B_3 + X_4)$. Let x^μ be a point on this orbit. Then the Lie derivative is given by the generator action:

$$\frac{dx^\mu}{d\alpha} = B_3 \cdot x^\mu + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -x \\ -t-z \\ 0 \\ x \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

giving the four linked ODEs;

$$\begin{aligned}\frac{dt}{d\alpha} &= \frac{1}{2} - x\\ \frac{dx}{d\alpha} &= -t - z\\ \frac{dy}{d\alpha} &= 0\\ \frac{dz}{d\alpha} &= \frac{1}{2} + x.\end{aligned}$$

Immediately, we have

$$y = \text{const}.$$

By differentiating the second equation, and substituting in the first and the fourth,

$$\frac{d^2x}{d\alpha^2} = -\frac{dt}{d\alpha} - \frac{dz}{d\alpha} = -1,$$

for which the general solution is

$$x = -\frac{1}{2}\alpha^2 + P\alpha + Q$$

with P and Q as yet to be determined constants. Substituting this result into the other equations and integrating gives

$$\begin{aligned}t &= \frac{1}{2}\alpha + \frac{1}{6}\alpha^3 - \frac{1}{2}P\alpha^2 - Q\alpha + S\\ z &= \frac{1}{2}\alpha - \frac{1}{6}\alpha^3 + \frac{1}{2}P\alpha^2 + Q\alpha + T\end{aligned}$$

with S and T as yet to be determined constants. Finally, we require that when $a = 0$, then $(t, x, y, z) = 0$, giving $P = 0, Q = 0, S = 0, T = 0$. The solution is presented in the following identity:

$$\exp\alpha\left(B_3 + X_4\right) \cdot \begin{pmatrix} 0\\ 0\\ 0\\ 0\end{pmatrix} = \begin{pmatrix} \frac{1}{2}\alpha + \frac{1}{6}\alpha^3\\ -\frac{1}{2}\alpha^2\\ 0\\ \frac{1}{2}\alpha - \frac{1}{6}\alpha^3\end{pmatrix}. \qquad \text{(B.15)}$$

B.2 Rank 0 group invariant fields and reduced equations

B.2.1 $P_{5,3}$: generated by $\cos\phi B_1 + \sin\phi B_2, B_3, B_4, X_1$

We abbreviate

$$\begin{aligned}c &= \cos\phi\\ s &= \sin\phi.\end{aligned}$$

Group invariant field

The condition $t + z = 0$ defines a singular orbit that divides $\mathbb{R}^4$ into two separate generic orbits. We choose $p_0 = (R, 0, 0, R)$ as a convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. One of the generic orbits will arise from R positive, the other from R negative. Consider the transformation of the convenient point

$$\exp(\alpha B_3)\exp(\beta B_4)\exp(\gamma(\cos\phi B_1 + \sin\phi B_2)) \cdot p_0 = \begin{pmatrix} \frac{1}{2}l_+ + \frac{\alpha^2+\beta^2}{2}l_+ \\ -\alpha l_+ \\ -\beta l_+ \\ \frac{1}{2}l_+ - \frac{\alpha^2+\beta^2}{2}l_+ \end{pmatrix}$$

where $2\gamma = \ln(2sR/l_+)$. Observe that by judicious choice of α, β and γ, followed by the use of the generator X_1, which changes l_- but leaves l_+, x and y unchanged, it is possible to transform the convenient point to any other point in $\mathbb{R}^4$ with the same sign of l_+. Therefore, $\alpha = -\frac{x}{l_+}$,$\beta = -\frac{y}{l_+}$, $2\gamma = \ln(2sR/l_+)$, and the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector field $V(x) = \Lambda(x)V(p_0)$ is

$$\begin{aligned}\Lambda(x) &= (\exp(\alpha B_3)\exp(\beta B_4))\exp(\gamma B_2) \\ &= \begin{pmatrix} 1+\frac{x^2+y^2}{2\,l_+^2} & \frac{x}{l_+} & \frac{y}{l_+} & \frac{x^2+y^2}{2\,l_+^2} \\ \frac{x}{l_+} & 1 & 0 & \frac{x}{l_+} \\ \frac{y}{l_+} & 0 & 1 & \frac{y}{l_+} \\ -\frac{x^2+y^2}{2\,l_+^2} & -\frac{x}{l_+} & -\frac{y}{l_+} & 1-\frac{x^2+y^2}{2\,l_+^2} \end{pmatrix} \\ &\times \begin{pmatrix} \frac{sR}{l_+}+\frac{l_+}{4sR} & 0 & 0 & -\frac{sR}{l_+}+\frac{l_+}{4sR} \\ 0 & c\cos\left(\ln\left(\frac{2sR}{l_+}\right)\right) & -c\sin\left(\ln\left(\frac{2sR}{l_+}\right)\right) & 0 \\ 0 & c\sin\left(\ln\left(\frac{2sR}{l_+}\right)\right) & c\cos\left(\ln\left(\frac{2sR}{l_+}\right)\right) & 0 \\ -\frac{sR}{l_+}+\frac{l_+}{4sR} & 0 & 0 & \frac{sR}{l_+}+\frac{l_+}{4sR} \end{pmatrix},\end{aligned}$$

giving the form of the invariant vector field :

$$V^\mu(x) = \begin{pmatrix} \left[\begin{array}{c} (A-D)\frac{R}{l_+} + (A+D)\left(\frac{l_+}{4R} + \frac{x^2+y^2}{l_+^2}\frac{l_+}{4R}\right) \\ +c\left(\frac{x}{l_+}B + \frac{y}{l_+}C\right)\cos\left(\ln\left(\frac{2sR}{l_+}\right)\right) - c\left(\frac{x}{l_+}C - \frac{y}{l_+}B\right)\sin\left(\ln\left(\frac{2sR}{l_+}\right)\right) \end{array}\right], \\ Bc\cos\left(\ln\left(\frac{2sR}{l_+}\right)\right) - Cc\sin\left(\ln\left(\frac{2sR}{l_+}\right)\right) + \frac{x}{l_+}\left((A+D)\frac{l_+}{2R}\right), \\ Bc\sin\left(\ln\left(\frac{2sR}{l_+}\right)\right) + Cc\cos\left(\ln\left(\frac{2sR}{l_+}\right)\right) + \frac{y}{l_+}\left((A+D)\frac{l_+}{2R}\right), \\ \left[\begin{array}{c} (-A+D)\frac{R}{l_+} + (A+D)\left(\frac{l_+}{4R} - \frac{x^2+y^2}{l_+^2}\frac{l_+}{4R}\right) \\ -c\left(\frac{x}{l_+}B + \frac{y}{l_+}C\right)\cos\left(\ln\left(\frac{2sR}{l_+}\right)\right) + c\left(\frac{x}{l_+}C - \frac{y}{l_+}B\right)\sin\left(\ln\left(\frac{2sR}{l_+}\right)\right) \end{array}\right] \end{pmatrix} \tag{B.16}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = -\frac{3(A+D)}{2R} \tag{B.17}$$

$$\Box^2 V^\mu(x) = \begin{pmatrix} -\frac{(A+D)}{R\,l_+} \\ 0 \\ 0 \\ +\frac{(A+D)}{R\,l_+} \end{pmatrix} \tag{B.18}$$

Reduced classical equations:

Checking (3.7) using the forms(B.16) and (B.18) for u^μ

$$\begin{aligned} -\frac{A+D}{R} &= \frac{\mu_0 q^2}{mc^2}\rho(A-D)R \\ A+D &= 0 \\ B &= 0 \\ C &= 0. \end{aligned}$$

Combining the first 2 of these equations, we reach the conclusion that either $\rho = 0$ or $(A-D) = 0$. $u^\mu u_\mu = 1$ forbids the latter and so we have no solution.

Reduced gauge independent M-D equations

It has been possible to rule out any non-zero solutions quite quickly from the continuity equation, seen by substituting J^μ into (B.17) giving

$$A + D = 0.$$

Therefore the length of the current vector $J^\mu J_\mu = A^2 - B^2 - C^2 - D^2$ at the convenient point, and therefore at all other points, must be zero. We call this '*zero current from anti-timelike continuity condition*'.

B.2.2 $P_{7,4}$: generated by B_2, B_3, B_4, X_1

Group invariant field

The condition $t + z = 0$ defines a singular orbit that divides $\mathbb{R}^4$ into two separate generic orbits. We choose $p_0 = (R, 0, 0, R)$ as a convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. One of the generic orbits will arise from R positive, the other from R negative. Consider the transformation of the convenient point

$$\exp(\alpha B_3)\exp(\beta B_4)\exp(\gamma B_2)\cdot p_0 = \begin{pmatrix} \frac{1}{2}l_+ + \frac{\alpha^2+\beta^2}{2}l_+ \\ -\alpha l_+ \\ -\beta l_+ \\ \frac{1}{2}l_+ - \frac{\alpha^2+\beta^2}{2}l_+ \end{pmatrix}$$

where

$$l_+ = 2\exp(-2\gamma)R.$$

Observe that by judicious choice of α, β and γ, followed by the use of the generator X_1, which changes l_- but leaves l_+, x and y unchanged, it is possible to transform the convenient point to any other point in $\mathbb{R}^4$ with the same sign of l_+. Therefore, the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector field $V(x) = \Lambda(x)V(p_0)$ is

$$\begin{aligned}\Lambda(x) &= (\exp(\alpha B_3)\exp(\beta B_4))\exp(\gamma B_2) \\ &= \begin{pmatrix} 1+\frac{x^2+y^2}{2\,l_+^2} & \frac{x}{l_+} & \frac{y}{l_+} & \frac{x^2+y^2}{2\,l_+^2} \\ \frac{x}{l_+} & 1 & 0 & \frac{x}{l_+} \\ \frac{y}{l_+} & 0 & 1 & \frac{y}{l_+} \\ -\frac{x^2+y^2}{2\,l_+^2} & -\frac{x}{l_+} & -\frac{y}{l_+} & 1-\frac{x^2+y^2}{2\,l_+^2} \end{pmatrix} \begin{pmatrix} \frac{R}{l_+}+\frac{l_+}{4R} & 0 & 0 & -\frac{R}{l_+}+\frac{l_+}{4R} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{R}{l_+}+\frac{l_+}{4R} & 0 & 0 & \frac{R}{l_+}+\frac{l_+}{4R} \end{pmatrix}\end{aligned}$$

giving the form of the invariant vector field :

$$V^\mu(x) = \begin{pmatrix} (A-D)\,\frac{R}{l_+} + \frac{(A+D)}{4R}\left(l_+ + \frac{x^2+y^2}{l_+}\right) + \frac{x}{l_+}B + \frac{y}{l_+}C \\ B + \frac{x(A+D)}{2R} \\ C + \frac{y(A+D)}{2R} \\ (-A+D)\,\frac{R}{l_+} + \frac{(A+D)}{4R}\left(l_+ - \frac{x^2+y^2}{l_+}\right) - \frac{x}{l_+}B - \frac{y}{l_+}C \end{pmatrix}. \qquad \text{(B.19)}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = \frac{3(A+D)}{2R} \qquad \text{(B.20)}$$

$$\Box^2 V^\mu(x) = \begin{pmatrix} -\frac{(A+D)}{R\,l_+} \\ 0 \\ 0 \\ +\frac{(A+D)}{R\,l_+} \end{pmatrix}. \qquad \text{(B.21)}$$

