

On divided-difference semigroups

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Abstract

The Newtonian divided-difference operators generate the nil-Coxeter algebra and semigroup. A bijective correspondence between the nil-Coxeter semigroup and the symmetric group is used to provide braid-like diagrams for the former, and corresponding Reidemeister-type moves for the relations. Conditions are given for similar relations to hold in a skew group ring. Interesting extensions of the nil-Coxeter semigroup are described and given diagrammatic representations.

Keywords: divided-differences, nil-Coxeter algebra, Bruhat order, diagram semigroups, presentations.

1 Introduction

1.1 Newton's divided-difference operators and the Nil-Coxeter semigroup relations

Let K be a field. The defining action of the symmetric group \mathbb{S}_n on $\mathbf{n} = \{1, 2, \dots, n\}$ induces actions on the vector space $V = K^n$ and on the field of rational functions $K[z_1, z_2, \dots, z_n]$. We write these as left actions defined, for $\pi \in \mathbb{S}_n$, by

$$\begin{aligned}\pi : i &\mapsto \pi i \quad (i \in \mathbf{n}), \\ \pi \mathbf{z} &= (z_{\pi i}) \quad (\mathbf{x} = (z_i) \in V), \\ \pi f(\mathbf{z}) &= f(\pi \mathbf{z}) \quad (f \in K[\mathbf{z}]);\end{aligned}$$

K is the fixed field, and the action on the polynomial ring $K(\mathbf{z})$ is a sub-act.

\mathbb{S}_n is generated by transpositions $\sigma_{ij} = (i \ j)$, $(i, j = 1 \dots n - 1)$, in fact by the simple transpositions $\sigma_i = \sigma_{i, i+1}$. Define $\partial_{ij} : K[\mathbf{z}] \longrightarrow K[\mathbf{z}]$ by

$$\partial_{ij} f = \frac{f - \sigma_{ij} f}{z_i - z_j}$$

and denote by ∂_i the *simple* divided-difference $\partial_{i, i+1}$. By the factor theorem, the ∂_{ij} map $K(\mathbf{z}) \longrightarrow K(\mathbf{z})$. As operators, $\partial_{ij} = (z_i - z_j)^{-1}(1 - \sigma_{ij})$, and so they interact with σ_{ij} according to

$$\sigma_{ij} \partial_{ij} = \partial_{ij}, \quad \partial_{ij} \sigma_{ij} = -\partial_{ij};$$

since, for any permutation π , $\pi\sigma_{ij}\pi^{-1} = \sigma_{\pi i, \pi j}$, we also have

$$(1) \quad \pi\partial_{ij}\pi^{-1} = (z_{\pi i} - z_{\pi j})^{-1} \pi (1 - \sigma_{ij}) \pi^{-1} = \partial_{\pi i, \pi j}.$$

Moreover, ∂_{ij} is a $(1, \sigma_{ij})$ -derivation in that

$$\partial_{ij}fg = f(\partial_{ij}g) + (\partial_{ij}f)(\sigma_{ij}g) = (\partial_{ij}f)g + (\sigma_{ij}f)\partial_{ij}g.$$

Importantly for our concerns here, the ∂_i satisfy, for all $i, j \in \{1, \dots, n-1\}$, the relations

$$\begin{aligned} \partial_i^2 &= 0, \\ \partial_i\partial_{i+1}\partial_i &= \partial_{i+1}\partial_i\partial_{i+1}, \\ \partial_i\partial_j &= \partial_j\partial_i \quad (|i-j| > 1), \end{aligned}$$

reminiscent of those obtaining for the simple transpositions $\sigma_i = (i \ i+1)$ which generate the symmetric group in its presentation as a Coxeter group of type A_{n-1} . The algebra generated by these ∂_i is thus called the *nil-Coxeter* algebra of type A_{n-1} ; logically the semigroup ND_n presented by the generators and relations given above could be called the nil-Coxeter semigroup of type A_{n-1} .

In the single-variable case, these operators have a distinguished history in an idea of Newton's: given data (\mathbf{x}, \mathbf{y}) with $\mathbf{y} = f(\mathbf{x})$, we can sequentially fit interpolating polynomials $p_k(x)$ of degree k by

$$p_k(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) + \dots$$

such that c_k can be calculated without recomputing the c_j of lower degree terms. The coefficients are given by

$$\begin{aligned} c_0 &= f(x_1), \\ c_1 &= (x_1 - x_2)^{-1}(y_1 - y_2) = \partial_1 f(x_1), \\ c_2 &= (x_2 - x_3)^{-1}(y_2 - y_3) = \partial_2 f(x_2), \quad \text{etc.}, \end{aligned}$$

in notation which uses the action of \mathbb{S}_n on (\mathbf{x}, \mathbf{y}) . So ND_n may also be referred to as the *Newton's* or *lesser* divided-difference semigroup on n points, because we study it here as a member of a suite of related semigroups arising from divided-difference operators.