Reduced classical equations:

Let u_μ take the form of the group invariant vector field V_μ and so $u_\mu(p_0) = (A, B, C, D)$. Checking (3.7) using the forms (B.19) and (B.21) leads to the following algebraic equations:

$$\begin{aligned} -\frac{A+D}{R} &= \frac{\mu_0 q^2}{mc^2}\rho(A-D)R \\ A+D &= 0 \\ B &= 0 \\ C &= 0. \end{aligned}$$

Combining the first 2 of these equations, we reach the conclusion that either $\rho = 0$ or $(A-D) = 0$. But the requirement $u^\mu u_\mu = 1$ forbids the latter and so we conclude we must adopt the zero current solution, of no further interest.

Reduced gauge independent M-D equations

It has been possible to rule out any non-zero solutions quite quickly from the continuity equation, seen by substituting J^μ into (B.20) giving

$$A + D = 0.$$

Therefore the length of the current vector $J^\mu J_\mu = A^2 - B^2 - C^2 - D^2$ at the convenient point, and therefore at all other points, must be zero. We call this '*zero current from anti-timelike continuity condition*'.

B.2.3 $\widetilde{P}_{7,7}$: generated by $B_2 + aX_3, B_3, B_4, X_1$

Group invariant field

The condition $t + z = 0$ defines a singular orbit that divides $\mathbb{R}^4$ into two separate generic orbits. We choose $p_0 = (R, 0, 0, R)$ as a convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. One of the generic orbits will arise from R positive, the other from R negative. Consider the transformation of the convenient point

$$\exp(\alpha B_3)\exp(\beta B_4)\exp(\gamma\,(B_2 + aX_3)) \cdot p_0 = \begin{pmatrix} \frac{1}{2}l_+ + \frac{\alpha^2+\beta^2}{2}l_+ + \alpha\alpha_1 a \\ -\frac{1}{2}\ln\frac{l_+}{2R}a - \alpha l_+ \\ -\beta l_+ \\ \frac{1}{2}l_+ - \frac{\alpha^2+\beta^2}{2}l_+ - \alpha\alpha_1 a \end{pmatrix}$$

where $\exp(-2\gamma)R = \frac{1}{2}l_+$. Observe that by judicious choice of α, β and γ, followed by the use of the generator X_1, which changes l_- but leaves l_+, x and y unchanged, it is possible to transform the convenient point to any other point in $\mathbb{R}^4$ with the same sign of l_+. Therefore, $\exp(-2\gamma)R = \frac{1}{2}l_+$, $\alpha = -\frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)}{l_+}$, $\beta = -\frac{y}{l_+}$ and the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector field $V(x) = \Lambda(x)V(p_0)$ is

$$\begin{aligned}\Lambda(x) &= (\exp(\alpha B_3)\exp(\beta B_4))\exp(\gamma B_2) \\ &= \begin{pmatrix} 1 + \frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)^2+y^2}{2\,l_+^2} & \frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)}{l_+} & \frac{y}{l_+} & \frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)^2+y^2}{2\,l_+^2} \\ \frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)}{l_+} & 1 & 0 & \frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)}{l_+} \\ \frac{y}{l_+} & 0 & 1 & \frac{y}{l_+} \\ -\frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)^2+y^2}{2\,l_+^2} & -\frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)}{l_+} & -\frac{y}{l_+} & 1 - \frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)^2+y^2}{2\,l_+^2} \end{pmatrix} \\ &\times \begin{pmatrix} \frac{R}{l_+} + \frac{l_+}{4R} & 0 & 0 & -\frac{R}{l_+} + \frac{l_+}{4R} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{R}{l_+} + \frac{l_+}{4R} & 0 & 0 & \frac{R}{l_+} + \frac{l_+}{4R} \end{pmatrix}\end{aligned}$$

giving the form of the invariant vector field :

$$V^\mu(x) = \begin{pmatrix} (A-D)\frac{R}{l_+} + (A+D)\frac{l_+}{4R}\left(1 + \frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)^2 + y^2}{l_+^2}\right) + \frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)}{l_+}B + \frac{y}{l_+}C \\ B + \left(x + \frac{1}{2}\ln\frac{l_+}{2R}a\right)\frac{(A+D)}{2R} \\ C + y\frac{(A+D)}{2R} \\ (-A+D)\frac{R}{l_+} + (A+D)\frac{l_+}{4R}\left(1 - \frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)^2 + y^2}{l_+^2}\right) - \frac{\left(x+\frac{1}{2}\ln\frac{l_+}{2R}a\right)}{l_+}B - \frac{y}{l_+}C \end{pmatrix} \tag{B.22}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = (A+D)\frac{3}{2R} \tag{B.23}$$

$$\Box^2 V^\mu(x) = \begin{pmatrix} -\frac{(A+D)}{R\,l_+} \\ 0 \\ 0 \\ +\frac{(A+D)}{R\,l_+} \end{pmatrix}. \tag{B.24}$$

Reduced classical equations:

Let u_μ take the form of the group invariant vector field V_μ and so $u_\mu(p_0) = (A, B, C, D)$. Checking (3.7) using the forms (B.22) and (B.24) requires

$$\begin{aligned} -\frac{A+D}{R} &= \frac{\mu_0 q^2}{mc^2}\rho(A-D)R \\ A+D &= 0 \\ B &= 0 \\ C &= 0. \end{aligned}$$

Combining the first 2 of these equations, we reach the conclusion that either $\rho = 0$ or $(A-D) = 0$. But the requirement $u^\mu u_\mu = 1$ forbids the latter and so we conclude we must adopt the zero current solution, of no further interest**.**

Reduced gauge independent M-D equations

It has been possible to rule out any non-zero solutions quite quickly from the continuity equation, seen by substituting J^μ into (B.23) giving

$$A + D = 0.$$

Therefore the length of the current vector $J^\mu J_\mu = A^2 - B^2 - C^2 - D^2$ at the convenient point, and therefore at all other points, must be zero. We call this '*zero current from anti-timelike continuity condition*'.

B.2.4 $P_{8,4}$: generated by B_2, B_3, X_1, X_2

Group invariant field

The condition $t+z=0$ defines a singular orbit that divides $\mathbb{R}^4$ into two separate generic orbits. We choose $p_0=(R,0,0,R)$ as a convenient point, and a general vector at the convenient point $V^\mu(p_0)=(A,B,C,D)$. One of the generic orbits will arise from R positive, the other from R negative. Consider the transformation of the convenient point

$$\exp(\beta B_3)\exp(\alpha B_2)\cdot p_0=\begin{pmatrix}\exp(-2\alpha)R+\beta^2\exp(-2\alpha)R\\ -2\beta\exp(-2\alpha)R\\ 0\\ \exp(-2\alpha)R-\beta^2\exp(-2\alpha)R\end{pmatrix}.$$

Observe that by judicious choice of α and β, followed by the use of the generators X_1, X_2 which leave not only $l_+=t+z$ but also x unchanged, it is possible to transform the convenient point to any other point in $\mathbb{R}^4$ with the same sign of l_+. We have $\beta=-\frac{x}{l_+}$ and $\exp(-2\alpha)R=\frac{1}{2}l_+$. Therefore, the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector field $V(x)=\Lambda(x)V(p_0)$ is

$$\begin{aligned}\Lambda(x) &= \exp(\beta B_3)\exp(\alpha B_2)\\ &= \begin{pmatrix}1+\frac{x^2}{2\,l_+^2} & \frac{x}{l_+} & 0 & \frac{x^2}{2\,l_+^2}\\ \frac{x}{l_+} & 1 & 0 & \frac{x}{l_+}\\ 0 & 0 & 1 & 0\\ -\frac{x^2}{2\,l_+^2} & -\frac{x}{l_+} & 0 & 1-\frac{x^2}{2\,l_+^2}\end{pmatrix}\begin{pmatrix}\frac{R}{l_+}+\frac{l_+}{4R} & 0 & 0 & -\frac{R}{l_+}+\frac{l_+}{4R}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\frac{R}{l_+}+\frac{l_+}{4R} & 0 & 0 & \frac{R}{l_+}+\frac{l_+}{4R}\end{pmatrix}\end{aligned}$$

giving the form of the invariant vector field :

$$V^\mu(x)=\begin{pmatrix}(A-D)\frac{R}{l_+}+(A+D)\left(\frac{l_+}{4R}+\frac{x^2}{l_+^2}\frac{l_+}{4R}\right)+\frac{x}{l_+}B\\ B+\frac{x(A+D)}{2R}\\ C\\ (-A+D)\frac{R}{l_+}+(A+D)\left(\frac{l_+}{4R}-\frac{x^2}{l_+^2}\frac{l_+}{4R}\right)-\frac{x}{l_+}B\end{pmatrix}. \tag{B.25}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x)=\frac{(A+D)}{R} \tag{B.26}$$

$$\Box^2 V^\mu(x)=\begin{pmatrix}-\frac{(A+D)}{2R\,l_+}\\ 0\\ 0\\ +\frac{(A+D)}{2R\,l_+}\end{pmatrix}. \tag{B.27}$$

This conforms with the form (B.25), as it should, if we take $A'+D'=0$, $(A'-D')R=-\frac{(A+D)}{2R}$, $B'=C'=0$ for the invariant form associated with $\Box^2V^\mu$ which must be an invariant vector field too.

Reduced classical equations:

Let u_μ take the form of the group invariant vector field V_μ and so $u_\mu(p_0) = (A, B, C, D)$. Checking (3.7) using the forms (B.25) and (B.27) leads to the following algebraic equations:

$$\begin{aligned} -\frac{A+D}{2R} &= \frac{\mu_0 q^2}{mc^2}\rho(A-D)R \\ A+D &= 0 \\ B &= 0 \\ C &= 0. \end{aligned}$$

Combining the first 2 of these equations, we reach the conclusion that either $\rho = 0$ or $(A-D) = 0$. But the requirement $u^\mu u_\mu = 1$ forbids the latter and so we conclude we must adopt the zero current solution, of no further interest.

Reduced gauge independent M-D equations

It has been possible to rule out any non-zero solutions quite quickly from the continuity equation, seen by substituting J^μ into (B.26) giving

$$A + D = 0.$$

Therefore the length of the current vector $J^\mu J_\mu = A^2 - B^2 - C^2 - D^2$ at the convenient point, and therefore at all other points, must be zero. We call this '*zero current from anti-timelike continuity condition*'.

B.2.5 $P_{8,6}$: generated by $B_2, B_3, X_1, X_2 + bX_3$

Group invariant field

This result uses the same transformations as the preceding, $P_{8,4}$, but the second transformation $\exp(\beta B_3)$ needs to be expressed now in terms of the l_+ and $x+by$ components of the mapped point, the latter component being the component that is now unchanged by the remaining 2 generators, X_1 and $X_2 + bX_3$. This gives $\beta = -\frac{x+by}{l_+}$, and the form of the invariant vector field is:

$$V^\mu(x) = \begin{pmatrix} (A-D)\frac{R}{l_+} + (A+D)\frac{l_+}{4R}\left(1 + \frac{(x+by)^2}{l_+^2}\right) + \frac{x+by}{l_+}B \\ B + \frac{x(A+D)}{2R} \\ C \\ (-A+D)\frac{R}{l_+} + (A+D)\frac{l_+}{4R}\left(1 - \frac{(x+by)^2}{l_+^2}\right) - \frac{x+by}{l_+}B \end{pmatrix}. \qquad \text{(B.28)}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = \frac{(A+D)}{R} \qquad \text{(B.29)}$$

$$\Box^2 V^\mu(x) = \begin{pmatrix} -\frac{(A+D)}{2R\,l_+} \\ 0 \\ 0 \\ +\frac{(A+D)}{2R\,l_+} \end{pmatrix}. \tag{B.30}$$

This conforms with the form (B.28), as it should, if we take $A' + D' = 0$, $(A' - D')R = -\frac{(A+D)}{2R}$, $B' = C' = 0$ for the invariant form associated with $\Box^2 V^\mu$ which must be an invariant vector field too.

Reduced classical equations:

Let u_μ take the form of the group invariant vector field V_μ and so $u_\mu(p_0) = (A, B, C, D)$. Checking (3.7) using the forms (B.25) and (B.27) leads to the following algebraic equations:

$$\begin{aligned} -\frac{A+D}{2R} &= \frac{\mu_0 q^2}{mc^2}\rho(A-D)R \\ A+D &= 0 \\ B &= 0 \\ C &= 0. \end{aligned}$$

Combining the first 2 of these equations, we reach the conclusion that either $\rho = 0$ or $(A - D) = 0$. But the requirement $u^\mu u_\mu = 1$ forbids the latter and so we conclude we must adopt the zero current solution, of no further interest.

Reduced gauge independent M-D equations

It has been possible to rule out any non-zero solutions quite quickly from the continuity equation, seen by substituting J^μ into (B.29) giving

$$A + D = 0.$$

Therefore the length of the current vector $J^\mu J_\mu = A^2 - B^2 - C^2 - D^2$ at the convenient point, and therefore at all other points, must be zero. We call this '*zero current from anti-timelike continuity condition*'.

B.2.6 $\widetilde{P}_{8,11}$: generated by $B_2 + aX_3, B_3, X_1, X_2$

Group invariant field

The condition $t + z = 0$ defines a singular orbit that divides $\mathbb{R}^4$ into two separate generic orbits. We choose $p_0 = (R, 0, 0, R)$ as a convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. One of the generic orbits will arise from R positive, the other from R negative. Consider the transformation of the convenient point (where identity (B.10) has been used to calculate the first transformation):

$$\exp(\beta B_3)\exp(\alpha\,(B_2 + aX_3))\cdot p_0 = \begin{pmatrix} \frac{1}{2}l_+ + \frac{\beta^2}{2}l_+ + \beta\alpha a \\ -\alpha a - \beta l_+ \\ 0 \\ \frac{1}{2}l_+ - \frac{\beta^2}{2}l_+ - \beta\alpha a \end{pmatrix}.$$

Observe that by judicious choice of α and β, followed by the use of the generators X_1, X_2 which leave not only $l_+ = t + z$ but also x unchanged, it is possible to transform the convenient point to any other point in $\mathbb{R}^4$ with the same sign of l_+. We have $\beta = -\frac{(x+\alpha a)}{l_+} = -\frac{\left(x - \frac{l_+}{4R}a\right)}{l_+} = \frac{-x}{l_+} + \frac{a}{4R}$ and $\exp(-2\alpha)R = \frac{1}{2}l_+$. Therefore, the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector field $V(x) = \Lambda(x)V(p_0)$ is

$$\begin{aligned} \Lambda(x) &= \exp(\beta B_3)\exp(\alpha B_2) \\ &= \begin{pmatrix} 1 + \frac{1}{2}\left(-\frac{x}{l_+} + \frac{a}{4R}\right)^2 & \left(\frac{x}{l_+} - \frac{a}{4R}\right) & 0 & \frac{1}{2}\left(-\frac{x}{l_+} + \frac{a}{4R}\right)^2 \\ \left(\frac{x}{l_+} - \frac{a}{4R}\right) & 1 & 0 & \left(\frac{x}{l_+} - \frac{a}{4R}\right) \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2}\left(-\frac{x}{l_+} + \frac{a}{4R}\right)^2 & -\left(\frac{x}{l_+} - \frac{a}{4R}\right) & 0 & 1 - \frac{1}{2}\left(-\frac{x}{l_+} + \frac{a}{4R}\right)^2 \end{pmatrix} \\ &\times \begin{pmatrix} \frac{R}{l_+} + \frac{l_+}{4R} & 0 & 0 & -\frac{R}{l_+} + \frac{l_+}{4R} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{R}{l_+} + \frac{l_+}{4R} & 0 & 0 & \frac{R}{l_+} + \frac{l_+}{4R} \end{pmatrix} \end{aligned}$$

giving the form of the invariant vector field :

$$V^\mu(x) = \begin{pmatrix} (A-D)\frac{R}{l_+} + (A+D)\frac{l_+}{4R}\left(1 + \left(-\frac{x}{l_+} + \frac{a}{4R}\right)^2\right) + \left(\frac{x}{l_+} - \frac{a}{4R}\right)B \\ B + \left(\frac{x}{l_+} - \frac{a}{4R}\right)\left((A+D)\frac{l_+}{2R}\right) \\ C \\ (-A+D)\frac{R}{l_+} + (A+D)\frac{l_+}{4R}\left(1 - \left(-\frac{x}{l_+} + \frac{a}{4R}\right)^2\right) - \left(\frac{x}{l_+} - \frac{a}{4R}\right)B \end{pmatrix}. \tag{B.31}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = \frac{(A+D)}{R} \tag{B.32}$$

$$\Box^2 V^\mu(x) = \begin{pmatrix} -\frac{(A+D)}{2R\,l_+} \\ 0 \\ 0 \\ +\frac{(A+D)}{2R\,l_+} \end{pmatrix}. \tag{B.33}$$

Reduced classical equations:

Let u_μ take the form of the group invariant vector field V_μ and so $u_\mu(p_0) = (A, B, C, D)$. Checking (3.7) using the forms (B.31) and (B.33) requires

$$\begin{aligned} -\frac{A+D}{2R} &= \frac{\mu_0 q^2}{mc^2}\rho(A-D)R \\ A+D &= 0 \\ B &= 0 \\ C &= 0. \end{aligned}$$

Combining the first 2 of these equations, we reach the conclusion that either $\rho = 0$ or $(A - D) = 0$. But the requirement $u^\mu u_\mu = 1$ forbids the latter and so we conclude we must adopt the zero current solution, of no further interest.

Reduced gauge independent M-D equations

It has been possible to rule out any non-zero solutions quite quickly from the continuity equation, seen by substituting J^μ into (B.32) giving

$$A + D = 0.$$

Therefore the length of the current vector $J^\mu J_\mu = A^2 - B^2 - C^2 - D^2$ at the convenient point, and therefore at all other points, must be zero. We call this '*zero current from anti-timelike continuity condition*'.