1.2 Coxeter groups, reduced words, Bruhat order.

However the story really begins with a Coxeter group G given by the presentation $\langle X \mid R \rangle \cong X^*/R^\sharp$, where R^\sharp is the congruence generated by R . For G , we take $X = \{x_1, x_2, \dots, x_{n-1}\}$ and $R = R_G = \{(x_i x_j)^{m_{ij}} = 1\}$, where (m_{ij}) is a symmetric $(n-1) \times (n-1)$ matrix with all $m_{ii} = 1$. It is convenient notation to define $(ab)^n$ to mean the initial segment of $(ab)^n$ of length n ; thus $(ab)^3 = aba$, etc. Then the relations R may be expressed as

$$\begin{aligned} (2) \quad & x_i^2 = 1 \\ (3) \quad & (x_i x_j)^{m_{ij}} = (x_j x_i)^{m_{ij}} \quad (i < j), \end{aligned}$$

in a form showing that (3) preserves the length of words in X^* . The values of m_{ij} are conventionally encoded in a Coxeter graph; the symmetric group has $m_{i,i+1} = 3$ and $m_{ij} = 2$ for $|i - j| > 1$. The words of minimum length in their R^\sharp -classes are called *reduced* words, and reduced words are related by transitions of the type (3). Any reduced word R^\sharp -equivalent to w may be denoted by $[w]$. The *length* $\ell(w)$ of a member of G is the length of any of its reduced words; $\ell(uv) \leq \ell(u) + \ell(v)$, with equality if and only if uv is already reduced (i.e., uv cannot be shortened by transitions of the kind (2)). There is a unique word w_0 of greatest length; in \mathbb{S}_n , $\ell(w_0) = n(n-1)/2$.

The *weak Bruhat* order \leq_l on G is the transitive closure of the covering relation, $u \prec v$ if and only if $x_i u = v$ and $\ell(u) + 1 = \ell(v)$. The *strong Bruhat* order on G , denoted \leq_s , is characterised (in [6], Ex. 2.1.8) by

$$(4) \quad u \leq_s v \iff v = w_1 u w_2 \text{ and } \ell(u) \leq \ell(v)$$

for some $w_1, w_2 \in G$. See also [5] for the Bruhat orders on \mathbb{S}_n ; they are of considerable importance in the combinatorial study of permutations.

1.3 Modifications; bijections

If we modify the relations for \mathbb{S}_n by retaining (3) but replacing (2) by

$$(5) \quad x_i^2 = x_i,$$

we get an idempotent-generated monoid M_n (or a semigroup, depending on the ambient variety); alternatively if we modify (2) to

$$(6) \quad x_i^2 = 0,$$

we get a nilpotent-generated semigroup ND_n in the variety of semigroups with zero. The corresponding sets of relations and congruences may be subscripted for disambiguation, thus: R_S, R_M, R_N . Transitions in R_S between reduced words may be chosen to be all of the type (3). So if u, v are reduced words, then $u R_S^\sharp v$ if and only if $u R_N^\sharp v$ if and only if $u R_M^\sharp v$; the R^\sharp -classes differ on non-reduced words, of course. But this means that the non-zero, non-identity elements of the three semigroups are in bijective correspondence; this extends to bijections between ND_n, \mathbb{S}_n , and M_n by associating $0, 1_S$ and 1_M . These bijections are not of course isomorphisms, but any representation of \mathbb{S}_n , for example by diagrams, transfers to a representation of ND_n or M_n . Only the multiplication rule has to be changed!

Divided-difference operators and their properties have been discussed extensively in the literature; in addition to references above, see [1], [2], [3] and [4]. As well, see [7], [10] for informal literature on the idempotent variant (5).

Before returning to consequences of these modified relations, concentrating on nilpotent-generated divided-difference semigroups, we detour to establish that they can be realised in certain operator-type rings. From now on, we consider only type A Coxeter groups, i.e. symmetric groups (so $m_{i,i+1} = 3$ and $m_{ij} = 2$ for $|i - j| > 1$).

2 Realisation in skew group rings

Let the symmetric group $\mathbb{S}_n \cong \langle s_i \rangle_{i=1}^{n-1}$ act (on the left) on a field F , and let $F\mathbb{S}_n$ be the corresponding skew group ring, to which the action extends by linearity and

$$s_j (a_i s_i) = (s_j a_i) s_j s_i.$$

See e.g. [8] for background on skew group rings.

Let us seek elements $t_i \neq 0$ of $F\mathbb{S}_n$, one for each i , satisfying the braid relations

$$(7) \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1},$$

$$(8) \quad t_i t_j = t_j t_i \quad (|i - j| > 1),$$

and with the reasonable side condition (which we dub “embeddability”) that these relations be preserved under the embeddings of any \mathbb{S}_m in \mathbb{S}_n ($m < n$) which result from the natural order-preserving embeddings of \mathbf{m} in \mathbf{n} . For example, \mathbb{S}_2 embeds in \mathbb{S}_n by each map $s_1 \mapsto s_i$ for $i = 1, \dots, n-1$, and $t_1 \in F\langle s_1 \rangle$, so $t_1 = a_1 1 + b_1 s_1$; thus

$$(9) \quad t_i = a_i 1 + b_i s_i,$$

i.e., $t_i \in F\langle s_i \rangle$, which we assume throughout.

We say that t_i has the *symmetrising* property if

$$(10) \quad s_i t_i = t_i,$$

which is equivalent to each of $s_i a_i = b_i$ or $s_i b_i = a_i$. The first proposition deals with the connection between (10) and a generalisation of the “squaring” relations (5) and (6).

Proposition 1 *t_i is symmetrising if and only if $t_i^2 = (a_i + b_i) t_i$.*

Proof. From (9) we have

$$t_i^2 = (a_i^2 + b_i (s_i b_i)) 1 + (a_i b_i + b_i (s_i a_i)) s_i,$$

while

$$(a_i + b_i) t_i = (a_i^2 + a_i b_i) 1 + (a_i b_i + b_i^2) s_i.$$

So these are equal if and only if $s_i a_i = b_i$. ■

It is also easy to see that $t_i t_j = t_j t_i$ is equivalent to $s_i s_j = s_j s_i$, $s_i a_j = a_j$ (and so, if t_i is symmetrising, to $s_i b_j = b_j$ and to $s_i t_j = t_j$). So (8) holds in $F\mathbb{S}_n$. Let us next turn to the braid relations (7).

Proposition 2 *If each t_i is symmetrising, then (7) is equivalent to the conjunction of*

$$(11) \quad s_{i+1} a_i = s_i a_{i+1} \quad (= c_i, \text{ say}),$$

$$(12) \quad a_i + b_i = q \in F, \text{ with } s_i q = q, \text{ and}$$

$$(13) \quad a_i a_{i+1} = c_i (a_i + a_{i+1} - q).$$

Proof. By expanding (7), equating coefficients, and cancelling with the use of $s_i a_i = b_i \neq 0$, we come to

$$\begin{aligned}
([1]) \quad & a_i a_{i+1} a_i + b_i (s_i a_{i+1}) a_i = a_{i+1} a_i a_{i+1} + b_{i+1} (s_{i+1} a_i) a_{i+1}, \\
([s]) \quad & a_i a_{i+1} + b_i (s_i a_{i+1}) = a_{i+1} (s_i a_{i+1}), \\
([s']) \quad & a_{i+1} a_i + b_{i+1} (s_{i+1} a_i) = a_i (s_{i+1} a_i), \\
([ss']) \quad & s_i s_{i+1} a_i = a_{i+1}, \\
([s's]) \quad & a_i = s_{i+1} s_i a_{i+1}, \\
([ss's]) \quad & b_i (s_i b_{i+1}) (s_i s_{i+1} b_i) = b_{i+1} (s_{i+1} b_i) (s_{i+1} s_i b_{i+1}).
\end{aligned}$$

From $([ss'])$ or the equivalent $([s's])$ we have (11). Now from $([s])$ and $([s'])$ we have $a_i a_{i+1} = (s_i a_{i+1}) (a_{i+1} - b_i) = (s_{i+1} a_i) (a_i - b_{i+1})$ and so $a_{i+1} - b_i = a_i - b_{i+1}$, i.e., $a_i + b_i = a_{i+1} + b_{i+1}$ for all i . Thus $q = a_i + b_i$ is also independent of i . Moreover, for all i , $s_i q = s_i a_i + s_i b_i = b_i + a_i = q$, so q is in the fixed field of the action of \mathbb{S}_n . So (12) holds. Returning to $([s])$ we have

$$a_i a_{i+1} = c_i (a_{i+1} - (q - a_i)),$$

giving (13).

Conversely, we can check by reversing these computations that (10), (11), (12) and (13) imply the coefficient equations $([s])$ to $([s's])$. Also from (13) we have

$$\begin{aligned}
a_i a_{i+1} (a_i - a_{i+1}) &= c_i (a_i + a_{i+1} - q) (a_i - a_{i+1}) = c_i (a_i - b_{i+1}) (a_i - a_{i+1}) \\
&= c_i (a_i (b_{i+1} - b_i) - b_{i+1} (a_i - a_{i+1})),
\end{aligned}$$

using $a_i - a_{i+1} = b_{i+1} - b_i$ from (12); so

$$a_i a_{i+1} (a_i - a_{i+1}) = c_i (-a_i b_i + b_{i+1} a_{i+1}),$$

which gives $([1])$. Finally

$$s_i s_{i+1} b_i = s_i s_{i+1} s_i a_i = s_{i+1} s_i s_{i+1} a_i = s_{i+1} a_{i+1} = b_{i+1},$$

giving $([ss's])$. ■

Thus we have a semigroup inside $F\mathbb{S}_n$ which is generated by $\{q1, t_1, t_2, \dots\}$ and satisfies (7,8) and a common generalisation of (5) and (6),

$$(14) \quad t_i^2 = q t_i, \text{ with } q = \mathbb{S}_n q \in F.$$

In particular, with $q = 1, 0$ we recover idempotent, resp. nilpotent, generators.