B.2.7 $\widetilde{P}_{8,13}$: generated by $B_2 + aX_2, B_3, X_1, X_2 + bX_3$

Group invariant field

The condition $t+z=0$ defines a singular orbit that divides $\mathbb{R}^4$ into two separate generic orbits. We choose $p_0 = (R, 0, 0, R)$ as a convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. One of the generic orbits will arise from R positive, the other from R negative. This result uses the same transformations as the preceding, $P_{8,4}$, but the second transformation $\exp(\beta B_3)$ needs to be expressed now in terms of the l_+ and $x+by$ components of the mapped point, the latter component being the component that is now unchanged by the remaining 2 generators, X_1 and $X_2 + bX_3$. This gives $\beta = \frac{-x-by}{l_+} + \frac{a}{4R}$, and the form of the invariant vector field is:

$$V^\mu(x) = \begin{pmatrix} (A-D)\frac{R}{l_+} + (A+D)\frac{l_+}{4R}\left(1 + \left(-\frac{x+by}{l_+} + \frac{a}{4R}\right)^2\right) + \left(\frac{x+by}{l_+} - \frac{a}{4R}\right)B \\ B + \left(\frac{x+by}{l_+} - \frac{a}{4R}\right)\left((A+D)\frac{l_+}{2R}\right) \\ C \\ (-A+D)\frac{R}{l_+} + (A+D)\frac{l_+}{4R}\left(1 - \left(-\frac{x+by}{l_+} + \frac{a}{4R}\right)^2\right) - \left(\frac{x+by}{l_+} - \frac{a}{4R}\right)B \end{pmatrix}. \tag{B.34}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = \frac{(A+D)}{R} \tag{B.35}$$

$$\Box^2 V^\mu(x) = \begin{pmatrix} -\frac{(A+D)}{2R}\frac{1+b^2}{l_+} \\ 0 \\ 0 \\ +\frac{(A+D)}{2R}\frac{1+b^2}{l_+} \end{pmatrix}. \tag{B.36}$$

Reduced classical equations:

Let u_μ take the form of the group invariant vector field V_μ and so $u_\mu(p_0) = (A, B, C, D)$. Checking (3.7) using the forms (B.34) and (B.36) requires

$$\begin{aligned} -\frac{A+D}{2R}(1+b^2) &= \frac{\mu_0 q^2}{mc^2}\rho(A-D)R \\ A+D &= 0 \\ B &= 0 \\ C &= 0. \end{aligned}$$

Combining the first 2 of these equations, we reach the conclusion that either $\rho = 0$ or $(A-D) = 0$. But the requirement $u^\mu u_\mu = 1$ forbids the latter and so we conclude we must adopt the zero current solution, of no further interest.

Reduced gauge independent M-D equations

It has been possible to rule out any non-zero solutions quite quickly from the continuity equation, seen by substituting J^μ into (B.32) giving

$$A + D = 0.$$

Therefore the length of the current vector $J^\mu J_\mu = A^2 - B^2 - C^2 - D^2$ at the convenient point, and therefore at all other points, must be zero. We call this '*zero current from anti-timelike continuity condition*'.

B.2.8 $\widetilde{P}_{10,7}$: generated by $B_3 + X_4, B_4 + bX_3, X_1, X_2$

Group invariant field

We choose $p_0 = (0, 0, 0, 0)$ as the convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. Consider the transformation $\exp(\beta\,(B_4 + bX_3))\exp(\alpha\,(B_3 + X_4)) \cdot p_0$ of the convenient point. Orbits that are generated by the first generator $B_3 + X_4$ and pass through the origin have been solved giving identity (B.15). The two components of $B_4 + bX_3$ commute, as in identity (B.12).

The result is

$$\exp(\beta\,(B_4 + bX_3))\exp(\alpha\,(B_3 + X_4)) \cdot p_0 = \begin{pmatrix} \frac{1}{2}\alpha + \frac{1}{6}\alpha^3 + \frac{1}{2}\beta^2\alpha \\ -\frac{1}{2}\alpha^2 - \beta b \\ -\beta\alpha \\ \frac{1}{2}\alpha - \frac{1}{6}\alpha^3 - \frac{1}{2}\beta^2\alpha \end{pmatrix}.$$

Observe that by judicious choice of α and β followed by the use of the generators X_1 and X_2 which change l_- and y but leaves l_+ and x unchanged, it is possible to transform the convenient point to any other point in $\mathbb{R}^4$. Therefore, the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector field $V(x) = \Lambda(x)V(p_0)$ is

$$\Lambda(x) = \begin{pmatrix} 1 + \alpha^2/2 + \beta^2/2 & -\alpha & -\beta & \alpha^2/2 + \beta^2/2 \\ -\alpha & 0 & 0 & -\alpha \\ -\beta & 0 & 0 & -\beta \\ -\alpha^2/2 - \beta^2/2 & \alpha & \beta & 1 - \alpha^2/2 - \beta^2/2 \end{pmatrix}$$

where $\alpha = l_+$, $\beta = -\frac{1}{b}\left(x + \frac{\alpha^2}{2}\right) = -\frac{1}{b}\left(x + \frac{l_+^2}{2}\right)$, giving the form of the invariant vector field

$$V^\mu = \begin{pmatrix} A + \frac{1}{2}(l_+^2 + \frac{1}{b^2}\left(x + \frac{l_+^2}{2}\right)^2)(A + D) - l_+B + \frac{1}{b}\left(x + \frac{l_+^2}{2}\right)C \\ B - l_+(A + D) \\ C + \frac{1}{b}\left(x + \frac{l_+^2}{2}\right)(A + D) \\ D - \frac{1}{2}({l_+}^2 + \frac{1}{b^2}\left(x + \frac{l_+^2}{2}\right)^2)(A + D) + l_+B - \frac{1}{b}\left(x + \frac{l_+^2}{2}\right)C \end{pmatrix}. \tag{B.37}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = \frac{dV_0}{dl_+}\frac{\partial l_+}{\partial t} + \frac{dV_3}{dl_+}\frac{\partial l_+}{\partial z} = 0$$

$$\Box^2 V^\mu(x) = \begin{pmatrix} -\frac{(A+D)}{b^2} \\ 0 \\ 0 \\ \frac{(A+D)}{b^2} \end{pmatrix}. \tag{B.38}$$

Reduced classical equations:

Let u_μ take the form of the group invariant vector field V_μ and so $u_\mu(p_0) = (A, B, C, D)$. Checking (3.7) using (B.37) requires that

$$\begin{aligned} 0 &= B\rho \\ 0 &= C\rho \\ 0 &= (A + D)\rho \\ -\frac{A + D}{b^2} &= \frac{\mu_0 q^2}{mc^2}\rho A \\ \frac{A + D}{b^2} &= \frac{\mu_0 q^2}{mc^2}\rho D. \end{aligned}$$

Combining the above, either (i) $\rho = 0$ or (ii) $B = 0$, $C = 0$, $A + D = 0$ $\implies$ $A = 0$ and $D = 0$. Therefore this is a zero solution.

Reduced gauge independent M-D equations

This case has been looked at using Mathematica. Amongst the cases that have been studied, this one was unusual in that the expression for the electromagnetic field tensor in terms of invariant four-current and axial currents gave non-zero curl($m^\mu \partial_\nu n_\mu$). Applying the general form (B.37) to u^μ and k^μ,

$$u^\mu = \begin{pmatrix} u_a + \frac{1}{2}(l_+^2 + \frac{1}{b^2}\left(x + \frac{l_+^2}{2}\right)^2)(u_a + u_d) - l_+ u_b + \frac{1}{b}\left(x + \frac{l_+^2}{2}\right) u_c \\ u_b - l_+(u_a + u_d) \\ u_c + \frac{1}{b}\left(x + \frac{l_+^2}{2}\right)(u_a + u_d) \\ u_d - \frac{1}{2}({l_+}^2 + \frac{1}{b^2}\left(x + \frac{l_+^2}{2}\right)^2)(u_a + u_d) + l_+ u_b - \frac{1}{b}\left(x + \frac{l_+^2}{2}\right) u_c \end{pmatrix} \tag{B.39}$$

and

$$k^\mu = \begin{pmatrix} k_a + \frac{1}{2}(l_+^2 + \frac{1}{b^2}\left(x + \frac{l_+^2}{2}\right)^2)(k_a + k_d) - l_+ k_b + \frac{1}{b}\left(x + \frac{l_+^2}{2}\right) k_c \\ k_b - l_+(k_a + k_d) \\ k_c + \frac{1}{b}\left(x + \frac{l_+^2}{2}\right)(k_a + k_d) \\ k_d - \frac{1}{2}({l_+}^2 + \frac{1}{b^2}\left(x + \frac{l_+^2}{2}\right)^2)(k_a + k_d) + l_+ k_b - \frac{1}{b}\left(x + \frac{l_+^2}{2}\right) k_c \end{pmatrix}.$$

The calculation of $\mathfrak{B}_\nu$ (6.1), followed by $qF^{\mu\nu}$(6.3) and $q\partial_\mu F^{\mu\nu}$, was enormously complicated, giving

$$q\partial_\mu F^{\mu\nu} = \begin{pmatrix} \left\{ \begin{array}{c} (u_a + u_d) m \cos\beta + ((u_a + u_d)k_c - u_c(k_a + k_d)) \\ \times((k_a + k_d)^2 - (u_a + u_d)^2) \end{array} \right\} b^{-2} \\ 0 \\ 0 \\ \left\{ \begin{array}{c} -(u_a + u_d) m \cos\beta + ((u_a + u_d)k_c - u_c(k_a + k_d)) \\ \times((k_a + k_d)^2 - (u_a + u_d)^2) \end{array} \right\} b^{-2} \end{pmatrix}.$$

From Maxwell's equations, we have that this must equal $\mu_0 q^2 \times$(B.39), from which we draw the following conclusions:

$$\begin{aligned} u_b &= 0 \\ u_c &= 0 \\ (u_a + u_d) &= 0. \end{aligned}$$

But we require

$$u^\mu u_\mu = u_a^2 - u_b^2 - u_c^2 - u_d^2 = 1$$

at the convenient point, and therefore no solution is possible in this case. We call this '*zero current from anti-timelike coupled condition*'.