Corollaries. (i) In the case where $F = K(z_1, z_2, \dots, z_n)$, the choices $a_i = -b_i = (z_i - z_{i+1})^{-1}$ satisfy (11) and (12) with $q = 0$, hence also $c_i^{-1} = a_i^{-1} + a_{i+1}^{-1} = (z_i - z_{i+2})$, which is (13). This gives (returning to the use of σ_i for generators) $t_i = (z_i - z_{i+1})^{-1} (1 - \sigma_i) = \partial_i$.

(ii) For a solution with $q = 1$, we take $a_i = z_i (z_i - z_{i+1})^{-1}$ and $b_i = s_i a_i = -z_{i+1} (z_i - z_{i+1})^{-1}$. So (11) is met with $c_i = z_i (z_i - z_{i+2})^{-1}$, and we may calculate that

$$a_i^{-1} + a_{i+1}^{-1} - (a_i a_{i+1})^{-1} = 1 - \frac{z_{i+2}}{z_i} = c_i^{-1},$$

giving (13). Then

$$\begin{aligned} t_i &= z_i (z_i - z_{i+1})^{-1} 1 - z_{i+1} (z_i - z_{i+1})^{-1} \sigma_i \\ &= (z_i - z_{i+1})^{-1} (z_i 1 - z_{i+1} \sigma_i) = (z_i - z_{i+1})^{-1} (1 - \sigma_i) z_i 1 = \partial_i z_i 1, \end{aligned}$$

which is the *isobaric divided-difference* operator.

(iii) Similarly, a third solution is given by $a_i = z_{i+1} (z_i - z_{i+1})^{-1}$ and $b_i = s_i a_i = -z_i (z_i - z_{i+1})^{-1}$, thus with $a_{i+1} = z_{i+2} (z_{i+1} - z_{i+2})^{-1}$, $c_i = z_{i+2} (z_i - z_{i+2})^{-1}$ and $q = -1$.

These particular solutions are noted by Lascoux and Schützenberger [4]. See also [1] or [2]. The “symmetric part” operator, also idempotent, $\text{ev } f := \frac{1}{2} (1 + \sigma_i) f$, is of the form (9) with $a_i = b_i = \frac{1}{2}$. It does not satisfy (7) but is related to the isobaric divided-difference operator, by

$$\begin{aligned} \partial_i (z_i f) &= \partial z_i \cdot f + \sigma_i z_i \cdot \partial f = f + z_{i+1} \cdot \partial_i f \\ &= z_i \cdot \partial_i f + \partial_i z_i \cdot \sigma_i f = \sigma_i \cdot f + z_i \partial_i f \\ &= \frac{1}{2} (1 + \sigma_i) f + \frac{1}{2} (z_i + z_{i+1}) \partial_i f = \text{ev } f + \text{ev } z_i \cdot \partial_i f. \end{aligned}$$

3 Diagram semigroups and moves

For present purposes, though the concept is much broader, a *diagram* is a graph with n vertices, aligned in a $2 \times n$ matrix, and n edges (or *strands*) each incident on one vertex of the top row and one of the bottom row. The graph is drawn in the plane, with crossings allowed, and the edges are marked or decorated to show crossings of various kinds. Homotopy-equivalent drawings are regarded as representing the same diagram. There is also a notion of *moves*, (which may embody homotopy equivalence in a larger ambient space, for example) and which (together with homotopy equivalence) establish an equivalence relation, denoted by \sim , on diagrams. (Such moves mimic the familiar Reidemeister moves of knot theory, and are usually drawn so that irrelevant strands are omitted.)

Diagrams u and v (say) are concatenated by the following steps: (i) juxtaposing diagrams for u and v , with the left factor u above; (ii) identifying the lower row of vertices in u with the upper row of vertices in v ; (iii) erasing these “interior” points; and then (iv) simplifying the composite diagram according to the given moves, resulting in another diagram. The concatenations $u(vw)$ and $(uv)w$ are homotopy- and \sim -equivalent, and so the \sim -classes form a semigroup, which we call a *diagram semigroup*.

A standard example is provided by a representation \mathcal{S}_n of the symmetric group. Refer to Fig. 1, which shows schematically the diagram which is the image of s_i , one of the standard generators in the Coxeter presentation, and the product $s_i s_{i+1}$ in S_5 . Edges join vertices πi (in an upper row) and i (in the lower row); note the reversal of the usual order since we are treating these as left mappings. Here they will be drawn with braid-like crossings, that is, as a projection of a braid, but it will be irrelevant whether one strand crosses another “above” or “below”. See also Fig.2, which shows the moves for \mathcal{S}_n .

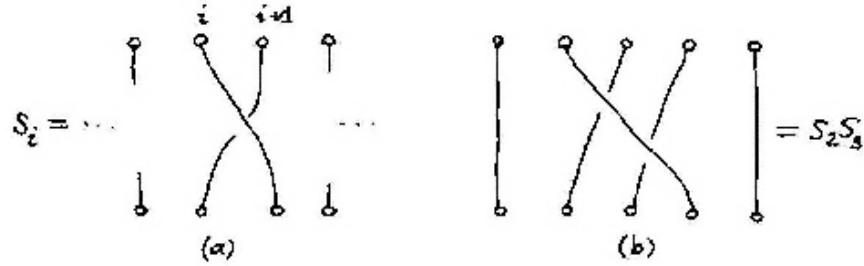


Figure 1: Diagrams for S_n : (a) the generator s_i ; (b) an example of a product in S_5 .

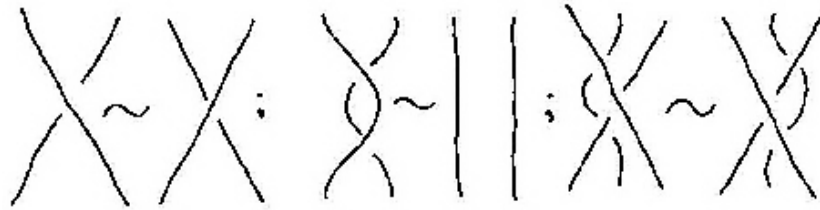


Figure 2: Moves for S_n .

3.1 More on presentations

Given a semigroup T , a set X , and $f : X \rightarrow T$, write f^+ for the unique homomorphism of X^+ to T which extends f . We say that relations $R \subseteq X^+ \times X^+$ are *satisfied in T* via f if for all $(u, v) \in R$, $f^+(u) = f^+(v)$, and as before write $R^\#$ for the congruence on X^+ generated by R . In this case, there is a unique *canonical* homomorphism $\phi : \langle X \mid R \rangle \rightarrow T$ such that $\phi|_X = f$; ϕ is surjective if and only if $f(X)$ is a set of generators for T . If T is a diagram semigroup, we say T has the *slice property* if every diagram in T may be drawn so that a finite number of horizontal lines, each intersecting the n strands, divides the diagram into layers such that each contains exactly one crossing. We say T has the *transition property* if for any move $s \sim t$ in T , and any valid assignment of vertices to the strands, there are $u, v \in X^+$ such that $f^+(u) = s$, $f^+(v) = t$, and u and v are connected by a sequence of elementary transitions in R . (Note that moves are diagrams of one or two layers in which vertices and irrelevant strands have been omitted, and thus represent collections of relations on “free” diagrams.) For a diagram semigroup T we have by standard methods

Proposition 3 *Let relations R be satisfied in T via $f : X \rightarrow T$, and let T have the slice property. Then the canonical homomorphism ϕ is*

- (i) *surjective if and only if each slice contains the image of a member of X ; and*
- (ii) *injective if and only if T has the transition property.*

Unfortunately this does not help to give a characterisation (e.g. geometric) of those diagrams which form the semigroup. At any rate, we will now be dealing with three kinds of semigroups, or rather, three levels of abstraction: at base, the level of operators

as expressed in section 1.1; above, the level of abstract presentations, as in section 1.2; and intermediate, the level of diagrammatic representations. These are distinguished notationally by the use of different typefaces: latin for the presentations (ND_n , etc.), calligraphic for the diagrams (\mathcal{ND}_n , etc.) and blackboard-boldface for the “real” (\mathbb{ND}_n , etc.).

4 The Newton’s divided-difference semigroup

We return to this semigroup, introduced in sections 1.1 and 1.3; it has presentation $\langle X \mid R \rangle$, with $X = \{x_i : i \in \mathbf{n} - 1\}$ and R composed of the relations (6), (7) and (8). Because of the bijection between S_n and ND_n noted in section 1.3, the members of ND_n may be represented by diagrams based on those for S_n . To distinguish the diagrams, we will use *nodes* on the crossings (drawn as filled circles) in place of the *bypasses* used for S_n . See Fig. 3 for the image of the generator d_i and corresponding moves for ND_n .