B.2.9 $P_{11,2}$: generated by $\cos\phi B_1 + \sin\phi B_2, X_1, X_2, X_3$

We abbreviate

$$\begin{aligned} c &= \cos\phi \\ s &= \sin\phi. \end{aligned}$$

Group invariant field

The condition $t+z=0$ defines a singular orbit that divides $\mathbb{R}^4$ into two separate generic orbits. We choose $p_0 = (R,0,0,R)$ as a convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A,B,C,D)$. One of the generic orbits will arise from R positive, the other from R negative. Consider the transformation of the convenient point (using identity (B.14)),

$$\begin{aligned} \exp\alpha(cB_1+sB_2)\cdot p_0 &= \begin{pmatrix} s\cosh 2\alpha & 0 & 0 & -s\sinh 2\alpha \\ 0 & c\cos 2\alpha & -c\sin 2\alpha & 0 \\ 0 & c\sin 2\alpha & c\cos 2\alpha & 0 \\ -s\sinh 2\alpha & 0 & 0 & s\cosh 2\alpha \end{pmatrix} \begin{pmatrix} R \\ 0 \\ 0 \\ R \end{pmatrix} \\ &= \begin{pmatrix} (\cosh 2\alpha - \sinh 2\alpha)sR \\ 0 \\ 0 \\ (\cosh 2\alpha - \sinh 2\alpha)sR \end{pmatrix} = \begin{pmatrix} \exp(-2\alpha)sR \\ 0 \\ 0 \\ \exp(-2\alpha)sR \end{pmatrix}. \end{aligned}$$

By the correct choice of α, followed by the application of the other 3 generators, it is possible to transform the convenient point, p_0, to *any* other point with the same sign of $l+$. The family of points reached by applying first this generator is characterised by the value of $t+z$, which is unchanged by the action of the other generators. So $\exp(-2\alpha)sR = \frac{t+z}{2} \implies -2\alpha = \ln\left(\frac{t+z}{2sR}\right) \implies 2\alpha = \ln\left(\frac{2sR}{t+z}\right)$.

Therefore, the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector field $V(x) = \Lambda(x)V(p_0)$ is

$$\Lambda(x) = \begin{pmatrix} \frac{sR}{t+z} + \frac{t+z}{4sR} & 0 & 0 & -\frac{sR}{t+z} + \frac{t+z}{4sR} \\ 0 & c\cos\left(\ln\left(\frac{2sR}{t+z}\right)\right) & -c\sin\left(\ln\left(\frac{2sR}{t+z}\right)\right) & 0 \\ 0 & c\sin\left(\ln\left(\frac{2sR}{t+z}\right)\right) & c\cos\left(\ln\left(\frac{2sR}{t+z}\right)\right) & 0 \\ -\frac{sR}{t+z} + \frac{t+z}{4sR} & 0 & 0 & \frac{sR}{t+z} + \frac{t+z}{4sR} \end{pmatrix}.$$

Then the invariant vector field is

$$V^\mu(x) = \begin{pmatrix} A\left(\frac{sR}{t+z} + \frac{t+z}{4sR}\right) + D\left(-\frac{sR}{t+z} + \frac{t+z}{4sR}\right) \\ Bc\cos\left(\ln\left(\frac{2sR}{t+z}\right)\right) - Cc\sin\left(\ln\left(\frac{2sR}{t+z}\right)\right) \\ Bc\sin\left(\ln\left(\frac{2sR}{t+z}\right)\right) + Cc\cos\left(\ln\left(\frac{2sR}{t+z}\right)\right) \\ A\left(-\frac{sR}{t+z} + \frac{t+z}{4sR}\right) + D\left(\frac{sR}{t+z} + \frac{t+z}{4sR}\right) \end{pmatrix}. \tag{B.40}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = (A+D)\frac{1}{2sR} \tag{B.41}$$

By (B.1),

$$\Box^2 V^\mu(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{B.42}$$

Reduced classical equations:

Since (B.42) is identically zero, it follows from (3.7) that the current must be zero.

Reduced gauge independent M-D equations

It has been possible to rule out any non-zero solutions quite quickly from the continuity equation, seen by substituting J^μ into (B.41) giving

$$A + D = 0.$$

Therefore the length of the current vector $J^\mu J_\mu = A^2 - B^2 - C^2 - D^2$ at the convenient point, and therefore at all other points, must be zero. We call this '*zero current from anti-timelike continuity condition*'.

B.2.10 $\widetilde{P}_{12,11}$ / $\widetilde{P}_{12,12}$: generated by $B_1 \pm X_4, X_1, X_2, X_3$

Group invariant field

We choose $p_0 = (0,0,0,0)$ as the convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A,B,C,D)$. Consider the transformation of the convenient point

$$\begin{aligned} \exp\alpha\left(B_1 \pm X_4\right)\cdot p_0 &= \exp\pm\alpha X_4\cdot\left(\exp\alpha B_1\cdot p_0\right)\ , \text{ since } [B_1, X_4] = 0 \\ &= \begin{pmatrix} \pm\frac{1}{2}\alpha \\ 0 \\ 0 \\ \mp\frac{1}{2}\alpha \end{pmatrix}. \end{aligned}$$

By the correct choice of α, followed by the application of the other 3 generators, it is possible to transform the convenient point, p_0, to *any* other point in $\mathbb{R}^4$. The family of points reached by applying first this generator is characterised by the value of $(t-z)$, which is unchanged by the action of the other generators.

So $\alpha = \pm\frac{1}{2}(t-z)$, and the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector $V(x) = \Lambda(x)V(p_0)$ is

$$\begin{aligned} \Lambda(x) &= \exp(\alpha B_1) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t-z) & \mp\sin(t-z) & 0 \\ 0 & \pm\sin(t-z) & \cos(t-z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

giving the form of the invariant vector :

$$V^\mu(x) = \begin{pmatrix} A \\ \cos(t-z)B \mp \sin(t-z)C \\ \sin(t-z)B \pm \cos(t-z)C \\ D \end{pmatrix}. \tag{B.43}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = 0.$$

By (B.1),

$$\Box^2 V^\mu(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{B.44}$$

Reduced classical equations:

Since (B.44) is identically zero, it follows from (3.7) that the current must be zero.

Reduced gauge independent M-D equations

In Mathematica, the general form of vectors (B.43) has been substituted for u^μ,k^μ, into the real part of the inversion for $\mathfrak{B}^\mu$ (6.1), and subsequently into the calculation of the electromagnetic field tensor $F^{\mu\nu}$ from $\mathfrak{B}^\mu$,j^μ, and k^μ(6.3). From this, the divergence of $F^{\mu\nu}$ has been calculated. This calculation has given a zero result for the zeroth component for all x:

$$\partial_\mu F^{\mu 0} = 0. \tag{B.45}$$

Therefore, the inhomogeneous Maxwell's equations (6.5) requires

$$J_0 = 0.$$

But

$$\rho^2 = J_0^2 - J_1^2 - J_2^2 - J_3^2$$

and so $J_1 = 0$, $J_2 = 0$, $J_3 = 0$ and $\rho = 0$. We call this '*zero current by Gauss's Law*'.

B.2.11 $\widetilde{P}_{12,13}$: generated by $B_1 + a(X_1 + X_4), X_1 - X_4, X_2, X_3$

Group invariant field

We choose $p_0 = (0,0,0,0)$ as the convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. Consider the transformation of the convenient point

$$\exp(\alpha\,(B_1 + a(X_1 + X_4))) \cdot p_0 = \begin{pmatrix} \alpha a \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Observe that by judicious choice of α followed by the use of the generators $(X_1 - X_4), X_2, X_3$ which leave t unchanged, it is possible to transform the convenient point to any other point in $\mathbb{R}^4$. Therefore, the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector field $V(x) = \Lambda(x)V(p_0)$ is

$$\Lambda(x) = \exp(\alpha B_1)$$

where

$$\alpha = \frac{t}{a},$$

giving the form of the invariant vector field :

$$V^\mu(x) = \begin{pmatrix} A \\ \cos\frac{2t}{a}B - \sin\frac{2t}{a}C \\ \sin\frac{2t}{a}B + \cos\frac{2t}{a}C \\ D \end{pmatrix}. \tag{B.46}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = 0.$$

$$\Box^2 V^\mu(x) = -\frac{4}{\alpha^2}\begin{pmatrix} 0 \\ \cos\frac{2t}{a}B - \sin\frac{2t}{a}C \\ \sin\frac{2t}{a}B + \cos\frac{2t}{a}C \\ 0 \end{pmatrix}. \tag{B.47}$$

This conforms with the form (B.46), as it should, if we take $A' = D' = 0$, $B' = -\frac{4B}{\alpha^2}$, $C' = -\frac{4C}{\alpha^2}$ for the invariant form associated with $\Box^2 V^\mu$ which must be an invariant vector field too.

Reduced classical equations:

Let u_μ take the form of the group invariant vector field V_μ and so $u_\mu(p_0) = (A, B, C, D)$. Checking (3.7) using using the forms (B.46) and (B.47) leads to

$$\begin{aligned} A &= 0 \\ D &= 0. \end{aligned}$$

But

$$\rho^2 = A^2 - B^2 - C^2 - D^2,$$

and so

$$\begin{aligned} B &= 0 \\ C &= 0 \\ \rho &= 0. \end{aligned}$$

This is another manifestation of *'Zero current by Gauss's law'.*

Reduced gauge independent M-D equations

In Mathematica, the general form of vectors (B.46) has been substituted for u^μ,k^μ, into the real part of the inversion for $\mathfrak{B}^\mu$ (6.1) and subsequently into the calculation of the electromagnetic field tensor $F^{\mu\nu}$ from $\mathfrak{B}^\mu$, j^μ, k^μ(6.3). From this, the divergence of $F^{\mu\nu}$ has been calculated. This calculation has given a zero result for the zeroth component for all x:

$$\partial_\mu F^{\mu 0} = 0. \tag{B.48}$$

Therefore, the inhomogeneous Maxwell's equations (6.5) requires

$$J_0 = 0.$$

But

$$\rho^2 = J_0^2 - J_1^2 - J_2^2 - J_3^2$$

and so $J_1 = 0$, $J_2 = 0$, $J_3 = 0$ and $\rho = 0$. We call this '*zero current by Gauss's Law*'.