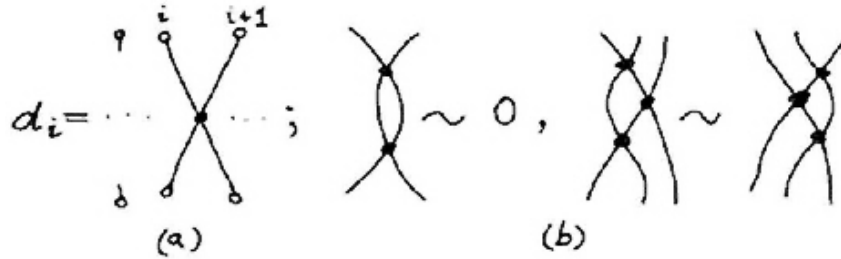


Figure 3: Diagrams and moves for ND_n : (a) the generator d_i ; (b) moves for ND_n .

Remark. We can produce a similar representation for the idempotent-generated monoid M_n , using (say) open circles at the crossings. The move shown in Fig. 4. replaces the first move in Fig. 3(b).



Figure 4: New move for M_n .

We may verify that the slice and transition properties hold in \mathcal{LD}_n , so the canonical map is an isomorphism, $LD_n \cong \mathcal{LD}_n$. The relations also hold in \mathbb{LD}_n , so $\phi : LD_n \rightarrow \mathcal{LD}_n$ is surjective. In fact it is bijective, since the Schubert polynomial $\mathfrak{S}_{w_0} = z_1^{n-1} z_2^{n-2} \dots z_{n-1}$ separates the members of \mathbb{ND}_n by

$$\partial_w \mathfrak{S}_{w_0} = \mathfrak{S}_{[w_0 w^{-1}]},$$

and the $\mathfrak{S}_{[w]}$ are distinct—in fact, a basis for P_{n-1} ([5], (4.13)).

4.1 Ideal structure and orders

ND_n is self-dual under the reversal of words (R^\sharp -equivalence is preserved because the words in (3) are symmetric under reversal). Denote by $\ell(u)$ the length function as described in section 1.2. So $\ell(u) \geq 1$ for all $u \neq 0$, and $\ell(uv) = \ell(u) + \ell(v)$ if $uv \neq 0$. Also observe that all non-reduced words in ND_n are equivalent to the zero, and if w is a reduced word, that $\ell(w)$ coincides with the (minimal) number of crossings in a diagram for w .

Proposition 4 *Let $u, v \in ND_n$. Then $(u, v) \in \mathcal{R}$ if and only if $u = v$.*

Proof. Suppose $(u, v) \in \mathcal{R}$. Then $u = 0$ if and only if $v = 0$, so suppose $u, v \neq 0$. Now $u = v$ or there exists $w \in ND_n$ such that $u = vw$, when $\ell(u) = \ell(v) + \ell(w) > \ell(v)$; similarly, $u = v$ or $\ell(v) > \ell(u)$. Hence the only possibility is that $u = v$. The reverse direction is trivial. ■

Hence the preorder \leq_R is actually a (partial) order; by lateral duality, \mathcal{L} is trivial and so also \mathcal{J} , and their associated preorders are really orders too. Define \leq'_R by

$$u \leq'_R v \text{ if and only if } 0 \neq u \leq_R v \text{ or } v = 0.$$

That is, the bottom element in \leq_R is moved to be the top element in \leq'_R . Similarly we define \leq'_J . The next proposition is immediate from the descriptions in section 1.2.

Proposition 5 *\leq'_R and \leq'_J are partial orders. Transferred to \mathbb{S}_n they coincide respectively with the weak and strong Bruhat orders.*

5 The greater divided-difference semigroup and its signed extension

Note that $\partial_{13}(z_1^2 z_2) = z_2(z_1 + z_3)$, which is symmetric with respect to neither σ_1 nor σ_2 ; thus $\partial_{13} \notin \mathbb{ND}_3$. By embeddability arguments, $\partial_{ij} \notin \mathbb{ND}_n$ for any $n \geq 3$ and $i \neq j \in \mathbf{n}$. This raises the question of studying the semigroup $\mathbb{GD}_n = \langle \partial_{ij} \mid i < j \in \mathbf{n} \rangle$. \mathbb{ND}_n is a subsemigroup, so (6), (7) and (8) still hold. Let $X = \{x_{ij} \mid i < j \in \mathbf{n}\}$ be a set of generators, and consider the relations

$$\begin{aligned} (15) \quad & x_{ij}^2 = 0, \text{ all } i < j, \\ (16) \quad & x_{ij}x_{kl} = x_{kl}x_{ij} \text{ for } i < j < k < l, \\ (17) \quad & x_{ij}x_{jk}x_{ij} = x_{jk}x_{ij}x_{jk} = x_{ij}x_{jk}x_{ik} \text{ for } i < j < k. \end{aligned}$$

Now let π be a permutation such that $\pi(1, 2, 3, 4) = (i, j, k, l)$. Then $\partial_{ij} = {}^\pi \partial_{12}$, $\partial_{jk} = {}^\pi \partial_{23}$, $\partial_{kl} = {}^\pi \partial_{34}$ and moreover

$$\begin{aligned} \partial_{12}\partial_{23}\partial_{13} &= \partial_1\partial_2\sigma_2\partial_1\sigma_2 = -\partial_1\partial_2\partial_1\sigma_2 \\ &= -\partial_2\partial_1\partial_2\sigma_2 = \partial_2\partial_1\partial_2. \end{aligned}$$

(These calculations take place in $\langle \mathbb{S}_n, \mathbb{N}\mathbb{D}_n \rangle$.) It follows that the relations 15 to 17, which we refer to collectively as R_G , hold in $\mathbb{G}\mathbb{D}_n$ via the map $x_{ij} \mapsto \partial_{ij}$. We may verify the slice and transition properties hold. It follows that there is a surjective homomorphism ϕ from $GD_n = \langle X | R \rangle$ to $\mathbb{G}\mathbb{D}_n$. There is also an isomorphism from GD_n to the semigroup \mathcal{GD}_n of diagrams generated by the set $\{d_{ij} \mid i < j \in \mathbf{n}\}$ (where d_{ij} is illustrated in Fig. 5(a)), subject to previous moves together with those illustrated in Fig. 5(b). What is

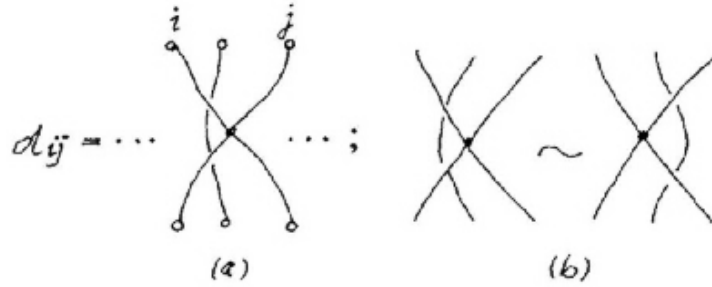


Figure 5: (a) Image of generator d_{ij} in \mathcal{GD}_n ; (b) extra move in \mathcal{GD}_n .

missing is a good geometric description of the diagrams which make up \mathcal{GD}_n , and a proof or counterexample of the injectivity of $\phi : GD_n \rightarrow \mathbb{G}\mathbb{D}_n$. Is there a set of polynomials which separate the members of $\mathbb{G}\mathbb{D}_n$? Perhaps the single Schubert polynomial \mathfrak{S}_{w_0} still does so in $\mathbb{G}\mathbb{D}_n$ as it does in $\mathbb{N}\mathbb{D}_n$.

We may remark that the transition in Fig. 6 is a non-move, as (calculating in $\mathbb{G}\mathbb{D}_n$) $\partial_{13}(z_1^2 z_3) = z_1 z_3$ and $\partial_1(z_1 z_3) = z_3 \neq -z_1 = \partial_2(z_1 z_3)$, so $d_1 d_{13} \neq \pm d_2 d_{13}$ in GD_n .

Structure. Define a length function in $\mathbb{G}\mathbb{D}_n$ as the usual length of a reduced word; for w reduced, $\ell(w)$ is also the number of *nodes* (not crossings) in the (reduced) diagram for w . Again we have, if $uv \neq 0$,

$$\ell(uv) = \ell(u) + \ell(v),$$

and the same argument as in Proposition 4 shows that $\mathbb{G}\mathbb{D}_n$ is \mathcal{J} -trivial.

There is another variant, the semigroup $\pm\mathbb{G}\mathbb{D}_n$ of *signed* divided-differences, in which (for $i < j$) ∂_{ji} is identified with $-\partial_{ij}$. The relations for $\pm\mathbb{G}\mathbb{D}_n$ include:

$$(18) \quad x_{ij} = -x_{ji}, \text{ all } i \neq j,$$

and we work in the variety of signed semigroups with zero, where there is a unary map $u \mapsto -u$ satisfying $-(-u) = u$ and $u(-v) = -uv$. Thus we may use generators x_{ij} with

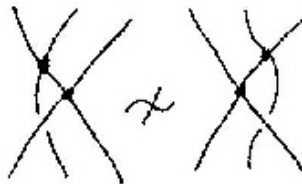


Figure 6: A non-move for \mathcal{GD}_n .

$i \neq j$, (18) is added, and the relations (15), (16) and (17) apply for $i \neq j$. The diagrams and moves are the same, but with minus signs included as appropriate to distinguish d_{ij} and d_{ji} , etc.