B.2.12 $\widetilde{P}_{12,14}$: generated by $B_1 + b(X_1 - X_4), X_1 + X_4, X_2, X_3$

Group invariant field

We choose $p_0 = (0, 0, 0, 0)$ as the convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. Consider the transformation of the convenient point

$$\exp(\alpha\,(B_1 + b(X_1 - X_4))) \cdot p_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\alpha b \end{pmatrix}.$$

Observe that by judicious choice of α followed by the use of the generators $X_1 + X_4, X_2, X_3$ which leave z unchanged, it is possible to transform the convenient point to any other point in $\mathbb{R}^4$. Therefore, the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector field $V(x) = \Lambda(x)V(p_0)$ is

$$\Lambda(x) = \exp(\alpha B_1)$$

where

$$\alpha = \frac{-z}{b},$$

giving the form of the invariant vector field :

$$V^\mu(x) = \begin{pmatrix} A \\ \cos\frac{2z}{b}B + \sin\frac{2z}{b}C \\ -\sin\frac{2z}{b}B + \cos\frac{2z}{b}C \\ D \end{pmatrix}. \tag{B.49}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = 0.$$

$$\Box^2 V^\mu(x) = -\frac{4}{b^2}\begin{pmatrix} 0 \\ \cos\frac{2z}{b}B - \sin\frac{2z}{b}C \\ \sin\frac{2z}{b}B + \cos\frac{2z}{b}C \\ 0 \end{pmatrix}. \tag{B.50}$$

This conforms with the form (B.49), as it should, if we take $A' = D' = 0$, $B' = -\frac{4B}{b^2}$, and $C' = -\frac{4C}{b^2}$ for the invariant form associated with $\Box^2 V^\mu$, which must be an invariant vector field too.

Reduced classical equations:

Let $u_\mu(p_0) = (A, B, C, D)$. Then checking (3.7) using the forms (B.49) and (B.50) leads to

$$\begin{aligned} A &= 0 \\ D &= 0, \text{ and so} \\ \rho &= 0. \end{aligned}$$

This is another manifestation of *'Zero current by Gauss's law'.*

Reduced gauge independent M-D equations

In Mathematica, the general form of vectors (B.49) has been substituted for u^μ,k^μ , into the real part of the inversion for $\mathfrak{B}^\mu$ (6.1) and subsequently into the calculation of the electromagnetic field tensor $F^{\mu\nu}$ from $\mathfrak{B}^\mu$,j^μ,k^μ(6.3). From this, the divergence of $F^{\mu\nu}$ has been calculated. This calculation has given a zero result for the zeroth component for all x:

$$\partial_\mu F^{\mu 0} = 0. \tag{B.51}$$

Therefore, the inhomogeneous Maxwell's equations (6.5) requires

$$J_0 = 0.$$

But

$$\rho^2 = J_0^2 - J_1^2 - J_2^2 - J_3^2$$

and so $J_1 = 0$, $J_2 = 0$, $J_3 = 0$ and $\rho = 0$. We call this '*zero current by Gauss's Law*'.

B.2.13 $P_{13,2}$: generated by B_2, X_1, X_2, X_3

Group invariant field

The condition $t + z = 0$ defines a singular orbit that divides $\mathbb{R}^4$ into two separate generic orbits. We choose $p_0 = (R, 0, 0, R)$ as a convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. One of the generic orbits will arise from R positive, the other from R negative. Consider the transformation of the convenient point

$$\exp(\alpha B_2)\cdot p_0 = \begin{pmatrix} \exp(-2\alpha)R \\ 0 \\ 0 \\ \exp(-2\alpha)R \end{pmatrix}.$$

By the correct choice of α, followed by the application of the other 3 generators, it is possible to transform the convenient point, p_0, to *any* other point with the same sign of l_+. The family of points reached by applying first this generator is characterised by the value of $(t+z)/2R$, which is unchanged by the action of the other generators.

So $\exp(-2\alpha)R = \frac{t+z}{2}$, and the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector field $V(x) = \Lambda(x)V(p_0)$ is

$$\begin{aligned} \Lambda(x) &= \exp(\alpha B_2) \\ &= \begin{pmatrix} \frac{R}{t+z}+\frac{t+z}{4R} & 0 & 0 & -\frac{R}{t+z}+\frac{t+z}{4R} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{R}{t+z}+\frac{t+z}{4R} & 0 & 0 & \frac{R}{t+z}+\frac{t+z}{4R} \end{pmatrix}. \end{aligned}$$

So the form of a vector at any point x^μ is determined from the reference vector at the convenient point p_0 by using this Lorentz transformation, dependent upon l_+ evaluated at x^μ:

$$\begin{aligned} V^\mu(x) &= \begin{pmatrix} A\left(\frac{R}{t+z}+\frac{t+z}{4R}\right)+D\left(-\frac{R}{t+z}+\frac{lt+z}{4R}\right) \\ B \\ C \\ A\left(-\frac{R}{t+z}+\frac{t+z}{4R}\right)+D\left(\frac{R}{t+z}+\frac{t+z}{4R}\right) \end{pmatrix} \\ &= \begin{pmatrix} (A-D)\frac{R}{t+z}+(A+D)\frac{t+z}{4R} \\ B \\ C \\ (-A+D)\frac{R}{t+z}+(A+D)\frac{t+z}{4R} \end{pmatrix}. \end{aligned} \tag{B.52}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = (A+D)\frac{1}{2R} \tag{B.53}$$

By (B.1),

$$\Box^2 V^\mu(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{B.54}$$

This conforms with the form (B.52), as it should, if we take $A' = B' = C' = D' = 0$ for the invariant form associated with $\Box^2 V^\mu$ which must be an invariant vector field too.

Reduced classical equations:

Since (B.54) is identically zero, it follows from (3.7) that the current must be zero.

Reduced gauge independent M-D equations

It has been possible to rule out any non-zero solutions quite quickly from the continuity equation, seen by substituting J^μ into (B.53) giving

$$A + D = 0.$$

Therefore the length of the current vector $J^\mu J_\mu = A^2 - B^2 - C^2 - D^2$ at the convenient point, and therefore at all other points, must be zero. We call this '*zero current from anti-timelike continuity condition*'.

B.2.14 $\widetilde{P}_{13,10}$: generated by $B_2 + aX_2, X_1, X_3, X_4$

Group invariant field

Set

$$\kappa \equiv \frac{2}{a}.$$

We choose $p_0 = (0,0,0,0)$ as the convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. Consider the transformation of the convenient point

$$\exp(\alpha(B_2 + aX_2)) \cdot p_0 = \begin{pmatrix} 0 \\ 0 \\ \alpha a \\ 0 \end{pmatrix}.$$

Observe that by judicious choice of α followed by the use of the generators X_1, X_3, X_4, which leaves y unchanged, it is possible to transform the convenient point to any other point in $\mathbb{R}^4$. Therefore, the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector field $V(x) = \Lambda(x)V(p_0)$ is

$$\begin{aligned} \Lambda(x) &= \exp(\alpha B_2) \\ &= \begin{pmatrix} \cosh\kappa y & 0 & 0 & -\sinh\kappa y \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\kappa y & 0 & 0 & \cosh\kappa y \end{pmatrix}, \end{aligned}$$

giving the form of the invariant vector field :

$$V^\mu(y) = \begin{pmatrix} V_a\cosh\kappa y - V_d\sinh\kappa y \\ V_b \\ V_c \\ -V_a\sinh\kappa y + V_d\cosh\kappa y \end{pmatrix}. \tag{B.55}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = 0$$

$$\Box^2 V^\mu = \begin{pmatrix} -\kappa^2(V_a \cosh \kappa y - V_d \sinh \kappa y) \\ 0 \\ 0 \\ -\kappa^2(-V_a \sinh \kappa y + V_d \cosh \kappa y) \end{pmatrix}. \tag{B.56}$$

Reduced classical equations:

Substitution of the above forms into the self-field classical approximation equations yields the following set of algebraic equations: Let u_μ take the form of the group invariant vector field V_μ and so $u_\mu(p_0) = (A, B, C, D)$. Checking (3.7)

$$\begin{aligned} \kappa^2 \begin{pmatrix} A \\ 0 \\ 0 \\ D \end{pmatrix} &= \frac{\mu_0 q^2 \rho}{mc^2} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \\ 1 &= A^2 - B^2 - C^2 - D^2 \end{aligned}$$

The solution is

$$\begin{aligned} A &= \cosh \alpha \\ B &= 0 \\ C &= 0 \\ D &= \sinh \alpha, \qquad \text{for some constant } \alpha, \end{aligned}$$

and

$$\rho = \frac{m\kappa^2}{\mu_0 q^2}.$$

The choice $\alpha = 0$ is the particular solution given at the end of chapter 3. Any other value of α is the same solution under a transformation

$$y \longrightarrow y - k,$$

and is therefore a group invariant solution under a group conjugate to $\widetilde{P}_{13,10}$. In any set of equations between vectors invariant under $\widetilde{P}_{13,10}$, it is always possible to make such a transformation so that *one* of the vectors is of this simpler general form,

$$V_1^\mu(y) = \begin{pmatrix} V_a \cosh \kappa y \\ V_b \\ V_c \\ -V_a \sinh \kappa y \end{pmatrix}. \tag{B.57}$$

Reduced gauge independent M-D equations

We substitute the above forms into the gauge invariant Maxwell Dirac equations. We choose to apply the simpler form (B.57) to u^μ

$$u^\mu(y) = \begin{pmatrix} u_a \cosh \kappa y \\ u_b \\ u_c \\ -u_a \sinh \kappa y \end{pmatrix}$$

and the general form (B.55) to k^μ:

$$k^\mu(y) = \begin{pmatrix} k_a \cosh \kappa y - k_d \sinh \kappa y \\ k_b \\ k_c \\ -k_a \sinh \kappa y + k_d \cosh \kappa y \end{pmatrix}.$$

The resulting set of reduced equations which have been given in chapter 6, along with interpretation of the solution, are:

$$0 = -2mc^2 \rho \sin \beta. \tag{B.58}$$

$$\mu_0 q^2 \rho u_a = +mc^2 \cos(\beta) \kappa^2 u_a + \frac{\hbar c}{2} \kappa^3 (u_b k_a - u_a k_b) \tag{B.59}$$
$$\mu_0 q^2 \rho u_b = 0 \tag{B.60}$$
$$\mu_0 q^2 \rho u_c = 0 \tag{B.61}$$
$$0 = \frac{1}{2} \kappa^3 u_b k_d \tag{B.62}$$

$$-k_c u_a + u_c k_a = 0 \tag{B.63}$$
$$u_c k_d = 0 \tag{B.64}$$
$$u_a k_d = 0. \tag{B.65}$$

$$1 = u_a^2 - u_b^2 - u_c^2 \tag{B.66}$$
$$-1 = k_a^2 - k_b^2 - k_c^2 - k_d^2 \tag{B.67}$$
$$0 = u_a k_a - u_b k_b - u_c k_c. \tag{B.68}$$

This has yielded the solution,

$$\begin{aligned} (u_a, u_b, u_c, 0) &= (1, 0, 0, 0) \\ (k_a, k_b, k_c, k_d) &= (0, \pm 1, 0, 0) \\ \beta &= 0 \\ \rho &= \frac{\kappa^2}{\mu_0 q^2} \left(mc^2 \mp \frac{\hbar c}{2} \kappa \right). \end{aligned}$$

The full derivation and interpretation has been presented in chapter 6, giving the reasons for ruling out $\cos \beta = -1$.

B.2.15 $\widetilde{P}_{14,10}$: generated by $B_3 + X_4, X_1, X_2, X_3$

Group invariant field

We choose $p_0 = (0,0,0,0)$ as the convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. Consider the transformation of the convenient point

$$\exp\alpha\,(B_3 + X_4)\cdot p_0 = \begin{pmatrix} \frac{1}{2}\alpha + \frac{1}{6}\alpha^3 \\ -\frac{1}{2}\alpha^2 \\ 0 \\ \frac{1}{2}\alpha - \frac{1}{6}\alpha^3 \end{pmatrix}.$$

By the correct choice of α, followed by the application of the other 3 generators, it is possible to transform the convenient point, p_0, to *any* other point in $\mathbb{R}^4$. The family of points reached by applying first this generator is characterised by the value of $(t+z)$, which is unchanged by the action of the other generators.

So $\alpha = (t+z)$, and (using (B.4)) the Lorentz transformation part can be written in terms of the coordinates of the point so reached

$$\begin{aligned} \Lambda(x) &= \exp(\alpha B_3) \\ &= \begin{pmatrix} 1+(t+z)^2/2 & -(t+z) & 0 & (t+z)^2/2 \\ -(t+z) & 0 & 0 & -(t+z) \\ 0 & 0 & 0 & 0 \\ -(t+z)^2/2 & (t+z) & 0 & 1-(t+z)^2/2 \end{pmatrix}. \end{aligned}$$

So the form of a vector at any point x^μ is determined from the reference vector at the convenient point p_0 by using this Lorentz transformation

$$V^\mu(x) = \begin{pmatrix} A + \frac{(t+z)^2}{2}(A+D) - (t+z)B \\ B - \frac{(t+z)}{2}(A+D) \\ C \\ D - \frac{(t+z)^2}{2}(A+D) + (t+z)B \end{pmatrix}. \tag{B.69}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu(x) = 0$$

By (B.1),

$$\Box^2 V^\mu(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{B.70}$$

Reduced classical equations:

Since (B.70) is identically zero, it follows from (3.7) that the current must be zero.

Reduced gauge independent M-D equations

In Mathematica, the general form of vectors (B.69) has been substituted for u^μ, k^μ, into the real part of the inversion for $\mathfrak{B}^\mu$ (6.1) and subsequently into the calculation of the electromagnetic field tensor $F^{\mu\nu}$ from $\mathfrak{B}^\mu$,j^μ,k^μ(6.3). From this, the divergence of $F^{\mu\nu}$ has been calculated. This calculation has given a zero result for the zeroth component for all x:

$$\partial_\mu F^{\mu 0} = 0. \tag{B.71}$$

Therefore, the inhomogeneous Maxwell's equations (6.5) requires

$$J_0 = 0.$$

But

$$\rho^2 = J_0^2 - J_1^2 - J_2^2 - J_3^2$$

and so $J_1 = 0$, $J_2 = 0$, $J_3 = 0$ and $\rho = 0$. We call this '*zero current by Gauss's Law*'.

B.2.16 $\widetilde{P}_{14,11}$ / $\widetilde{P}_{14,12}$: generated by $B_3 \pm X_2, X_1, X_3, X_4$

Group invariant field

We choose $p_0 = (0, 0, 0, 0)$ as the convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. Consider the transformation of the convenient point:

$$\begin{aligned}\exp(\alpha\,(B_3 \pm X_2)) \cdot p_0 &= \exp(\pm\alpha X_2) \cdot (\exp \alpha B_3 \cdot p_0)\ \text{, since } [B_3, X_2] = 0 \\ &= \begin{pmatrix} 0 \\ 0 \\ \pm\alpha \\ 0 \end{pmatrix}.\end{aligned}$$

Observe that by judicious choice of α, followed by the use of the translation generators X_1, X_3, X_4, which leave y unchanged, it is possible to transform the convenient point to any other point in $\mathbb{R}^4$. The family of points reached by applying first this generator is characterised by the value of y, which is unchanged by the action of the other generators. So $\alpha = \pm y$, and the Lorentz transformation $\Lambda(x)$ that gives the form of an invariant vector field $V(x) = \Lambda(x)V(p_0)$ is

$$\begin{aligned}\Lambda(x) &= \exp(\alpha B_3) \\ &= \begin{pmatrix} 1 + y^2/2 & \mp y & 0 & y^2/2 \\ \mp y & 0 & 0 & \mp y \\ 0 & 0 & 0 & 0 \\ -y^2/2 & \pm y & 0 & 1 - y^2/2 \end{pmatrix}\end{aligned}$$

giving the form of the invariant vector field :

$$V^\mu(x) = \begin{pmatrix} A + \frac{y^2}{2}(A + D) \mp yB \\ B \mp y(A + D) \\ C \\ D - \frac{y^2}{2}(A + D) \pm yB \end{pmatrix}. \tag{B.72}$$

Forms of certain derivatives:

$$\partial_\mu V^\mu = 0$$

$$\Box^2 V^\mu(x) = \begin{pmatrix} (A+D) \\ 0 \\ 0 \\ -(A+D) \end{pmatrix}. \tag{B.73}$$

This conforms with the form (B.72), as it should, if we take $A' = -D' = A + D$, $B' = C' = 0$ for the invariant form associated with $\Box^2 V^\mu$, which must be an invariant vector field too.

Reduced classical equations:

Let u_μ take the form of the group invariant vector field V_μ and so $u_\mu(p_0) = (A, B, C, D)$. Checking (3.7) using the forms (B.72) and (B.73) leads to the following algebraic equations:

$$-(A+D) = \frac{\mu_0 q^2}{mc^2}\rho A \tag{B.74}$$

$$(A+D) = \frac{\mu_0 q^2}{mc^2}\rho D \tag{B.75}$$

$$B = 0 \tag{B.76}$$

$$\pm(A+D)y = 0 \; \forall \; y \tag{B.77}$$

$$C = 0 \tag{B.78}$$

From (B.77), we get

$$(A+D) = 0$$

which when substituted into (B.74) and (B.75) gives either

$$A = 0$$
$$\text{and } D = 0, \tag{B.79}$$

or

$$\rho = 0. \tag{B.80}$$

But the requirement $u^\mu u_\mu = 1$ forbids (B.79), and so we conclude we must adopt the *zero current solution, of no further interest*.

Reduced gauge independent M-D equations

In Mathematica, it has been derived that the inhomogeneous Maxwell's equations are true if and only if

$$\begin{aligned} J_b &= 0 \\ J_a + J_d &= 0 \\ J_c &= 0 \\ J_a &= 0. \end{aligned}$$

Therefore the length of the current vector $J^\mu J_\mu = J_a^2 - {J_b}^2 - {J_c}^2 - {J_d}^2$ at the convenient point, and therefore at all other points, must be zero. We call this '*zero current from anti-timelike coupled condition*'.

B.2.17 $P_{15,0}$: generated by X_1, X_2, X_3, X_4

This group is the 4 dimensional abelian group of translations, $T(4)$.

Group invariant field

We choose $p_0 = (0, 0, 0, 0)$ as the convenient point, and a general vector at the convenient point $V^\mu(p_0) = (A, B, C, D)$. The action of any $g \in P_{15,0}$ on V^μ is to leave it unchanged. Therefore it comes as no surprise that group invariant vectors are constant in space:

$$\begin{aligned} V^\mu(x) &= V^\mu(0,0,0,0) \\ &= (A, B, C, D). \end{aligned}$$

Therefore, all derivatives of vectors vanish.

Reduced classical equations:

It follows that the D'Alembertian must be identically zero, so it follows from (3.7) that the current must be zero.

Reduced gauge independent M-D equations

From Maxwell's equations, the current is zero, and from the relation between $\mathfrak{B}^\mu$ and $F^{\mu\nu}$, the electromagnetic field is also zero. The only $P_{15,0}$ group invariant solution is the zero solution.

$$\rho = 0$$

We call this '*zero currents from zero derivatives'*.

Bibliography

[1] D J Arrigo, J M Hill, and P Broadbridge. Nonclassical symmetry reductions of the linear diffusion equation with a non linear source. *IMA Journal of Applied Mathematics*, 52:1–24, 1994.