6 The permutation-divided-difference monoid

The discussion in the previous section in turn suggests the investigation of the operator monoid $\mathbb{P}\mathbb{D}_n = \langle \mathbb{S}_n, \mathbb{N}\mathbb{D}_n \rangle$ which we call the *permutation-divided-difference* monoid. The relevant variety now is that of signed monoids with zero and we work with the generating set $\{\sigma_i, \partial_i \mid i \in \mathbf{n} - \mathbf{1}\}$. We revert to notation which uses s_i and d_i for generators in the abstract version.

$\mathbb{N}\mathbb{D}_n$ and \mathbb{S}_n are subsemigroups, so the relations (2) and (3) for the s_i , and (7) and (8) for the d_i , continue to hold via the mapping $s_i \mapsto \sigma_i, d_i \mapsto \partial_i$. As well, for all $i \in \mathbf{n} - \mathbf{1}$, the extra relations

$$\begin{aligned} s_i d_i &= d_i = -d_i s_i, \\ s_i d_j &= d_j = d_j s_i \text{ for } |i - j| \geq 2, \text{ and} \\ s_i d_{i+1} s_i &= s_{i+1} d_i s_{i+1}, \end{aligned}$$

hold in $\mathbb{P}\mathbb{D}_n$; we call the set of all these relations R_{PD} . The (real) permutation-divided-difference semigroup is thus a homomorphic image of $\langle s_i, d_i \mid R_{PD} \rangle = PD_n$. We define $\mathcal{P}\mathcal{D}_n$ to be the set of all diagrams with crossings either nodes or bypasses. The moves of $\mathcal{S}_n, \mathcal{N}\mathcal{D}_n$ and $\mathcal{G}\mathcal{D}_n$ still hold, together with new ones shown in Fig. 7. Under composition of diagrams and use of these moves, $\mathcal{P}\mathcal{D}_n$ forms a semigroup, in fact a signed monoid with zero.

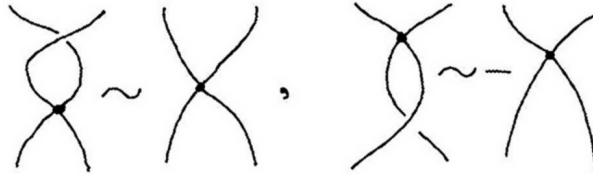


Figure 7: Extra moves for $\mathcal{P}\mathcal{D}_n$.

Then relations R_{PD} hold in $\mathcal{P}\mathcal{D}_n$ too, and, since the slice property also holds in $\mathcal{P}\mathcal{D}_n$, it is a homomorphic image of $\langle s_i, d_i \mid R_{PD} \rangle$. By inspection, the transition property holds, so $\mathcal{P}\mathcal{D}_n \cong PD_n$. Note that the group of units of PD_n is $\pm S_n = \{\pi, -\pi : \pi \in S_n\}$. We may extend from GD_n the definition of the length of a word in PD_n as the number of d -factors in w .

Structure. We shall say a monoid T is GJ^1 -factorisable if there are a subgroup G and a J -trivial subsemigroup J such that $T = GJ^1$.

Proposition 6 (i) if $w \in GD_n$, and $\pi \in \pm S_n$, then $\pi w \pi^{-1} \in GD_n$;
(ii) PD_n is GJ^1 -factorisable, with $G = \pm S_n$ and $J = GD_n$.

Proof. (i) Eqn. (1) confirms that the claim is true for words of length 1. Its truth for words of length k implies that if w has length $k + 1$, so $w = \mu w_1 \pi d_{ij} \rho$ with $\mu, \pi, \rho \in \pm S_n$ and w of length k , then

$$w = \mu w_1 \pi d_{ij} \rho = \mu \pi \rho \cdot (\pi \rho)^{-1} w_1 \pi \rho \cdot (\rho^{-1} d_{ij} \rho),$$

and thus implies its truth for $k + 1$.

(ii) Again by induction on the length of $a \in PD_n$. The inductive step relies on

$$a_1 = a d_i \rho \implies a_1 = \rho (\rho^{-1} a \rho) (\rho^{-1} d_i \rho),$$

where a, a_1 have lengths n and $n + 1$. ■

7 Summary

So what we know about the relations between these semigroups may be summarised in this diagram:

$$\begin{array}{ccccc} LD_n & \hookrightarrow & GD_n & \hookrightarrow & PD_n \\ \text{iso} \downarrow & & \text{iso} \downarrow & & \text{iso} \downarrow \\ \mathcal{L}\mathcal{D}_n & \hookrightarrow & \mathcal{G}\mathcal{D}_n & \hookrightarrow & \mathcal{P}\mathcal{D}_n \\ \text{iso} \downarrow & & ? \downarrow & & ? \downarrow \\ \mathbb{L}\mathbb{D}_n & \hookrightarrow & \mathbb{G}\mathbb{D}_n & \hookrightarrow & \mathbb{P}\mathbb{D}_n \end{array}$$

Clearly there is much to find out about their structure, combinatorics and relationships.

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