[2] J Beckers, J Harnad, and P Jasselett. Spinor fields invariant under space-time transformations. *J. Math. Phys.*, 21:2491–2499, 1980.

[3] J Beckers, M Harnad, M Perroud, and P Winternitz. Tensor fields invariant under subgroups of the conformal group of space-time. *J. Math. Phys.*, 19:2126–2153, 1978.

[4] I Bialynicki-Birula. Canonical formulation of relativistic hydrodynamics. *Rep Mathematical Phys*, 4:139–151, 1973.

[5] I Bialynicki-Birula. Field theory of photon dust. *Acta Phys Polon B*, 23:553–559, 1992.

[6] I Bialynicki-Birula. Hydrodynamic form of the Weyl equation. *Acta Phys Polon B*, 26:1201–1208, 1995.

[7] G W Bluman and S Kumei. *Symmetries and Differential Equations*. Springer, New York, 1989.

[8] D Bohm and B Hiley. *The Undivided Universe*. Routledge, London, 1993.

[9] G Bond and E Short. Schrödinger's wrong equation. *Chemistry and Industry*, pages 12–16, 3 June 2002.

[10] H Booth, G Legg, and P Jarvis. Algebraic solution for the vector potential in the Dirac equation. *J Phys A : Math Gen*, 34:5667–5677, 2001.

[11] H Booth and C Radford. The Dirac-Maxwell equations with cylindrical symmetry. *J Math Phys*, 38:1257–1268, 1997.

[12] J Chadam. Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac equations in one space dimension. *Journal of Functional Analysis*, 13:173–184, 1973.

[13] J Chadam and R Glassey. On the Maxwell-Dirac equations with zero magnetic field and their solution in two space dimensions. *Journal of Mathematical Analysis and Applications*, 53:495–507, 1976.

[14] J Crawford. On the algebra of Dirac bispinor densities: Factorisation and inversion theorems. *J Math Phys*, 26:1439–1441, 1985.

[15] J Crawford. Dirac equation for bispinor densities. In Common A Chisholm J, editor, *Clifford Algebras and Their Applications in Mathematical Physics*, pages 353–361. D Reidel Publishing Company, 1986.

[16] A Das. General solutions of Maxwell-Dirac equations in 1+1 - dimensional space-time and a spatially confined solution. *J Math Phys*, 34:3986–3999, 1993.

[17] A Das and D Kay. A class of exact plane wave solutions of the Maxwell-Dirac equations. *J Math Phys*, 30:2280–2284, 1989.

[18] C Eliezer. A consistency condition for electron wave functions. *Camb Philos*, 54:247–250, 1958.

[19] M Flato, J Simon, and E Taflin. Asymptotic completeness, global existence and the infrared problem for the Maxwell-Dirac equations. *Memoirs of the American Mathematical Society*, 606:1–312, 1997.

[20] W I Fushchich, W M Shtelen, and N I Serov. *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics*. Kluwer Academic Publishers, Dordrecht, Boston, 1993.

[21] W I Fushchich and I M Tsifra. *Teor. Mat. Fiz.*, 64:41, 1985.

[22] W I Fushchich and R Z Zhdanov. Symmetry and exact solutions of nonlinear spinor equations. *Phys. Rep.*, 172:123–174, 1989.

[23] G Gabrielse. The magnet in the electron. *Physics World*, 20, no 2:32–36, February 2007.

[24] G. Gabrielse, D. Hanneke, T. Kinoshita, M. Nio, and B. Odom. New determination of the fine-structure constant from the electron g-value and QED. *Phys Rev Lett*, 97:30802, 2006.

[25] P Gerard, P Markowich, N Mauser, and F Poupaud. Homogenization limits and Wigner transforms. *Communications on Pure and Applied Mathematics*, I:323–379, 1997.

[26] R Glassey and W Strauss. Conservation laws for the classical Maxwell-Dirac and Klein-Gordon-Dirac equations. *J Math Phys*, 20:454–458, 1979.

[27] F Gliozzi. String-like topological excitations of the electromagnetic field. *Nuclear Physics B*, 141:379–390, 1978.

[28] L Gross. The Cauchy problem for the coupled Maxwell and Dirac equations. *Communications of Pure and Applied Mathematics*, 19:1–15, 1966.

[29] F Güngör. Exact solutions of a (2+1)-dimensional nonlinear Klein-Gordon equation. *Phys. Scripta.*, 61:385–390, 2000.

[30] F Gürsey. On two-component wave equations. *Phys Rev*, 77:844, 1950.

[31] F Gürsey. On a conform-invariant spinor wave equation. *Nuovo Cimento*, 3:988–1006, 1956.

[32] D Hestenes. *Space-Time Algebra*. Gordon and Breach, New York, 1966.

[33] D Hestenes. Real spinor fields. *Journal of Mathematical Physics*, 8:798–808, 1967.

[34] N H Ibragimov. *CRC Handbook of Lie Group Analysis of Differential Equations, Volume I: Symmetries, Exact Solutions, and Conservation Laws*. CRC Press, Boca Raton, Florida, 1993.

[35] N H Ibragimov. *CRC Handbook of Lie Group Analysis of Differential Equations, Volume II:Applications in Engineering and Physical Sciences.* CRC Press, Boca Raton, Florida, 1995.

[36] C Itzykson and J Zuber. *Quantum Field Theory.* McGraw-Hill Book Company, New York, 1980.

[37] F Kaempffer. Spinor electrodynamics as a dynamics of currents. *Physical Review D*, 23:918–921, 1981.

[38] E G Kalnins, W Miller, and G C Williams. Matrix operator symmetries of the Dirac equation and separation of variables. *J. Math. Phys.*, 27:1893–1900, 1986.

[39] J Kijowksi and G Rudolph. Spinor electrodynamics in terms of gauge invariant quantities. *Lett Math Phys*, 29:103, 1993.

[40] V F Kovalev, S V Krivenko, and V V Pustovalov. Symmetry group of Vlasov-Maxwell equations. *Nonlinear Mathematical Physics*, 3:175–180, 1996.

[41] G Legg. A tensor formulation of the Dirac equation. In David Neilson, editor, *Australian Institute of Physics 15th Biennial Congress 2002 CD ROM ISBN 0-9598064-4-X*, pages 162–164, www.causal.on.net, July 2002. Causal Productions.

[42] S Lie. Theorie der partiellen differentialgleichungen beliebeger ordnung. *Leipz Berichte*, 47:53–128, 1895.

[43] A Lisi. A solitary wave solution of the Maxwell-Dirac equations. *Journal of Physics A: Math and Gen*, 28:5385–5392, 1995.

[44] P Lounesto. Clifford algebras and spinor operators. In Baylis W, editor, *Clifford (Geometric) Algebras with Applications to Physics, Mathematics, and Engineering*, pages 5–35. Birkhauser, Boston, 1996.

[45] N Mauser and S Selberg. Convergence of the Dirac-Maxwell system to the Vlasov-Poisson system. *Communications in PDE*, 32:503–524, 2007.

[46] E Merzbacher. *Quantum Mechanics, Second Edition.* John Wiley and Sons, New York, 1970.

[47] J Mickelsson. On a relation between Maxwell and Dirac theories. *Letters in Mathematical Physics*, 6:221–230, 1982.

[48] J Mickelsson. On the unification of electroweak interactions with gravity. *Czech J Phys B*, 32:521–528, 1982.

[49] B. Odom, D. Hanneke, B. D'Urso, and G. Gabrielse. New measurement of the electron magnetic moment using a one-electron quantum cyclotron. *Phys Rev Lett*, 97:030801, 2006.

[50] P J Olver. *Applications of Lie Groups to Differential Equations.* Springer, New York, 1991.

[51] L V Ovsiannikov. *Group Analysis of Differential Equations.* Academic Press, New York London, 1982.

[52] J Patera, P Winternitz, and H Zassenhaus. Continuous subgroups of the fundamental groups of physics. 1 General method and Poincaré group. *J. Math. Phys.*, 16:1597–1614, 1975.

[53] J Patera, P Winternitz, and H Zassenhaus. Continuous subgroups of the fundamental groups of physics. 2 The similitude group. *J. Math. Phys.*, 16:1615, 1975.

[54] C Radford. Localized solutions of the Dirac-Maxwell equations. *J Math Phys*, 37:4418–4433, 1996.

[55] C Radford. The stationary Maxwell-Dirac equations. *J Phys A : Math. Gen.*, 36:5663–5681, 2003.

[56] C Radford and H Booth. Magnetic monopoles, electric neutrality and the static Maxwell-Dirac equations. *J Math Phys A: Math Gen*, 32:5807–22, 1999.

[57] F Reifler and R Morris. Unification of the Dirac and Einstein lagrangians in a tetrad model. *J Math Phys*, 36:1741–1752, 1995.

[58] T Takabayasi. Relativistic hydrodynamics of the Dirac matter. *Suppl Prog Theor Phys*, 4:1–80, 1957.

[59] T Takabayasi. Theory of the Dirac field as a continuous assembly of small rotating bodies. *Nuovo Cimento*, 7:118–121, 1958.

[60] Y Takahashi. Reconstruction of a spinor via Fierz identities. *Physical Review D*, 26:2169–2171, 1982.

[61] Y Takahashi. The Fierz identities - a passage between spinors and tensors. *J Math Phys*, 24:1783–1790, 1983.

[62] S A Vladimirov. *Symmetry Groups for Differential Equations and Relativistic Fields*. Atomizdat, Moscow, 1979.

[63] M Wakano. Intensely localized solutions of the classical Dirac-Maxwell field equations. *Progress of Theoretical Physics*, 35:1117–1141, 1966.

[64] E Whittaker. On the relations of the tensor-calculus to the spinor-calculus. *Proceedings of the Royal Society (of London)*, A 158:38–46, 1937.

[65] V Zhelnorovich. Spinor as an invariant. *J App Math Mech*, 30:1289–1300, 1966.